Introduction to Probabilistic Methods in PDE
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Lecture 38
Geometric Brownian motion

We have already seen in earlier lectures that the Brownian motion and the operator that Laplacian operator associated with that. And we have also seen the Feynman-Kac formula etc and we recall few things already we have seen and few things, you know which are immediate, but we have not discussed but which gives the power of calculations, etc.

So, these are the things we are going to discuss today and also we are going to understand why do we need to study some other stochastic processes beyond Brownian motion.

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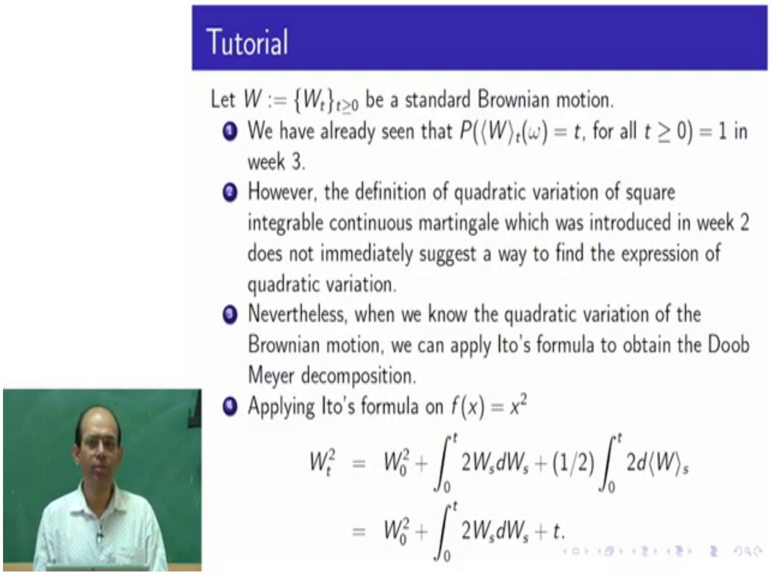

So, here assume that W is the standard Brownian motion as we do all the time. So, here W t, t greater or equals to 0, we have already seen in week 3 that probability that quadratic variation of W t is equal to t for all t positive, 0 or positive okay. So, the quadratic variation process is actually the identity function of time okay.

So, this thing already we have seen and there how did you compute that for computing that we have taken a sequence of partitions, and there, we have taken the increments. So, we have added square or the increments and we use the theorem that quantity, the summation of square of increments converging probability to the quadratic variation function.

Okay so, that was the thing what we have used to compute this thing okay, however, the definition of quadratic variation of square integrable continuous martingale which was introduced in week 2. So, that was done using the Doob Meyer decomposition correct. If you have a square integrable continuous martingale M, then M square is the sub martingale and that M square has Doob Meyer decomposition.

And in the Doob Meyer decomposition the non-martingale part, the part which is increasing part, so that constitutes the natural version of that constitutes to the quadratic variation of the process. So, that thing does not give you any indication how to compute quadratic variation, correct. So, that thing does not immediately suggest, we have to find the expression of quadratic variation.

Nevertheless, when we know the quadratic variation, I mean from this point also we have already obtained it of the Brownian motion. We can apply Ito's lemma to obtain the Doob Meyer decomposition, okay. So, the Doob Meyer decomposition we can obtain how do we do? We just take f of x is equal to x square, okay this function, okay. And then we apply f of W t apply Ito's formula f of W t.

So f W t, is W t square, so f of W t is equal to f of W 0 that is W 0 square plus first order derivative of f that is 2x, so 2 times W s okay evaluated at W s and (ds) d W s is running from 0 to t. Then Ito's formula also have second order term. So, plus half times 0 to t, second derivative of f, so what is second derivative? The 2x derivative, is this 2, correct? 2 and then the quadratic variation would appear or Brownian motion appear, okay.

