Introduction to Probabilistic Methods in PDE
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Kac's Theorem on the stochastic representation of solution to second-order linear ODE
Part 1
Lecture 36

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A second order linear ODE

1. Consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + k(x)u(t,x) = \frac{1}{2}\Delta u(t,x), \\ u(0,x) = f(x) \end{cases}$$

where $(t,x) \in (0,\infty) \times \mathbb{R}^d$. This is a special case of the Corollary 10 of the last lecture, where $g \equiv 0$. Let u be the solution of the above problem s.t. for some $\alpha > 0$ and $\forall\, x \in \mathbb{R}^d$

i. $\int_0^\infty e^{-\alpha t}|u(t,x)|dt < \infty$

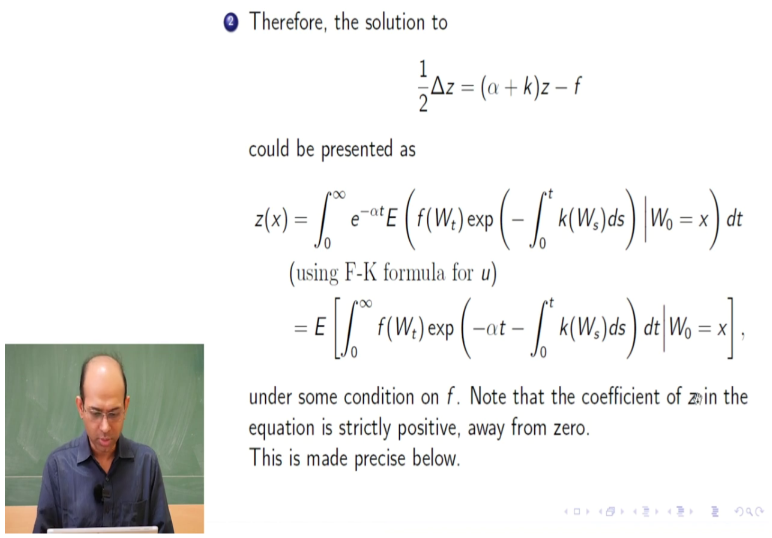
ii. $\lim_{t\to\infty} e^{-\alpha t}u(t,x) = 0$ and

iii. $\Delta \int_0^\infty e^{-\alpha t}u(t,x)\,dt = \int_0^\infty e^{-\alpha t}\Delta u(t,x)\,dt.$

Then $z_\alpha(x) := \int_0^\infty e^{-\alpha t}u(t,x)dt$ satisfies

$$\begin{aligned} \frac{1}{2}\Delta z_\alpha(x) &= \int_0^\infty e^{-\alpha t}\frac{1}{2}\Delta u(t,x)dt \\ &= \int_0^\infty e^{-\alpha t}\left(\frac{\partial u}{\partial t} + ku\right)dt \\ &= \int_0^\infty \left[\left(e^{-\alpha t}\frac{\partial u}{\partial t} + (-\alpha)e^{-\alpha t}u\right) + (k+\alpha)e^{-\alpha t}u\right]dt \\ &= \left(e^{-\alpha t}u(t,x)\right)\Big|_0^\infty + (k+\alpha)\int_0^\infty e^{-\alpha t}u(t,x)dt \\ &= 0 - u(0,x) + (k(x)+\alpha)z_\alpha(x) \\ &= (k(x)+\alpha)z_\alpha(x) - f(x). \end{aligned}$$

Recall that f and k are continuous and $k(x) \geq 0$, $\alpha > 0$.



Welcome. Today we are going to study a Second Order Linear ODE. We would actually derive the suitable ODE which, for which we can write down one stochastic representation. We would start with considering some PDE which we have already seen before, for which we have already obtained stochastic representation.

When I say stochastic representation I mean solution reaching in terms of conditional expectation of some function of Brownian motion, okay, so here we start with this Cauchy problem, okay. So this is Cauchy problem what we have seen in earlier class when we studied the Feynman-Kac formula, okay Feynman-Kac Theorem.

So there we had one more extra term, g term which is absent here. So this is the special case of the class what we have already studied. So consider such initial value problem. So remember that we have written initial value problem as a corollary of the Feynman-Kac Theorem because Feynman-Kac Theorem was presented as terminal value problem, correct. So this we called the corollary 10 in the last lecture, okay.

So here u is the function of t and x, k is the function of x, okay and f is the initial data, this is also function of x. We assume that k and f are continuous and what we do is that let u be the solution of the above problem such that for some alpha positive the following are true, okay.

So here we would assume appropriate condition on f etc okay, so that these following conditions are true. First is that integration 0 to infinity e to the minus alpha t mod of u t x dt okay, is finite for each x and limit t tends to infinity e to the power minus alpha t u t x goes to 0, this is equal to 0, we know that e to the power minus alpha t goes to 0 as t tends to infinity.

But here by stating that what we are saying is that the growth of u would be dominated by the decay by this exponential function and another thing is that we also assume that u is such that the Laplacian of the integration of this product of the function is the same as integration of the Laplacian of these product functions, okay.

