Introduction to Probabilistic Methods in PDE
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Kac's Theorem on the stochastic representation of solution to second order linear ODE
Part 2
Lecture 37

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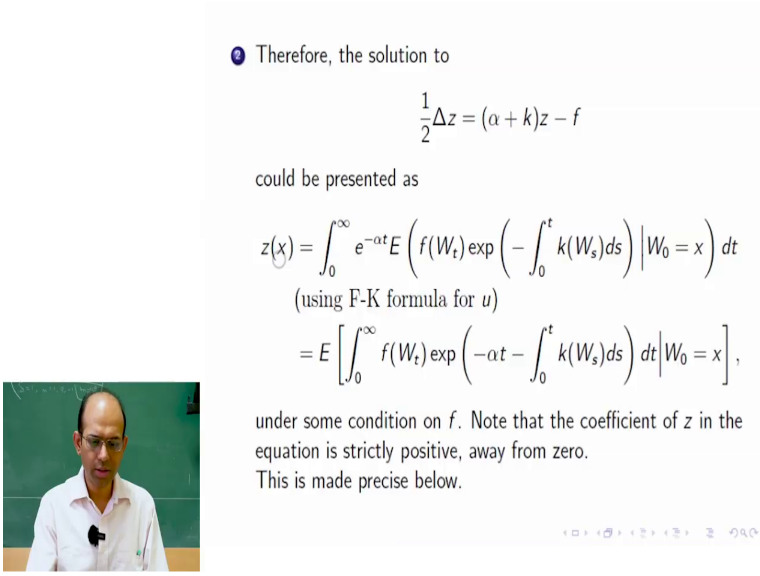

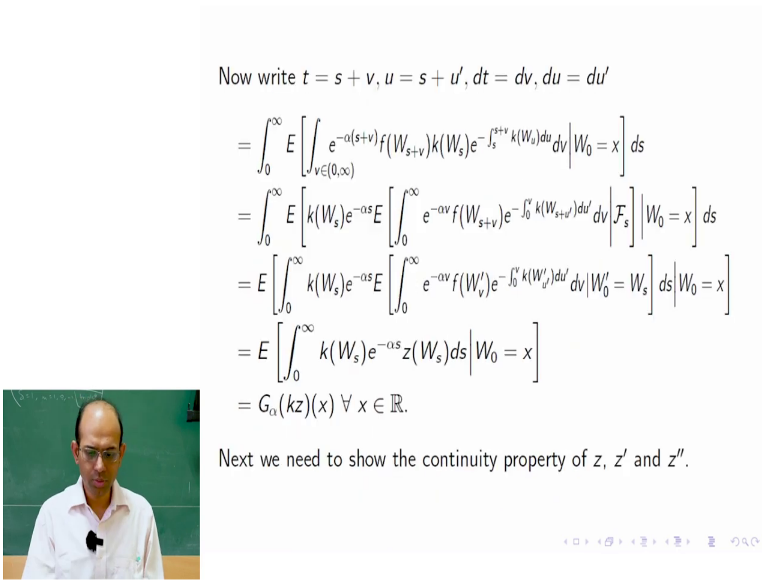

So we have proved yet that the function z as defined here z of x is equal to expectation of integration 0 to infinity f of Wt e to the power of minus alpha t minus 0 to t k Ws ds, so this thing, this zx satisfies the equation half of Laplacian z.

So since we have considered only one-dimensional case here, so we have done that half of z double prime, the second order derivative of z is equal to alpha plus k times of z minus f, okay.

So the only thing is that in this proof, something is remaining that we have not proved that, I mean this z is continuous, z prime is also continuous and z double prime is only piecewise continuous. We have not proved that, that we need to do here. So we need to show the continuity properties of this z, z prime and z double prime.