And this already we have computed that these can be taken as just t here because this whole integration okay in this path, this is t, I mean okay so, I can write down s here, so half and 2 would cancel each other, so integration 0 to 2 of ds is t okay, t and here we have 2 W s ds, okay. So, W t square is now written as some constant, so W 0 is actually 0 because standard Brownian motion is 0 is nothing is here and here we have into stochastic integration of 2 times Brownian motion with this Brownian motion, okay.

So, this stochastic integration gives me a local martingale okay, and here this is the remaining term, okay. So this is also that the Doob Meyer decomposition of W t square okay. So this is obtained using the Ito's formula, okay.

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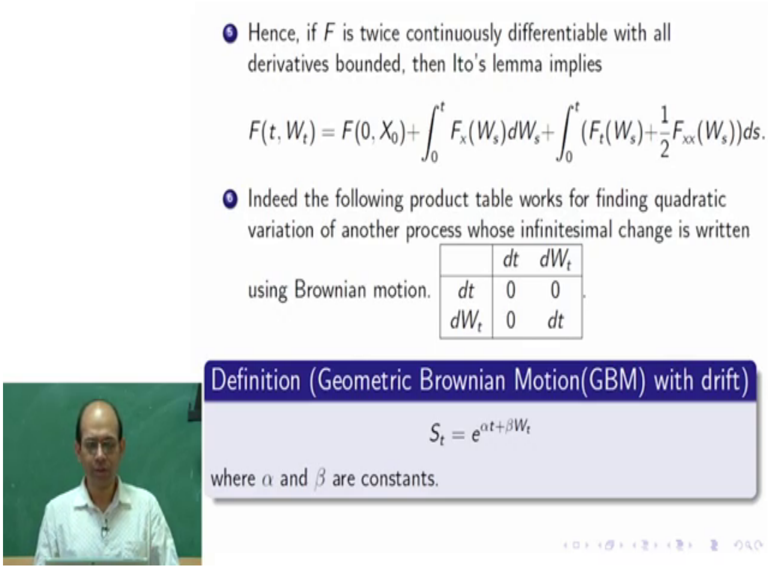

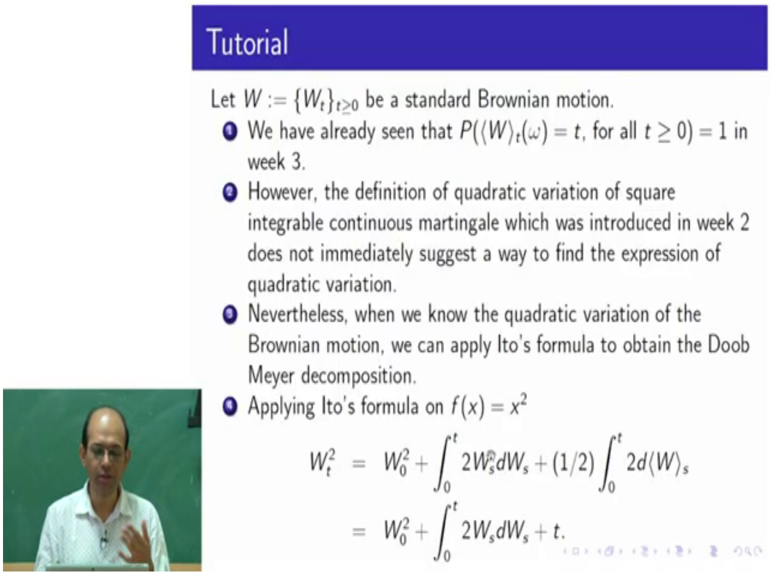

So, after talking about Brownian motion, now we also so this is just Ito's formula of function of x but function of x and t if we have, so this is a special case. If f is twice continuously differentiable with all derivatives bounded then Ito's lemma also implies this also recollection of Ito's lemma what we have already seen.