So that is, these are the three conditions we assume that u satisfy these conditions, okay, I mean this would be true for some, you know assumptions of k and f which we are not specifying now, we are just saying that let u be a solution of this Cauchy problem such that these three conditions are true, okay.

So this slide is basically derivation of the, of an appropriate second order ODE for which we are going to write down the stochastic representation. So to this end for derivation we would require this 1, 2, 3 properties okay. Under these three conditions only we can do that. So we are assuming these three conditions here.

Okay so we assume, I mean define this function z alpha. For a fixed given alpha, okay as alpha I have mentioned here, so for this alpha, z alpha is a function of x. It is not a function of t. So we integrate this e to the minus alpha t u t x with respect to t and then this right hand side is independent of t, okay. So we get a function of x here.

So this is basically Laplace transformation, correct, Laplace transformation of this function for each x, correct. So this satisfies now we consider half times Laplacian of z alpha x and already those three conditions we are going to use now at this stage.

We can take this Laplacian inside this integration, so left hand is equal to integration 0 to infinity e to minus alpha t half times Laplacian u dt this is equal to, now we are going to use the fact that u solves the Cauchy problem. So Laplacian of u is equal to del u del t plus k u, so half times Laplacian of u is equal to del u del t plus k times u.

So that is the fact we are going to use here. We are not writing you know x dependence at each and every state but this should be understood. So integration 0 to infinity e to the minus alpha t del u del t plus k of x times u of t comma x okay, dt. So this we again what we do is here we add and subtract some factor for simplification. So we, you know subtract this factor alpha times e to the minus alpha t times u and here we add alpha times e to the minus alpha t u, okay.

So e to the power minus alpha t del u del t, this term is here and then we have this subtraction minus alpha times e to the minus alpha t u, so this together we combine to write down, say t derivative of e to the power minus alpha t u t x, correct? t derivative of e to the minus alpha t u t x would be minus e to the minus alpha t del u del t plus minus alpha because derivative of this minus alpha times e to the minus alpha t and u, okay. And then this part already k was there, k plus alpha is there, so that also you keep, okay.

So now we perform the integration. For performing integration, this is t, we can take integration t here inside because k is function of x. It is not a function of t. So you can take k outside k of x plus alpha okay, and integration 0 to infinity e to the power minus alpha t u t x dt, okay.

So at this stage I am writing that expression of t and x explicitly, and here as I have explained that okay, this integration when you perform since it is partial, t derivative of this thing and then integration with respect to t would give me just, you know anti-derivative e to the minus alpha t u t x. But then at, u should be evaluated at t equal to infinity and then subtract with t is equal to 0, okay so these cases.

So that means okay we need to take the limit t tends to infinity of e to the minus alpha t u t x, okay but that we have already assumed here in point 2 that this is 0, okay. So we can write down 0 here and this, you know when we put t is equal to 0 we would get e to the 0 that is 1, so we would get u of 0 comma x. So we get k x plus alpha, that is this whole thing, this is z alpha x as defined, so we write down that as z alpha x, okay, this whole thing is z alpha x.

So we have obtained one equation of z alpha x here, I mean so here we write down, I mean more clean manner, so here u 0 x is basically initial condition that is f, it is already given so minus f of x, okay and then k of x plus alpha z alpha x. So it is now a basically since x is from Rd, so this is actually a PDE okay, so Laplacian z alpha is equal to some, you know a particular function f, okay, minus f and then some another function here.

Here it is not any arbitrary function because k is non-negative function but alpha is a positive number. So k is, you know, so k plus alpha would be strictly positive. So some strictly positive function, so here we can also have f plus k continuous etc as we have assumed earlier we can just keep those things, okay so this type of differential equation this z alpha would satisfy.

Thankfully we have stochastic representation of u because u is the solution of Cauchy problem and for that, you know under some conditions on f we have established that okay, it can be written as conditional expectation of some function of Brownian motion. So that we can actually use here to write down z alpha also in terms of some conditional expectation, correct? So that is the idea.

So therefore the solution to this half times Laplacian z is equal to alpha plus k of x times z minus f could be represented as 0 to infinity integration of e to the minus alpha t and then u is replaced by this, okay. So exactly that what appears in the Feynman-Kac formula, expectation of f of Wt e to the power of minus 0 to t k Ws ds given W0 is equal to x, the whole thing is a function of t, but you integrate with respect to t, you get just the function of x, okay.

So this whole thing we can also rewrite in the following manner, so this is just, you know f satisfies certain conditions then you can interchange the integration and the expectation, okay so, because this part is bounded by 1, okay so only this part. So here this will be, under some condition f we can do that and note that the coefficient of z here is strictly positive, okay.

So whenever you would have similar kind of equation, okay that half times Laplacian z and here you have strictly positive coefficient z minus f you have a hope that you would be able to write down the solution in terms of this, okay. But till now this is not completely rigorous way so we are going to state this as theorem and under what condition I mean we can perform everything as we wish.

So here this much is actually showing us that derivation of this equation for which you can hope to get this type of representation, okay. So note that the coefficient, so this is made precise below.