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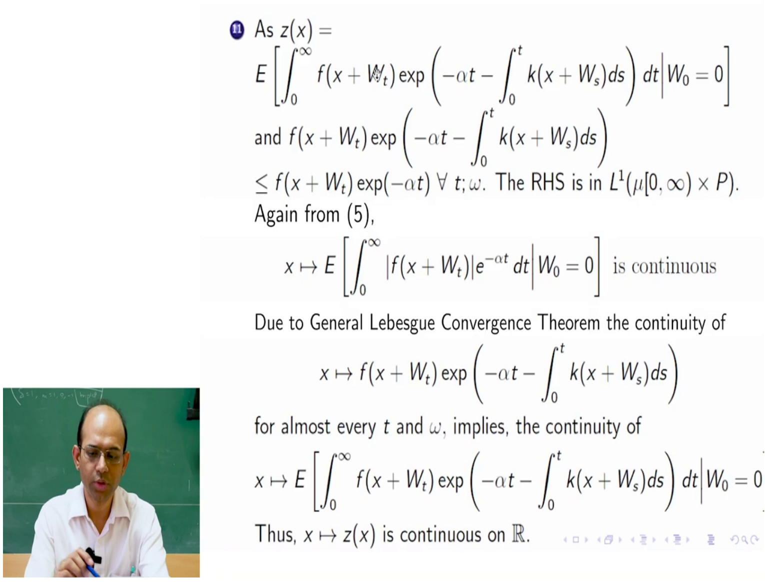

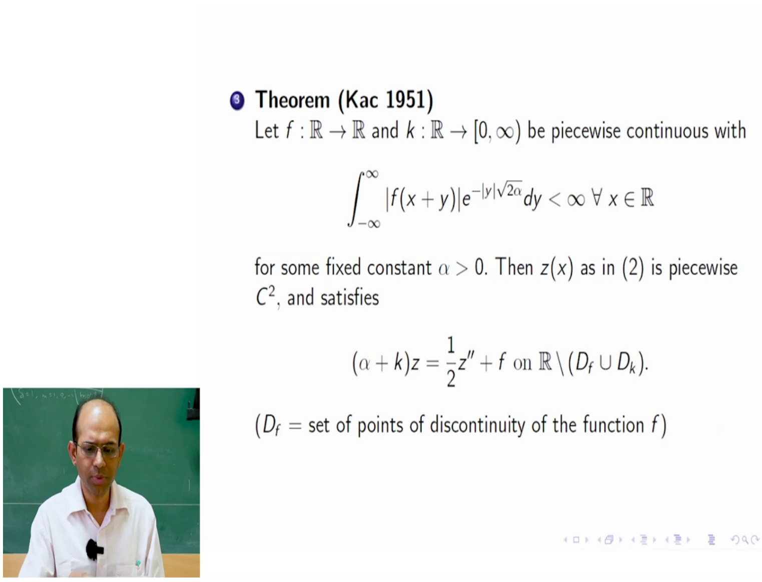

So here, exactly the earlier expression, only instead of W0 is equal to x, I am writing W0 is equal to 0 and f of Wt becomes f of x plus Wt. So expectation of integration 0 to infinity f of x plus Wt into e to the power of minus alpha t minus integration 0 to t k of x plus Ws ds, okay this whole thing is integrated with respect to t variable dt, okay.

So this is the expression of zx, we are recollecting it here and here this thing, f of x plus Wt into e to the power minus alpha t minus integration 0 to t k of x plus Ws ds, this whole quantity is less than or equal to f of x plus Wt into e to the minus alpha t, why? Because k is non-negative, okay.

So here this right hand side, this thing is L1 function with respect to this Lebesgue measure, with respect to t because this is integration with respect to Lebesgue measure on t time variable and probability (measure) integration with respect to probability measure, so here this whole thing is in L1 of the, with respect to this product measure. So what did we obtain?

We have obtained that this left hand side, this integrant is dominated by some particular L1 function, okay and again we see that, okay the map x to expectation of 0 to infinity f of x plus Wt e to the minus alpha t dt, this thing is continuous. How do you do that? This continuity is obtained writing in limit inside these integrations and that can be done only if that you can get, you know Dominated Convergence Theorem.

And that is due to the condition on f which is quoted in the statement of the theorem that this thing is finite and so we are here, so here this map x to this thing is continuous and hence due to the Dominated Convergence Theorem, the continuity of, so here x to f of x plus Wt into e to the power minus alpha t minus 0 to t k of x plus Ws ds for almost every t and omega, that implies the continuity.

I mean why that is continuous with respect to time, because W is Brownian motion, it is continuous path process so it is continuous with respect with to t. Here also we have continuity with respect to t, here also we have continuity with respect to t, okay, so but we are not looking at that, we are looking at for every t and every omega that continuous map of x to f of x this thing.

So from here what we get is that this map from x to expectation of integration 0 to infinity f of x plus Wt into e to the power of minus alpha t minus 0 to t k of x plus Ws ds okay, integration W0 is equal to 0, so this thing would be continuous, okay. Now what is this? This is exactly zx. So from here what we are going to get is that x to zx this map is continuous, okay.