That f of t, W t is equal to f of 0, X 0. So, here it should be W 0 okay, so this is a typo, plus integration 0 to t first derivative of f with respect to first derivative of f okay, with respect to the second variable. So, that is why I am writing x that means the second variable. For first variable I will write t always. So f x, W s dW s 0 to t plus then first the derivative with respect to the first variable appears and then here I mean there I should have just small s W s.

This is missing here also small s Ws, the function of 2 variables correct. So, this s, s is missing here, d s and then second order term that would give me half times double derivative of f with respect to the second variable and then quadratic variation of W that is ds.

So, quadratic variation of Brownian motion is s, okay so that is the reason that these both we can club together okay, because club together can write this manner, okay. So these are the things which is more or less recollection but this is something we are going to use also for some interesting examples.

So, here we now address a question that how can we find out quadratic variation of some other processes? For Brownian motion we have obtained right, the definition of quadratic variation did not help much but we had to actually use the property of Brownian motion that normal random variables etc and then the fourth order moment of the normal random variable is 3 times of you know of the square of the variance etc.

All these things have used there. So, here indeed the following product table works for finding quadratic variation of another process whose infrared signal changes or increment if you say increment is retained is in Brownian motion okay. So, for that, so if I have a process X, so dX t is written in terms of Brownian motions, increments, etc. So, dX t whole square I can then find out okay.

And then actually I would have you know product of terms which has increment infinitesimal increments and then one should know that what are the, I mean what are the terms and how to manipulate those terms okay, this is not very clear. However, this table really works that whenever one has those type of products, so dt into dt one can put 0 dt and dW t one can put 0 and dW t into dWt one can put dt okay, so this product table really works for calculating quadratic variation of some other processes, okay.

So, here we first talk about some process which is other than Brownian motion, say S t okay, we define S t is equal to e to the power of alpha t plus beta times W t okay, where alpha and beta are some constants, okay. So, this has a name this is called geometric Brownian motion with drift okay. Why geometric? Because there is no geometry after all I mean it is just that product because Brownian motion has additive increment etc here that is in the power of e.

So like here, I mean the product type of things do appear here. So, that is why we call it geometric Brownian motion. So, we are going to study this S t, okay. So, we would like to write down this S t, the infinitesimal increment of S t in terms of infinitesimal increment of W t okay, so that is the kind of thing we want to do.

So, one can actually directly apply Ito's formula over this, this is a function of Brownian motion correct. One can apply this Ito's formula correct and by applying Ito's formula one can get dS t correct. Okay so, that one can get and one can get the formula, okay. So, that is one way to find out the change of dS t etc and then dS t square one can find out, okay. There is another way, okay.

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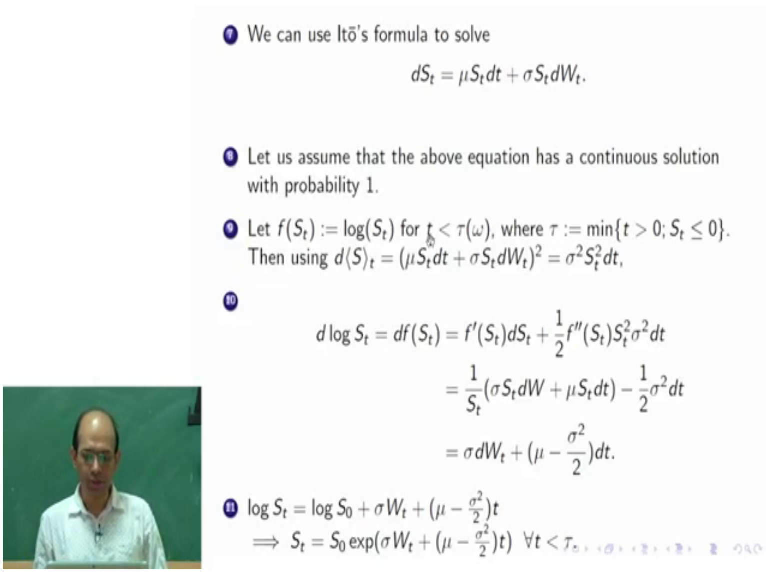