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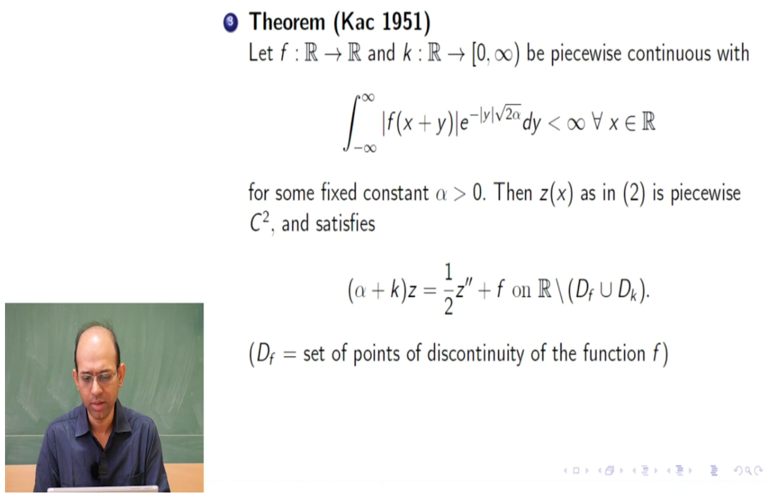

So this is the theorem by Kac in 1951 that let f and k be piecewise continuous functions, okay. They do not need to be continuous but at least piecewise continuous function. f may be positive or negative but k is non-negative, okay, and for on f we have this particular condition that mod of f of x plus y into e to the power minus mod y square root 2 alpha dy, this is finite for all x in R, okay.

This condition looks little different than the condition, the initial conditions what we have seen for example for Feynman-Kac Formula, okay. So there we require that e to the power minus a x square mod of f of x dx is, I mean, that should be finite where, for some positive a. But here it looks little different. I mean that would be explained in successive slides why do we need little different and what is the use of those.

Okay, so for some fixed constant alpha positive, okay so this is finite, fixed alpha positive, then for this alpha whatever it is, okay so then z x as in 2 okay, in 2 we have already defined this z x, okay so this is the expression, okay. So here we have this particular alpha also. So whatever alpha you would get for which this is finite, you take that alpha there and then you would get expression of z from here.

Okay and that z satisfies the following differential equation. Here we note that we are considering only one dimensional case. So earlier f is function from the space. So space was always considered as, considered as d-dimensional Euclidean space but here we are just considering only one-dimensional case, for the statement and the proof of the theorem, okay.

So, here since it is one-dimensional, so Laplacian is just the second derivative of z, okay. So alpha plus k times z is equal to half times double derivative of z plus f, okay. So this would be satisfied if not on the full real line but at least on the points where f and k, I mean either of them are, I mean both of them are continuous, okay.

So here I am writing that, I mean, set of points of discontinuity of function f is written by D subscript f. So, D subscript k is set of all of all points of discontinuity of the function k. Since f and k are not continuous functions, piecewise continuous, there could be, you know point of discontinuity.

What is the meaning of piecewise continuous? That means that you would be able to, get you know I mean points okay and basically you would be able to get subintervals on the real line, okay, partitions and then inside the subintervals it will be continuous. Okay from that this equation will be satisfied.

Now a special case, k and f both are continuous then this Df and Dk would be empty set so then this equation will be satisfied on the whole R. So this is little general version of that thing, okay.

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Proof of (3) starts

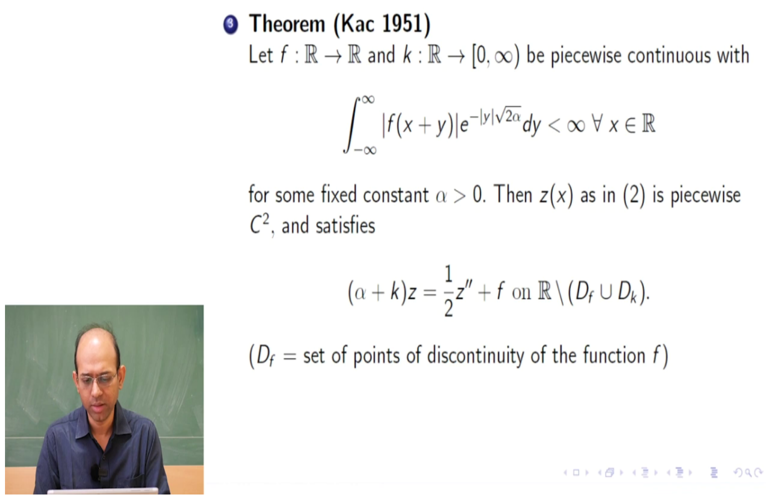
- For a piecewise continuous function g satisfying