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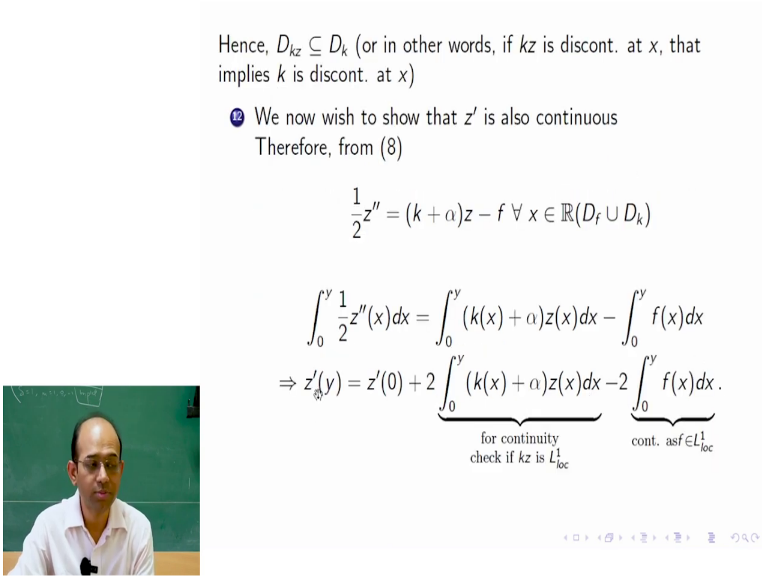

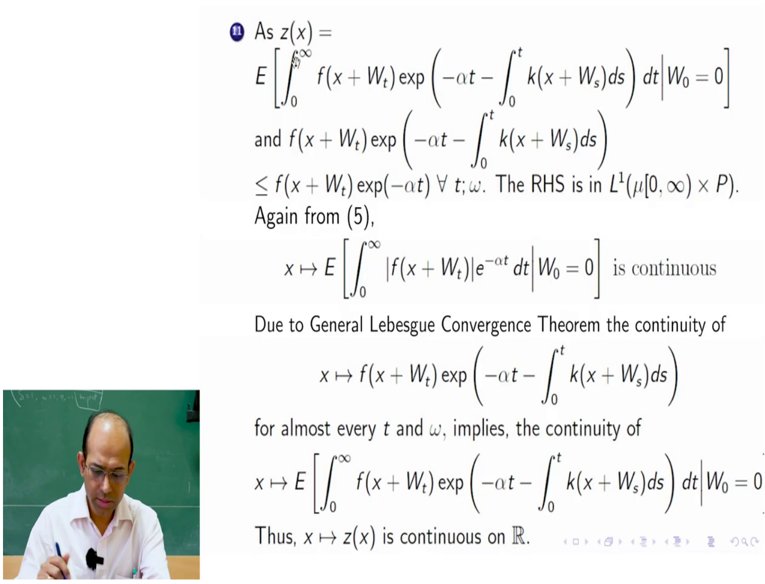

Then what do we do is that we consider that D kz, this is set of all discontinuities of the function kx zx okay. So here since we have obtained the z function is continuous, so it has no point of discontinuities. So it means that if k into z has any point of discontinuity that is coming due to the discontinuity of k. So the set of all point of discontinuities of the function of kz is subset of set of all discontinuities of the function k.

We cannot just say that these things are equal because sometimes point of discontinuities of k I mean, may not be discontinuities of kz because say for example there is a say, z is 0 or something, okay. So you may not get that discontinuity there but we can say that this is a subset of this Dk. So here we write in words that kz is discontinuous at x that implies that k is discontinuous at that point x.

We now show that, we wish to show that the z prime that is the almost everywhere derivative, okay so here because we cannot talk about derivative everywhere, correct because it is z itself need not be you know differentiable, I have not proved here, just I have proved z is continuous.

So this z prime okay, so we are now showing that okay it is also continuous. So z prime is also continuous. How are we doing that?