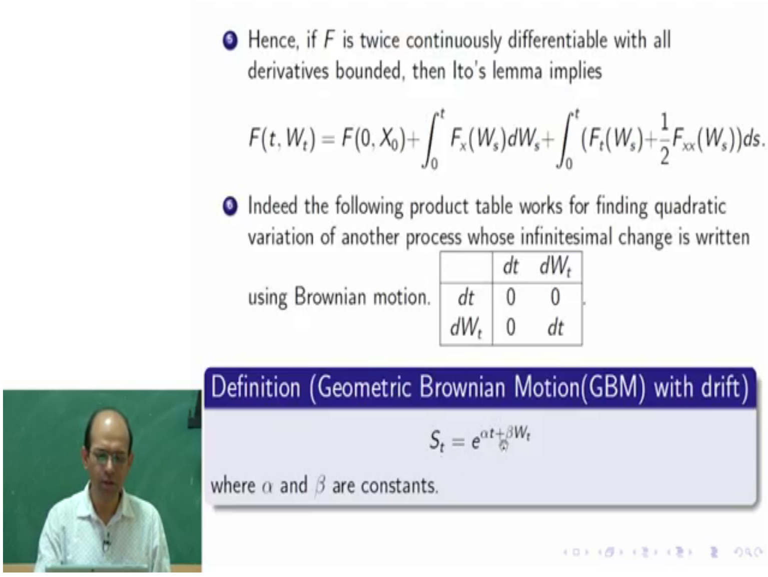

So, basically I am writing down what one would get. So, in general, so not with this alpha this beta but some other constant mu and sigma considered this type of description of a stochastic process where increment of the process, stochastic process at time t, dS t is equal to mu a constant times S t dt plus sigma S t dW t okay.

How are you going to understand this? We can understand this as a integral equation, correct. Because say for example, the integration sign is there, 0 to t, etc but that is I have just omitted and then S t's are unknown which appears both sides okay. So one can understand that this description as an equation. Okay so, we are going to see that, okay how are we going to solve these equation and what are you going to do?

But before that, let me explain that if dS t has this form, then the quadratic variation of S t would be just you take dS t whole square, then you get mu S t dt plus sigma S t dW t whole square term and then you use that product rule that product table, and then you would see the dt square term would not appear dt and dW t that cross terms would also not appear only this term would appear and then there will be sigma S t dW t whole square term would survive that would give me sigma square S t square dt, okay.

So, the quadratic variation of S t would become integration of sigma square S t square with respect to time t okay. So, that is the kind of thing however, we need to build the connection between this and this without building this connection, the whole discussion is baseless, correct? So, we are going to build that connection okay.

So, one connection as I told that okay, you use Ito's lemma on this and get it, so that you can do as an exercise another because there is nothing to do much it is just in direct application you would get it I mean one can see from here directly that okay, one would get this type of terms because the derivative of these would you get partial derivative, this x variable would get beta terms here and same e to the power S t dW s okay.

And then for time here, you get alpha terms. From here you would get half of beta square okay. So, you are going to get here, so we are not bothering much in that direction but the reverse direction. So, as you know that I do not have that I just have this thing okay. Then how to come up with that okay that solution, we can call them this thing as a solution of this equation. Okay so, we proceed this way.

So for the time being because we do not have any way to actually assert that this has a solution for the time being, we assume that the above equation has a continuous solution with probability one, okay. We assume that it has and when we assume and then you obtain that form and then we can do the reverse thing. We can actually use Ito's formula to cross verify that okay this is indeed the solution, okay.

So, assume that it has a continuous solution, I mean, this kind of treatment what I am doing now, is very similar as you have done in your college in your graduation level. When given an ODE ordinary differential equation, you try to find out the solution. Try to find out the form of the solution, correct? Without bothering much whether the solution exists or not okay, you believe that it exists and then applied some formula to solve it.