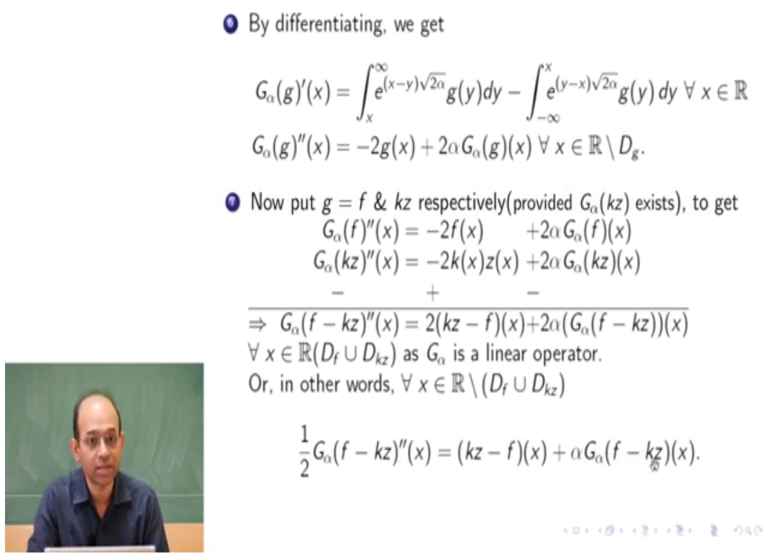
$$\int_{-\infty}^{\infty} |g(x+y)| e^{-|y|\sqrt{2\alpha}} dy < \infty,$$

the resolvent

$$\begin{aligned} G_\alpha(g)(x) &:= E\left[\int_0^\infty e^{-\alpha t} g(W_t)dt \,\middle|\, W_0 = x\right] \\ &= \int_{-\infty}^{\infty}\int_0^\infty e^{-\alpha t} g(y+x) \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dt\, dy \\ &= \int_{-\infty}^{\infty} g(y+x) \frac{1}{\sqrt{2\alpha}} e^{-|y|\sqrt{2\alpha}} dy \text{(by integrating wrt } t) \\ &= \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} g(y) e^{-|y-x|\sqrt{2\alpha}} dy \\ &= \frac{1}{\sqrt{2\alpha}} \left(\int_{-\infty}^{x} g(y) e^{(y-x)\sqrt{2\alpha}} dy + \int_x^\infty g(y) e^{(x-y)\sqrt{2\alpha}} dy \right). \end{aligned}$$

- Above is absolutely continuous in x and so a.e. differentiable.





So we start the proof here. Okay before proof we clarify that what we are going to do. We are not here trying to prove that this equation has, you know, I mean what we are trying to do here is that the z what is defined here is going to satisfy this equation, okay.

Okay so, and also z x is piecewise C2, okay so like you know, I mean otherwise I cannot talk about that, you know the second order derivative and that is, you know here. So at least those points of continuity, okay, so this is the thing.

So today we are going to see that the z what is defined earlier which is conditional expectation of integration of this thing is going to satisfy this equation. Okay so for that we first, okay for piecewise continuous function g satisfying this condition we can write down resolvent function, okay.

So this mod of g of x plus y e to the power minus mod y square root of, here it is typo, 2 times alpha dy is finite. Under this condition we can define this G alpha g. So let us see, I mean what is the, why do we need this condition. So here G alpha g is defined as expectation of integration e to the minus alpha t g Wt dt. So it looks like, you know, like Laplace transformation of this function of t, okay for every omega.

So here what we do, we find out this conditional expectation because Wt is a normal random variable with mean 0 and variance t. So we can find out this expectation by multiplying this integration by the probability density function and we integrate with respect to y minus infinity to infinity.

So that we do here, e to the minus alpha t g of y plus x, okay so this x is as before but y is, you know the Wt value. So yt, y and 1 over square root 2 pi e to the power minus y square by 2t okay because t is the variance and dt dy. Okay, here since we have this condition with us, okay so what we need, what we are going to do is that we are going to compute integration of this with respect to t variable. g does not depend on t, g comes outside.

And then this is e to the minus alpha t of 1 over square root 2 pi t e to the minus y square by 2t , okay. So this is Laplace transformation of this, you know this PDF function, okay and for that we know the expression. So one can also, you know use some integral calculator, so that integration turns out to be 1 over square root 2 alpha times e to the power of minus mod y square root of 2 alpha, okay.

So here you have square root 2 pi t. So this function is function of t. This also depends on t. This also depends on t. All these three things together you integrate from 0 to infinity you are going to get this, okay. If you want to perform this integration by yourself, you would see that, okay error function, I mean actually complementary error function will appear there and then you know the behavior of the complementary error function as x tends to infinity or minus infinity those things are used to get this expression.

Okay, so now this expression is independent of t. There is no t here. Now here we see that above things appear here. So g of y plus x, you know e to the power of minus mod y this, you know, square root of 2 alpha dy obtained here. So that is the reason we had to assume this, okay. We had to assume this, I mean otherwise I am not sure whether this would exist, okay.

Now since this whole thing is in L1, okay, this integrant so we can find out this integration and we write down here this way, and mod of y minus x we can write down, because just change of variable formula, so instead of y plus x we did, just write y here so that we will get minus of mod y minus x and now here y from minus infinity to x, there y is smaller than x so y minus x is negative.

So instead of minus mod y minus x, we just write y minus x for minus infinity to x range and the complementary range x to infinity, on the other hand x minus y becomes negative. So we write down e to the minus, e to the power x minus y square root 2 alpha, okay, simple. So this thing is written as sum of these two integrals. So here we have understood quite nicely that what is my G alpha g x, okay.