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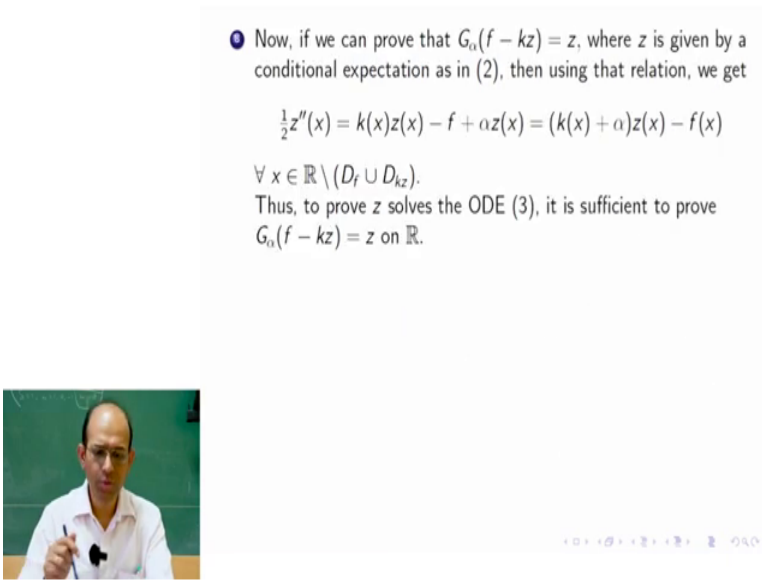

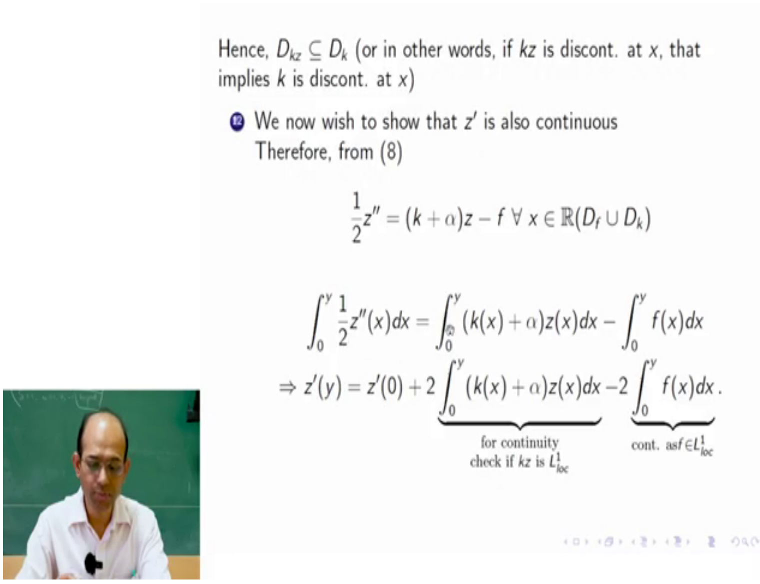

So first we look at 8 so 8 here we see that half of z double prime is written in this way, okay. So this can be integrated out, so we want to do that here. So half z double prime is equal to k of x plus alpha times z minus f, okay for all x this is true when R, okay set minus is missing, set minus Df Dk.

So for those cases this is true, and then we integrate out. So we integrate from 0 to y half z double prime and here on the right hand side what we see is that this 0 to infinity k x plus alpha times zx dx, okay. So what we know only that z is continuous and of course for k we know that is non-negative etc, and piecewise continuous, ofcourse. And f is the function, given function what is mentioned earlier.

So here when we see that f is piecewise continuous and that 0 to y we are integrating, here we are using the L1 loc property of function f, so what we do is that left hand side, the integration of z double prime is z prime. Okay this is second derivative, so integrate so we get to the first derivative here.

So z prime y minus z prime of 0, so that minus z prime of 0 I take on the right hand side, so z prime of y is equal to z prime of 0 plus 2, this half comes here on the right hand side so 2 times integration 0 to y, okay k of x plus alpha zx dx minus 2 times 0 to y f of x dx. Now here if we want to conform the continuity of z prime from here, we really need the integrant is locally integrable, okay. If integrant is locally integrable, L1 loc we say in short, then this integration as a function of y is indeed continuous.

However that is not yet clear so we need to check for continuity of this term with respect to y we need to check that indeed that this product is in L1 loc, okay. So for alpha z we do not need to bother because z is a continuous function. Any continuous function is in L1 loc, correct because you take any compact set, okay and on a compact set continuous function anyway bounded, and there it is therefore, you know integrable.