And this is actually a correct occasion to clarify the importance of Ito's lemma, because when you try to solve ordinary differential equation, what you actually have done is that you try to integrate, okay you try to integrate. You tried to apply the fundamental theorem of calculus that you wanted to see that it is an anti-derivative of some other function.

But now for stochastic calculus here that fundamental theorem of calculus does not hold because I mean in that form, so what is substituting is the Ito's formula because that actually, you know it shows that okay, what is the correct thing to look at. Okay so Ito's formula is the thing what you are going to use for this.

Okay so, assuming that S t has a I mean this has a solution then we use exactly the way we used to think in for solving ODE that since you know the unknown is multiplied here, so we are going to take log of these things, okay. So, we do exactly the same manner f of S t we define, f of S t is log of S t and then this is not defined when S t is negative correct. So, we can only define these for time t which is less than the exit time of the positive domain, okay.

So, the first time when S t becomes negative, I mean non-positive 0 or negative, okay so exit of the upper half plane okay exit time of the upper half plane. So, for all t which is before the exit time then it is well defined then I can define it okay, so after assuming that this equation has a solution I am considering f of S t and apply and I am going to apply Ito's formula on this okay.

So, here d S, the quadratic variation of S t also I am calculating as I have indicated earlier that it would be sigma square S t square dt. So, the increment of quadratic variation of S is sigma squared S t square dt okay. Now I am since I know all these things, so I can apply now Ito's formula. You remember that for applying Ito's formula of a function of a process I need quadratic variation of the process, correct?

So, far I just knew the quadratic variation of Brownian motion okay, but using that product table, I now write down this is the quadratic variation of S t okay, so we are going to now use the Ito's formula here, d log S t is equal to okay, now f the Ito's formula, the derivative comes f prime S t dS t plus half times double derivative of f S t, and then the quadratic variation, so quadratic variation is sigma square S t square dt. Okay this is just direct application of Ito's formula on this.

So, now here we compute. So, f of S t is log S t, so derivative is 1 over S t therefore, and here double derivative is minus 1 over S t square okay, so that thing would come, but S t square and this S t square would cancel each other, but I would retain the negative sign, so minus sigma square dt half is also there, half is there.

So, from this I am going to get this and from here this dS t, I already know this is the form, so I am going to write down dS t is equal to mu S t dt sigma S t dW t okay, so take here, okay. Here this subscript t is missing, 1 over S t sigma S t dW t plus mu S t dt. Now here I have S t and I have S t here and anyway my S t's are positive. So, you know these things all makes sense we can cancel these S t's, etc.

So, we get a very nice formula here. So, very nice thing, so here I just have sigma dW t from this part, from this part I have mu and from here minus half sigma square, so mu minus half sigma square by 2 dt. So, that is the form. Okay so, now I can integrate both sides from 0 to t. So, if we integrate this from 0 to t, I would get log S t minus log S 0. That log S 0 I transfer to the right hand side.

So, log of S t is equal to log of S 0 plus integration of this is very easy, the sigma times W t integration of this, this is just constant mu minus sigma square by 2 times t. And now, I take exponential both sides. So, it is S t is equal to S 0 times e to the power of this term, sigma W t plus mu minus sigma square by 2 t okay. But t is running from 0 to tau, okay.

I cannot assure that this is the expression for whole positive real line because this thing is only makes sense when t is less than tau. Now, if I can prove that tau is infinity with probability 1, we would be done. So, we need to do that okay, we need to do this.