But why are we using this function G alpha g that will be clear in the subsequent slides. Here at least what we have obtained that G alpha x as defined this manner is integral of some, you know functions here okay. So this is a measurable function okay, and then since we are integrating this with respect to y, y is from here minus infinity to x and here so we can obtain that okay that this is absolutely continuous in x, because this is integration of a measurable function here.

So what I am going to get that this you know whole integral is a absolutely continuous, this is L1 function also, correct. Not only measurable, this is L1 function. We are integrating so we are going to get absolutely continuous function in x and we know that absolutely continuous function is almost everywhere differentiable. So G alpha g you know this whole, this function is differentiable in x almost everywhere sense.

Since it is differentiable so we differentiate, okay with respect to x and then, so this is Leibniz rule correct, for integration when you have this, you know the function is given as integral and also, the x variable also appears inside. So we do that.

So here we do not know that whether g is continuous or what, okay. However if it is continuous then we know that okay its derivatives, one part would be g of x times e to the power of x minus x that is e to the 0, and then plus then this integral but here you do the derivative with respect to x, correct?

But even if is not continuous, even if g is not continuous we can observe that this coefficient appears here and here and when you put y is equal to x so both these factors are same like 1, and here it, x is in the lower limit, here x is the upper limit so both should actually cancel each other. So we do not need to worry about the, I mean the derivative with respect to these variables. We just need to worry about the inside variable, okay because those would cancel, okay. So from here whatever you are going to get, you get the negative sign there.

Okay so when you do that, inside you take the derivative here, so x is multiplied with minus square root of 2 alpha here, here it is multiplied with square root of 2 alpha and here also you have square root of 2 alpha in the denominator, so that would go.

So you have just, you are going to have only integration x to infinity e to the power x minus y square root of 2 alpha g y dy minus integration minus infinity to x e to the power y minus x square root of 2 alpha g y dy. This is true for all x in R. Now next step here we are going to see that here what is the second derivative of the resulting function, okay.

So for that what we do is that with respect to x if we take derivative then this is g of x, e to the power x and this is 0, e to the 0 is 1. And here this side you are going to get here e to power of y is equal to x is e to the 0 that is 1, and here also g of x here.

However here we have a minus sign, a negative sign. Since it is a negative sign so they would not cancel because here we have minus gx, here we have got minus gx, so minus 2 gx, okay. So that is the reason, since they do not cancel so to get this we really need that g to be continuous at that point x because we are not going to get this thing okay.

So only the point x which is not a point of discontinuity, for that we are going to get this factor and then derivative with respect to x inside, that part we are going to deal with here, that is x to infinity here, again square root of 2 alpha would appear here, here also we are going to get square root of 2 alpha and then when we club these two things together okay, and here you see the negative sign so plus sign would appear.

So that would resemble with exactly the definition of G alpha g, okay because what we have just divided in 2 parts, we are going to get those 2 parts which we can again combine to get minus infinity integration of e to the power of mod of y minus x, okay g y dy. So that we do, so square root of 2 alpha we get square root of, is there a typo? Yeah, there is no typo because G alpha includes also 1 over square root of 2 alpha, okay so this also, this is also there so we are going to get 2 times alpha times G alpha of g x, okay.

So we have obtained this thing, okay. Till now it is not clear how these functions are relevant for us. Actually this is obtained for a general g okay, but we are going to plug in f, k etc at the place of g. So we do that. So put g is equal to f here, okay and also g is equal to kz, okay where z is the given function written in terms of conditional expectation.

Okay first line is the line obtained for g is equal to f case. We can get this thing only for those x where x is not a point of discontinuity of f, okay so G alpha f double prime of x is equal to minus 2 times of f plus 2 alpha times G alpha f x. For kz we are going to write down G alpha of kz double prime, second derivative is equal to, again from minus, minus 2k times zx okay plus 2 alpha was there, 2 alpha times G alpha kz x okay?

Okay, so now we subtract these two lines, okay minus, minus becomes plus minus here so when you subtract so G alpha of f minus kz, see here I am using the linearity property of G alpha. Why? Because look at the definition of G alpha as a function of g, it is linear. It is a linear function of g.

So G alpha of f minus kz double prime x is equal to 2 times kz, k times z correct, kz minus f of x, okay and then 2 alpha times G alpha of f minus kz x. So we have obtained this equation. This equation looks quite familiar. So let us see how are we going to do this because second derivative appears here and then f minus kz also appears here, so we are going to take advantage of that G alpha, I mean this, this quantity is, double derivative appears and it appears here also.

Okay, so here this is obtained for all x which are not point of discontinuous of either f or kz. So if we write down this cleanly then half times G alpha of f minus kz double prime of x is equal to kz minus f of x plus alpha times G alpha of f minus kz of x, okay. Now we would like to relate this equation, okay with the original equation, okay.

We note that if we can establish that this function okay, this is function of x, correct, if this function is exactly equal to z then we are going to get half times double prime of z is equal to alpha times z here and k times z here and minus f, correct?