So only thing is that, this you know, this product etc is in L1 loc, that requires proof. And for f what happens that the condition what we have on f that, I mean that integration that is finite, okay. So from that we understand that whenever we consider, I mean that f is in L1 loc. If f is not in L1 loc that we would not have got that condition because we had multiplied something but that is e to the minus alpha t okay.

So that thing is also, you know a non-zero quantity, so we get continuity of this term with respect to y but for this we need to something more. But if you do this then we would be able to ensure that the first derivative of z is also continuous with respect to the y variable.

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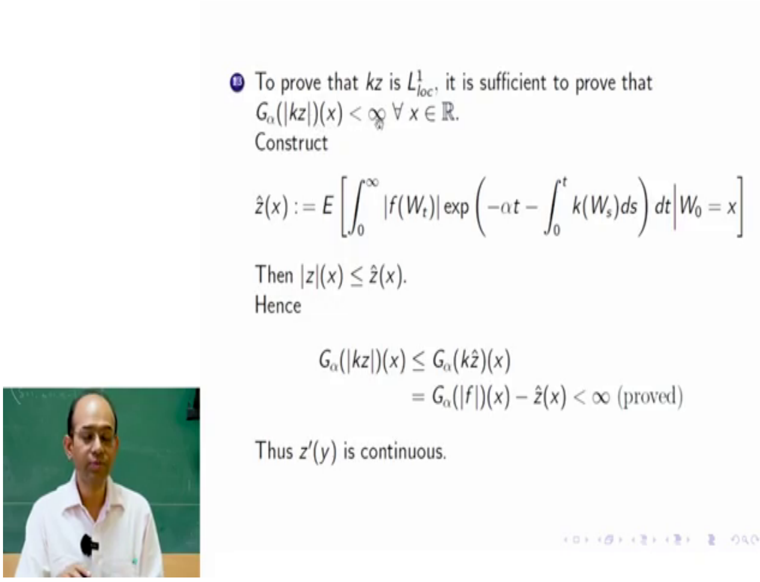