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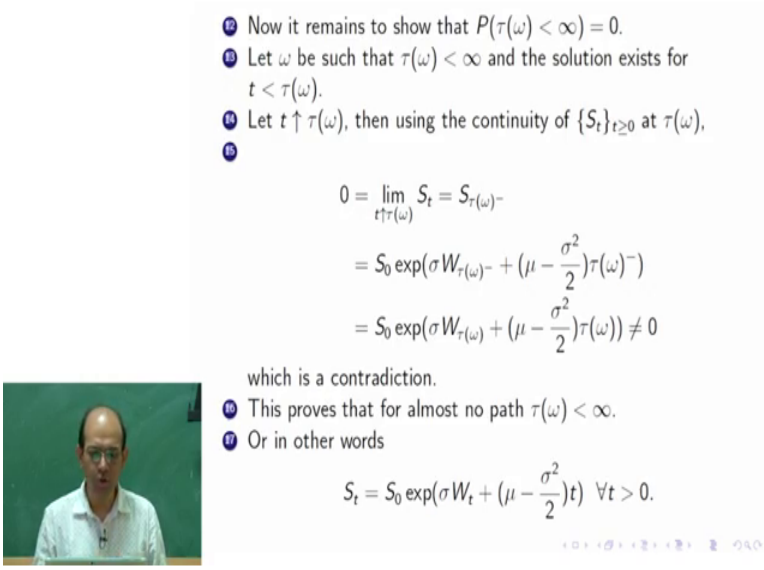

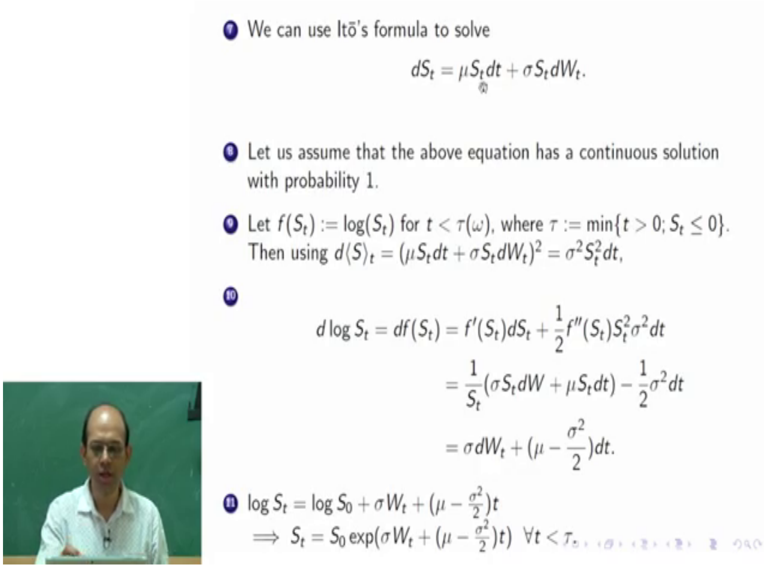

So, it remains to show that probability that tau omega is finite is 0 okay. So, you are going to show that okay tau omega finite that event would happen with probability 0. So, now let omega be such that tau omega is finite, okay. So, I would like to show that this class of omega is coming for 0 measure set. Let omega be such that tau omega is finite and the solution exists for, you know up to tau omega.

So, I cannot say t is equal to tau up to tau omega. So, let t, now we are letting t tends to tau omega okay, we are going to take the limit okay and then we are also going to use continuity of S t. So, here we are actually do not need to do much here. So, let us see what we do. So, here since S t is a continuous path solution. So, tau omega is the time when S t becomes 0 or negative, but before that it was positive.

So, if it has a continuous path at tau omega it should be 0, correct, it cannot be negative, okay. So, here limit t approaches to tau omega of S t that is S t omega minus. So, that is 0. So, now for this we know the expression of S t, we are writing that and allowing this thing t is represent tau omega.

Since all these functions are continuous functions because e exponential function okay, these are all continuous function. So, as t tends to t tau omega, this converges to here for that each and every omega. So, it S0t into e to the power of sigma W tau omega minus and also Brownian motion is continuous actually I can write down tau omega also here. So, mu minus sigma square by 2 tau omega.