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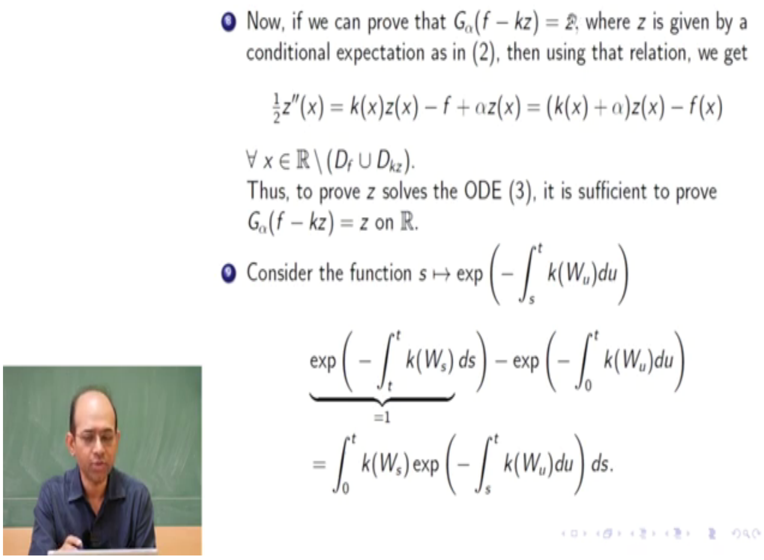

So if you can prove that this, you know whole thing is z where z is given by the conditional expectation as in 2, okay then using that relation we get that half times z double prime is equal to, you know, k times z minus f plus alpha z, okay. So this we can rewrite again by taking z, you know common k plus alpha times z, so this alpha and this k together, minus f which looks exactly same as the linear ODE what we have started with, okay.

So to prove that the function z what we have started with the conditional expectation, okay that is going to solve this equation, the original equation what we have proposed that the linear second order ODE only if we can prove this one, so this is the gap. So we need to show now that this is indeed true, okay.

So this is the thing. If we can show that this is true, because for G alpha already we have proved that okay, that I mean the relation already you have established here, correct, for G alpha we have already established here. So only thing is that if this is equal to z then we know that okay, z is going to satisfy this ODE. So it is sufficient to prove this is equal to z.

So now for that we consider this function s to e to the power of minus s to t k Ws du, this is just, you know a continuous function of s, okay and we know, we can use the Fundamental Theorem of Calculus and this function is actually function of bounded variation. So we can do this, so this is differentiable.

So e to the power of minus t to t, I mean because I am now using Fundamental Theorem of Calculus so we are writing down that s is equal to t case, s is equal to 0 case, this difference as derivative of this and integration 0 to t, okay. So why are we doing this? We are just doing this because to obtain a relation which we are going to use in the next slide.

So this is saying that t to t is 0, so e to the 0 is 1, minus, e to the power minus 0 to t k W u du is equal to the derivative of the whole thing with respect to s is k Ws times e to the power of s k Ws in terms of, okay, k Ws times e to the minus s to t k Wu du, okay this is the derivative, so integration of that appears, okay. So this is just application of Fundamental Theorem of Calculus.

Okay, so that you can do when your function is absolutely continuous function. Here is actually differentiable function.

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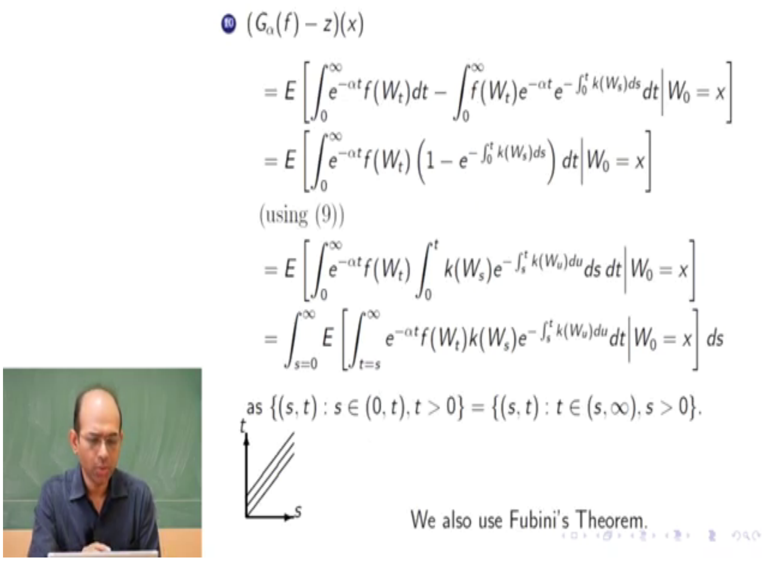