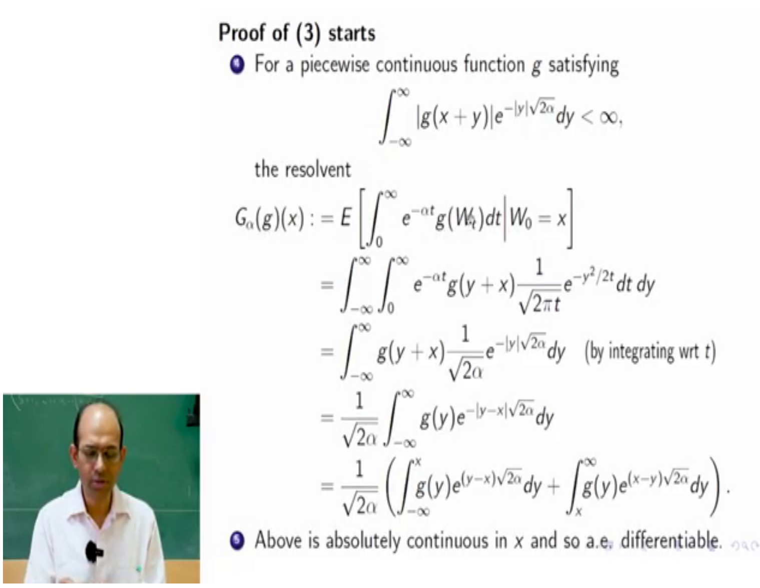

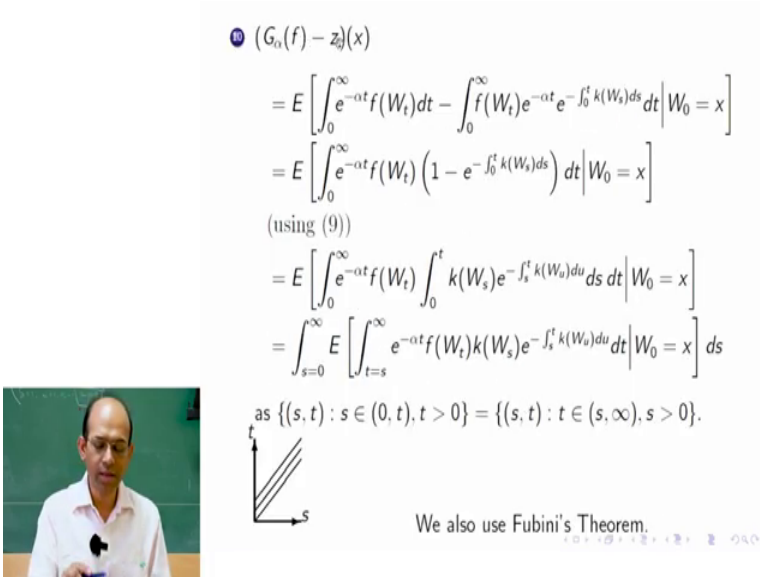

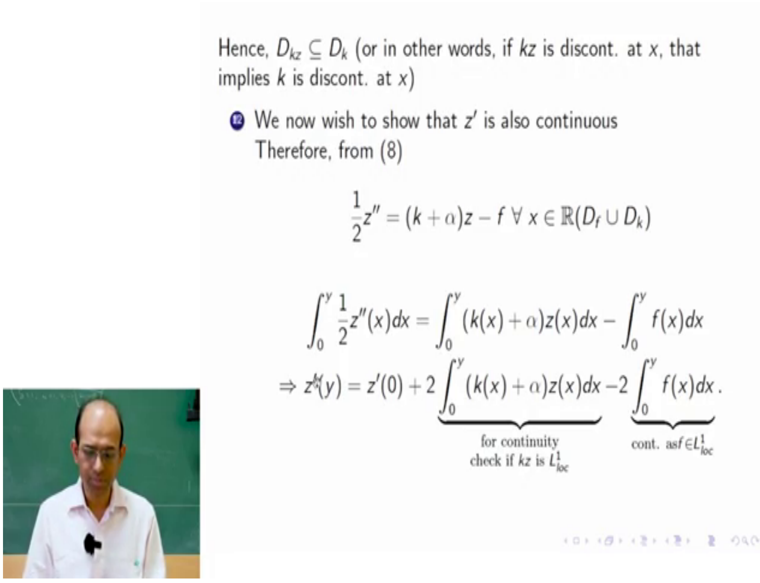

Now to prove that k times z is in L1 loc okay, it is sufficient to prove that the G alpha of mod kz is finite. Why is it so? Because G alpha mod kz as you know we have done the G alpha of f is finite that implies that f is in L1 loc, that you know the integration, so let us recall quickly what is G alpha definition, so this is G alpha definition.

G alpha of g is equal to this function, right, correct, so here g is there and e to the minus alpha t is there or if you want to remove this expectation etc then this is without expectation, correct, g of y into e to the power of some number, some function which is strictly positive function, okay and then dy. So if this is finite for all x that ensures that okay that this function g is in L1 loc, okay.

So we do this here. We want to prove that kz is in L1 loc, we want to prove that G alpha of mod kz is finite for each and every x here, okay. So for doing this I mean we have not treated mod of kz before because we have treated G alpha of kz, okay so here, for the first time we are doing it. So we are taking z hat of x.

So this z hat of x is defined as expectation of mod of f Wt e to the minus alpha t k Ws ds, okay so now here it is clear that I have mod here so this is not less than or equals to zx, okay because zx has just f of Wt without any modulus.

So we know that zx is less than or equals to mod zx and therefore now we can take modulus both sides, but right hand side is always non-negative, so mod z is still less than or equals to mod zx, okay, you understand? That zx, first that prove that less than this, and then this is non-negative so mod of zx is less than or equals to z hat of x.

Okay hence so here actually mod of zx is like, mod is outside, okay so you take mod inside, you get less than equal to sign correct. So mod of z is equal to, is less than z hat of x. So hence G alpha of mod kz what is this? This is where you are going to have mod here, okay, so that is less than or equals to G alpha of kz hat.

Okay because k is anyway non-negative, okay so when we want to write down this mod kz in this whole expression of the integration so here let us go back to the definition of this.

So this expression we look at, so here if you mod, so then k is anyway non-negative and if you have kz so mod z would appear here, okay so there you have z hat here, okay. So mod z is less than or equals to z hat, so that we can write down here, z hat okay, z hat k alpha of k z hat x. So monotonicity we are using of G alpha, okay.

Now this G alpha of k z hat is exactly equal to G alpha of mod f minus z hat. How do you do that? We do that because of earlier equation what we have established here. G alpha of f minus z is equal to G alpha of kz, does not matter what f I choose. If I would have chosen here mod f itself then I would have got g alpha of mod f minus this thing, so instead of z I would have got z hat there is equal to G alpha of k z hat here.