So, this basically this goes to tau omega, the minus sign is not important here is makes no sense tau omega. So, here is this tau omega. So, this is all I mean same thing is written actually without minus sign okay. Now, this thing we have obtained, but we have assumed that tau is finite, so e to the power of some quantity here okay, some quantity which is almost surely, you know Brownian motion for every omega you get a finite value, correct.

Okay so, here this is non-zero surely because here this is just e to the some number, so that is that cannot be 0. So, we get a contradiction. So, what does it imply? So, this contradiction proves that so for almost no path we can see that tau omega is finite. We have arrived at this contradiction by assuming that tau omega is finite, correct. So, then I have arrived at this contradiction.

So, tau omega is obtained by the exit time of the solution. So, almost no solution paths are such that tau omega can be found finite or in other words when you are looking for solution S t is equal to S 0 into e to the power of sigma W t plus mu minus sigma square by 2 t for all t positive.

So, this is see, I mean this is geometric Brownian motion by definition because here I have Brownian motion times some constant here and here I have a drift term okay, so geometric

Brownian motion with drift. So, the definition when we have visited that alpha and beta, so this is alpha and this is beta, okay this is beta, good.

Now, when you, one obtains this one can say that okay I have obtained these by assuming that it has a solution, but how can you be sure that it has actually solution, but okay now I have this formula. Now from these you use Ito's formula on this function because this exists of course because Brownian motion is there it means the step here given okay and then we obtain that this solves that SDE is very straightforward to check that this actually solves that SDE.

Okay so, this gives you an idea that how one can solve a given SDE, okay. Just it is not very different from the things what you have done in your ODE course in undergrad level. But only thing is that the formula is different. And, I mean, here is some kind of you know, analysis which one has to be careful, one takes log function correct. So, this is nothing other than that.

Okay otherwise it is not much difficult thing. Okay so, next what we do is that, we would check what is the utility of this process, okay. So, this process is actually very popular for its application in modeling stock price market okay and that is also a field which gives rise to some other processes for which one may not actually find out such kind of closed solution, correct. So, this is no solution is in closed form.

And then for those applications, it also becomes very important to actually develop a theory for stochastic differential equations. So, we call these equations as stochastic differential equation because this is of course differential equation with some differentials involved, but it is random correct because the Brownian motions are there.

So, then it becomes important to develop a theory for stochastic differential equation that even one cannot find out one cannot solve the equation in known elementary in terms elementary function. But still can one assure existence of the solution, okay. So, that also becomes important. So, we are going to see those type of examples also.

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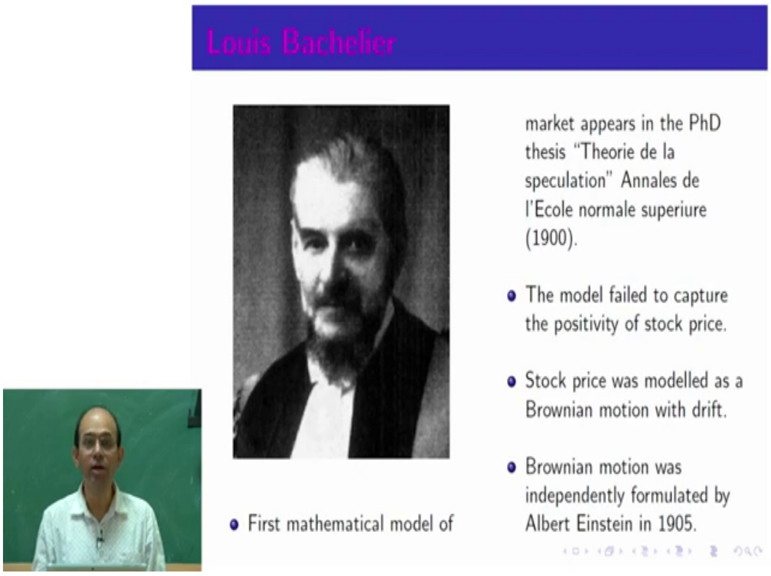