Proof of (3) starts

- For a piecewise continuous function g satisfying

$$\int_{-\infty}^{\infty} |g(x+y)| e^{-|y|\sqrt{2\alpha}} dy < \infty,$$

the resolvent

$$\begin{aligned} G_\alpha(g)(x) &:= E\left[\int_0^\infty e^{-\alpha t} g(W_t) dt \middle| W_0 = x\right] \\ &= \int_{-\infty}^{\infty}\int_0^\infty e^{-\alpha t} g(y+x) \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dt\, dy \\ &= \int_{-\infty}^{\infty} g(y+x) \frac{1}{\sqrt{2\alpha}} e^{-|y|\sqrt{2\alpha}} dy \text{(by integrating wrt } t) \\ &= \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} g(y) e^{-|y-x|\sqrt{2\alpha}} dy \\ &= \frac{1}{\sqrt{2\alpha}} \left(\int_{-\infty}^{x} g(y) e^{(y-x)\sqrt{2\alpha}} dy + \int_x^\infty g(y) e^{(x-y)\sqrt{2\alpha}} dy \right). \end{aligned}$$

- Above is absolutely continuous in x and so a.e. differentiable.

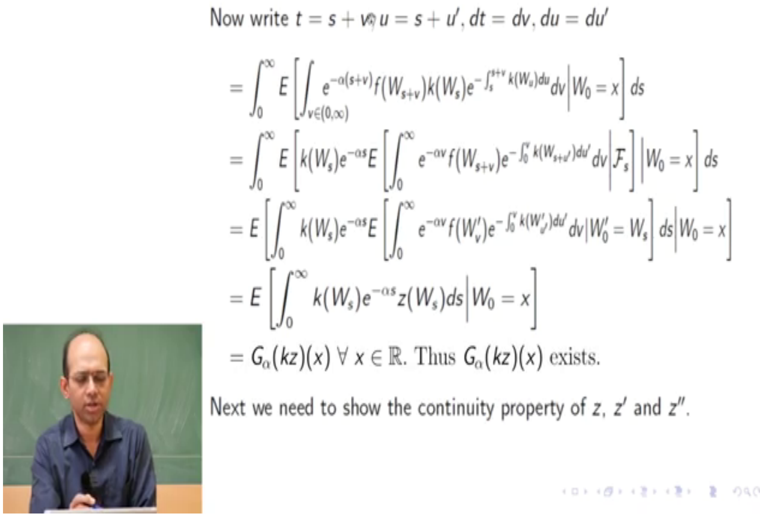
- **Theorem (Kac 1951)**
Let $f : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R} \to [0, \infty)$ be piecewise continuous with

$$\int_{-\infty}^{\infty} |f(x+y)| e^{-|y|\sqrt{2\alpha}} dy < \infty \ \forall\, x \in \mathbb{R}$$

for some fixed constant $\alpha > 0$. Then $z(x)$ as in (2) is piecewise C^2, and satisfies

$$(\alpha + k)z = \frac{1}{2} z'' + f \text{ on } \mathbb{R} \setminus (D_f \cup D_k).$$

(D_f = set of points of discontinuity of the function f)



Okay now we consider G alpha f minus z. Why do we consider that? Because we indeed want to show G alpha of f minus kz is equal to z. For that we are considering G alpha of f, okay bracket closed minus z that we need to show that G alpha of kz. Okay if we can show that then we are done, correct?

So instead of writing G alpha f minus kz which turns out to be very complicated we are writing now G alpha of f minus z and that we show as G alpha of kz. So G alpha of f minus z appears here and after several steps we would at the end show that G alpha of kz, okay. So now our task is to just, you know obtain that step by step manner.

So G alpha of f, okay here we are using the definition of G alpha of f, okay so e to the power of minus alpha t f of Wt dt given W0 is equal to x, that is the definition of G alpha of f, remember, yes? So this is equivalent, this you know, g, this is the definition that you just evaluate the function, evaluate the Brownian motion, I mean the function at the Brownian motion and you take Laplace transformation, okay of that.

So here we write down, therefore e to the minus alpha t, f of Wt dt minus, so now we are writing the full expression of z here, that is f of Wt e to the minus alpha t into e to the power minus 0 to t k Ws ds, okay. This whole thing is a function of t and then we integrate with respect to t, this is from 0 to infinity.

So now this part we take common. We take e to the minus alpha t times f of Wt common here so then I get 1 minus e to the power of minus 0 to t k of Ws ds, okay the whole thing, dt. Now this is the term which appears, we have already obtained, 1 minus e to the power of minus 0 to t k Wu du is integration 0 to t this thing, okay.

So k times e to the power of minus integration of k, okay so we are going to write down that this difference is integration of k times e to the minus integration of k, okay so and other terms are as it is, e to the minus alpha t f of Wt, okay. So this integration is with respect to the s variable so I have ds. So actually I have 3 integrations 1, 2, 3 okay. This is with respect to t. This is with respect to s and this is with respect to u, okay.

So now what we do, we observe certain type of rearrangement of integration. Imagine that here we have that s is running from 0 to t and t is running from 0 to infinity, okay. So that is giving me some triangular region. So if s axis is horizontal, t axis is vertical so what we are getting is that s is from 0 to t, so that means from here to here, s is from 0 to t and t is running from 0 to infinity.

So basically the upper part of this triangle, okay is the domain on which I am integrating here, s-t plane. So this whole thing can also be viewed as t is ranging from 0 to s and s, I mean t is ranging from s to infinity, t is ranging from s to infinity and s is ranging from 0 to infinity.