So that is the thing we are using here, that this G alpha of k z hat is equal to G alpha of mod f minus z hat x. Okay I am using that earlier result which you have already proved, okay. So now this quantity, is it finite? Yes it is finite because z hat, you know I mean coming from here, here the condition on f is such that this is finite, okay so z hat is finite because here we have even some quantity which is even less than 1.

So our condition was that mod of f of Wt e to the minus alpha t, only that part okay that is finite. Here we have again some you know negative term here. So this is smaller than 1, so that is finite and here this part is also finite because of that assumption of f so all these are finite.

So what we have obtained is that G alpha of mod kz of x is finite for each and every x and that was the thing we were looking for, then we can say that okay kz in L1 loc. Since kz is in L1 loc so this thing is continuous with respect to y variable or in other words z prime is continuous with respect to y variable.

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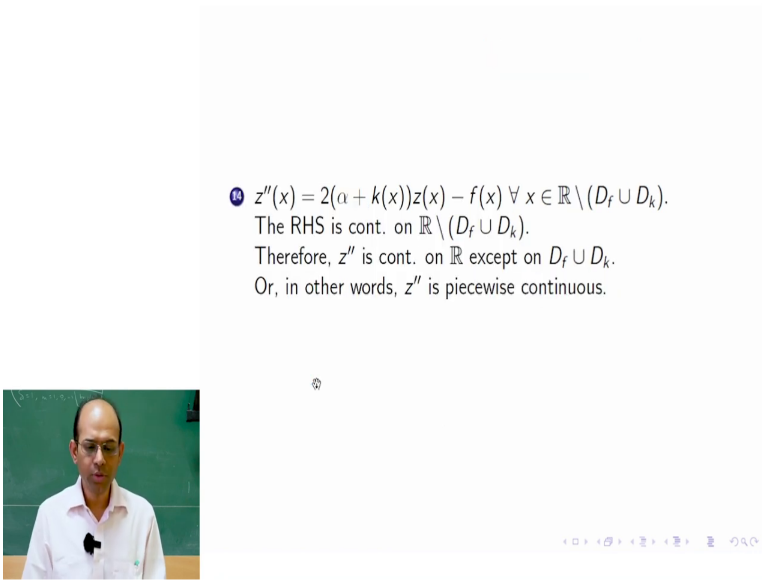

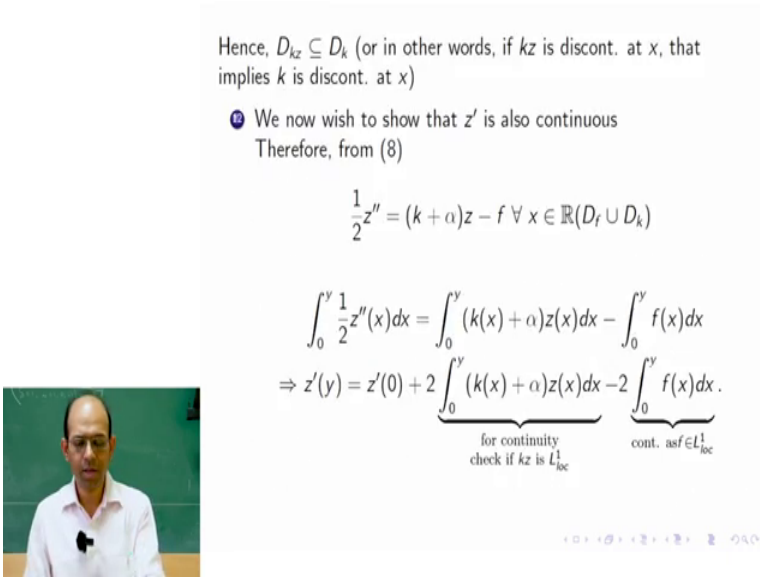

Okay, so till now I talked only about z prime. Now we should look at the z double prime. So here z double prime is coming from the equation itself okay, because half z double prime is equal to this, but now you are multiplying 2 okay, so g multiplied by 2 properly, half z double prime is equal to k plus alpha z minus f, so I should got 2 times f here, 2 times, here is 2 missing, so z double prime of x is equal to 2 times of alpha plus kx zx minus 2 times of f of x, okay.