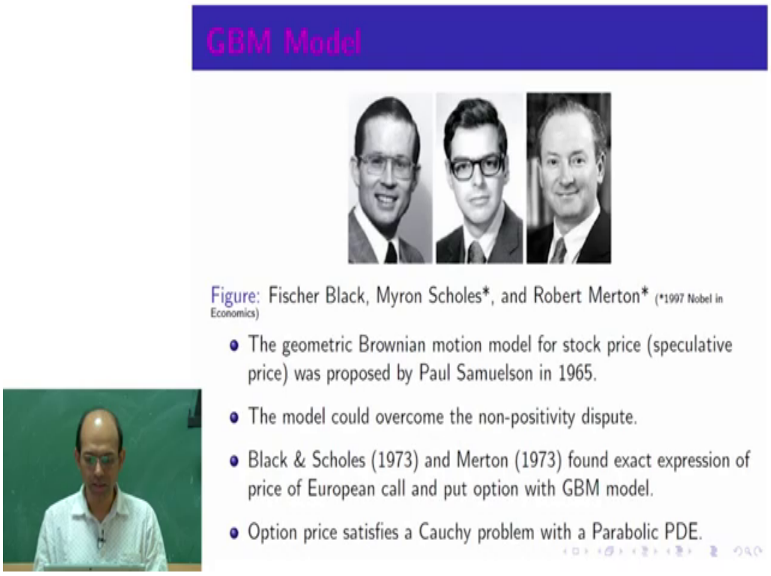

First we just recall that what was the starting of the applications of stochastic processes. So, the here the first mathematical model of market appears in PhD thesis of Louis Bachelier. So, it is some known very old picture of Louis Bachelier. This appeared in 1900. So, more than a century before correct, 120 years back.

So, there he actually when he used this, you know I mean he studied market using stochastic process that time no mathematical tools for Brownian motion was existing, okay. I mean the name of Brownian motion was also not common there. However, he used a random walk etc.

Actually he used random walk and then actually he came up with correct formulation Brownian motion in his PhD thesis as a limit of random works okay.

This is actually 5 years before Einstein's paper on Brownian motion. So, the model, however failed to capture the positive stock price Brownian motion in any way I mean, can go negative. So, as a model, it was pretty much useless for application. However, that was beginning of the effort, that using stochastic process or the tools in probability theory for modeling stock price.

So, he modeled the stock prices using Brownian motion intrigued. Brownian motion was you know independently formulated by Albert Einstein in 1905 and then a long gap okay. So, the geometric Brownian motion what we have just seen now with greater details we have seen actually the SDE for geometric Brownian motion, we have seen the differential geometry Brownian motion, we have seen also how to derive the solution of the equation.

So, these geometric Brownian motion was actually proposed by Paul Samuelson in 1965 to model stock price okay. So, it is nearly, you know 65 years after the work of Louis Bachelier. So, Louis Bachelier try to use this Brownian motion and then geometric Brownian motion was proposed by Paul Samuelson in 1965.

Okay I mean, the long gap is mainly because the Louis Bachelier work was mostly ignored because nobody thought that okay that can be improvised okay

And also the, I mean required tools mathematical tools were also vastly unpopular during this time period, the model could overcome the non-positivity dispute what Bachelier had. But then in 1973 paper, actually this work was done by Black and Scholes early. So, but they published that in 1973 that they used this geometric Brownian motion to model stock price and then did some stochastic analysis to find out price of options. But that is a different thing, what is option, we are not going to details of the definition of these derivatives of stock price.

But that was the first breakthrough in this field and then many researchers become interested to study stochastic process because that gives better insight of pricing of tradable assets, so which is very much real life scenario. So, in the work of Black-Scholes and also followed by

Merton's work, so they have shown that option price satisfies a Cauchy problem with a parabolic PDE, so that they have done.