So this whole triangle can also be written as, so this is the thing I am trying to write down here. So s comma t, this set, this upper triangle, this s comma t such that this s is between 0 to t and t is positive. This is same as t is from s to infinity and s is positive, okay, s is positive and t is from s to infinity. So we do that.

So here this integration, okay, this whole integration, this looks like repeated integration so we write down that as, I mean so here actually I have jumped a step, I mean this expression should be outside first, okay. So here this repeated integration I am writing as t from s to infinity and s is from 0 to infinity.

However you would also use Fubini’s Theorem of f because you know we have assumed that integration of this, this thing is finite, okay so that is nothing but integration, if I write down in terms of, you know, if you replace y by Wt because this is like, you know integration with respect to the Brownian motion expectation.

So then we are going to get, this is nothing but saying that integration 0 to infinity of expectation of f of Wt, okay given W0 is equal to x, that is finite. So that is saying that okay this function f of Wt is L1 function with respect to t and omega together.

So we can apply the, so here as you can see that okay this thing is same as this thing, correct. So here I have written. So that is going to show me that okay this is L1. So we can use Fubini's Theorem. So by applying Fubini's Theorem we take expectation inside and this integration outside. So here we have e to the minus alpha t f of Wt k of Ws okay as it is as before, okay, only thing is that here t is ranging from s to infinity, s is ranging from 0 to infinity.

Okay so here the reason that I want to take expectation inside because I would like to do further calculation. I would like to use Tower property of expectation in the next slide. So here since our t is from s to infinity we can very well write down t as s plus some non-negative variable, okay. So we are going to do that. t we are going to write down t is equal to s plus v. v is 0 to infinity.

Okay we are also going to use other transformations, okay one after another, so we first do this thing here. Integration 0 to infinity expectation of, so v is from 0 to infinity, e to the power of minus alpha s plus v. So where I had e to the minus alpha t that I am writing as e to the minus alpha times s plus v where I have f of Wt I am writing f of Ws plus v, k s this is as before so k Ws is as before, okay and here I have s to t, so that we are going to write down as s to s plus v, okay k Wu du, okay.

So this is exactly the earlier integration just by this small change of variable here. So dt is same as dv so I can write dv here. Okay so now here also we are going to change, we would be able to change because u is running from s to s plus v. So we would write down u is equal to s plus u prime then u prime would run from 0 to v, correct. And u we are going to write down as s plus u prime, okay.

Good, so nothing else is done here. So e to the minus alpha s is just taken, I mean outside of this, okay so here we apply Tower property of expectation. Expectation of this is conditional expectation at 0, so we do conditioning at the time s here in between, okay. So with respect to the filtration, till, generated by the Brownian motion till time s, and then this k Ws is measurable with respect to this filtration. So it comes out.

We also take e to the power minus alpha s outside this here, and rest is inside, e to the power minus alpha v is here, f of s Ws plus v is also here and this part as I have already explained that this is the same thing. Okay, so now here this part we are going to use the Markov property of Brownian motion. That the information of the past and present, conditional expectation given the information past and present is same as the conditional expectation given the present information, okay?

So only at the time s, so here only Ws is required. Now since Ws is required so we can imagine that possibly s is the starting point of another Brownian motion, okay and so far expectation is concerned, okay so only distribution law of Brownian motion would be utilized here. So we generate another Brownian motion at time, I mean when Ws is there so from s onwards okay, we call that W prime.

So imagine W prime 0 is equal to Ws here and then everything would be like W of s plus u prime would W prime of u prime, right. W of s plus v would be W prime of v, okay. So we are going to, you know generate a Brownian motion, okay imagining s is the starting point. So it is expectation of 0 to infinity e to the minus alpha v f of W prime v, e to the power of minus 0 to v k of W prime u prime du prime, okay dv. Okay so this much is okay.

But we can recognize this expectation. This is what; this is e to the power minus alpha v, okay so this together, e to the alpha, minus alpha v and e to the minus this thing is there. So this is precisely the Feynman-Kac Formula for the solution of the Cauchy problem what I have stated in the beginning.

So this is basically z, okay this is basically z but evaluated at W0, at Ws, correct. So evaluated at Ws. So this is z of Ws, okay. This is z of Ws, now we have k of Ws as before, e to the minus alpha s and z of Ws. So k and z multiplication is here, e to the minus alpha s and integration 0 to infinity is here and with respect to s the integration is done. So, and then expectation is also taken.

So this is precisely the definition of G alpha. G alpha is defined exactly this manner. Here that expectation of integration of e to the minus alpha t, g of Wt, okay. So instead of g here we have obtained kz, k times z, k times z evaluated at Wt okay, e to the power minus alpha t dt 0 to infinity, okay.

So this is G alpha of kz x, so this true, this we have obtained for all x in R, okay. So here we stop here. So here we have obtained that okay, this z satisfies this equation but we have not yet shown that z, z prime or z double prime they are piecewise continuous. That we have not yet shown. That we are going to show in the next lecture.