So this is true for all x, which x? Which are in R but not in Df union Dk. What happens that, because we did not get this equation for all R but we have obtained only for these points which is not in the point of discontinuities of f and k, actually kz we had but we are dropping kz by k here, okay, it is just because that this is a larger set, so since we are subtracting by larger set so still this equation holds, okay.

So still this equation holds. So this equation we know for this type of point x. Now this right hand side we look at this, so this is piecewise continuous, f is piecewise continuous, and k is also piecewise continuous, z is continuous of course but then this multiplication we can only talk about piecewise continuous.

So this right hand side is continuous on these R subtract the point of discontinuities of f and k, okay, so on this region, okay, on this set and since this is continuous on this, so left hand side is also continuous on this set. Okay so we can say that z double prime is continuous on R except Df union Dk, okay. This is trivially true, so or in other words that z double prime is piecewise continuous, okay. So this proves this theorem. So what is the theorem?

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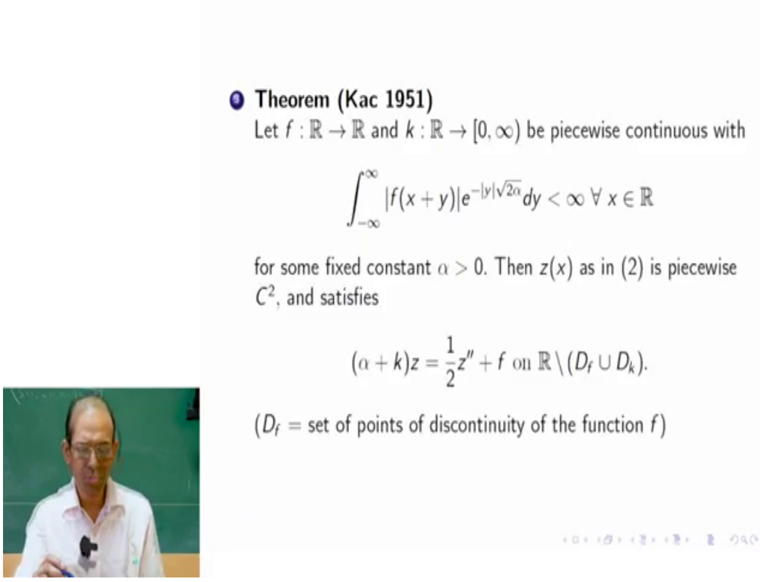

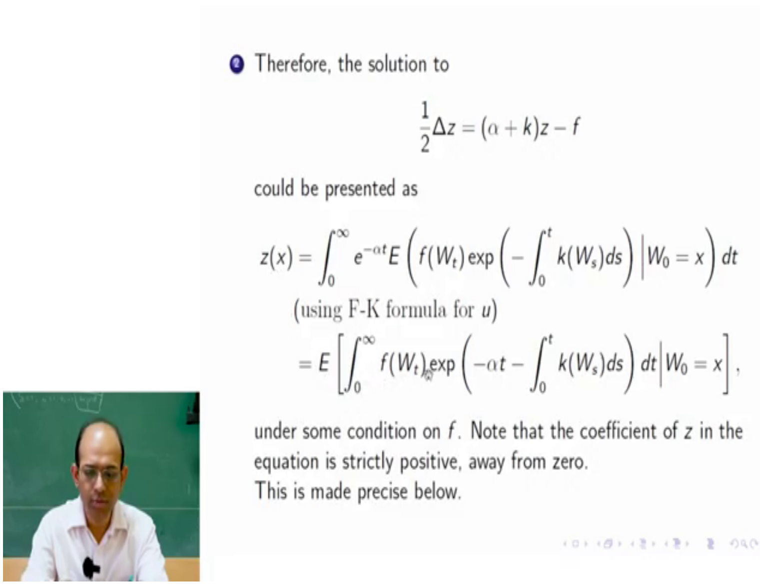

That what we started in the last lecture, this is Kac's Theorem which says that if f satisfies this condition, okay this integrability condition and k is of course non-negative and piecewise continuous then for alpha positive, okay, I mean this alpha positive, so then z as defined in earlier I mean, integration of f, so this is this one, that z is equal to f of Wt this thing, okay, integration of f because to this t and expectation is Wt.

So that expression of z is piecewise C2 that we have proved, that z double prime is piecewise continuous and that satisfies this equation that how z double prime is equal to alpha plus kz minus f, okay, that is proved. Okay thank you very much.