Introduction to Probabilistic Methods in PDE
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Lecture 35
The Feynman-Kac formula

So, today we are going to see the formula by Feynman and Kac, and we are also going to see the proof of the formula.

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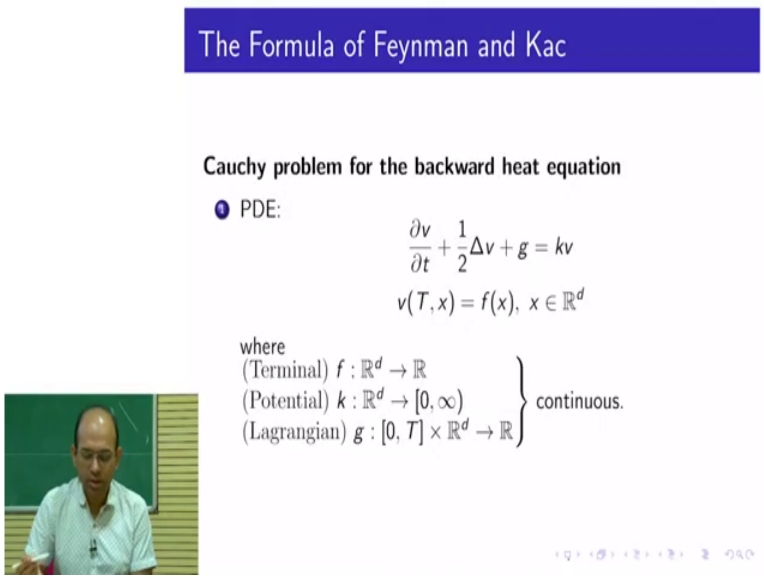

So, this is the theorem, this is saying that given a backward heat equation so here the heat equation appears that v is a function of time and space x, so we assume x is a member of d dimensional Euclidean space. Here, del v del t appears so partial derivative of v with respect to time.

And then the Laplacian of v multiply with the half, half Laplacian v plus g this is another function, is equal to k times v. So the unknown v appears here, here and here so it appears first order term, it is time derivative, second order space derivative and a 0th order term. And this equation we would like to solve this equation with the terminal condition capital T at time capital T, this v capital T is if function, f.

Where this function g, k, f all are continuous so here more better description is given. So here we assume f which is the terminal condition is a continuous function which is taking point x to real number, so Rd to R, the function k which we call potential function k is also function of x I have not written the dependency of k on x, as I have not written dependency of v with respect to time and x that is but these are understood.

So k is also a function of Rd to real number but here non-negative real number 0 or positive, so that is the assumption on k and k is also continuous. Again, this g function this is also a function of time and space, so g is also a continuous function, so here this time variable is restricted to 0 to capital T compact set.

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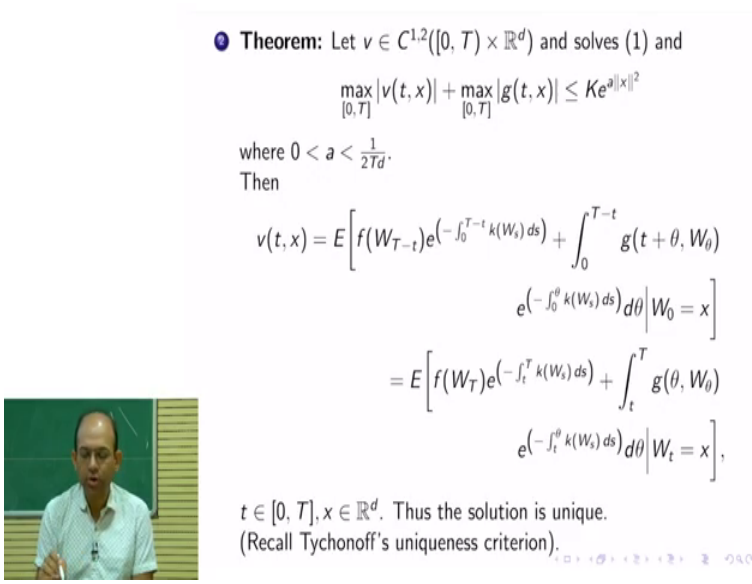

So this is a theorem, this theorem is saying that let v be a function C 1, 2 that means once differentiable with respect to time and this v function, the solution v we are going to look at this. So v is 1s differentiable with respect to time and twice differentiable with respect to space, and solves the equation 1 and here the growth property that v satisfy this growth property and also g. So mod of v and mod of g supremum over all possible time.

So that is less than or equal to k times this capital K is different from this small k, so this capital K times e to the power a norm x square and here this a is positive number which is 1 over 2 times capital Td. Remember, what is my capital T? Capital T is this terminal time and d is the dimension. So this a is like that, so we have upper bound of a, a should be less than 1 over 2 Td. That means we know that if capital T or d is large then a upper bound is smaller, so with that a, this norm of mod of v and mod of g this addition should be less than or equals to this number, so that is the assumption on the solution space, so this is the solution space we are looking at and we are looking at that this theorem says that, if that is the case then the solution v can be written as conditional expectation of some function of Brownian motion.

What is this? So we just read this, when v of t comma x, so now here I am writing down the explicit dependence on v on t and x that is conditional expectation of small f which is the terminal condition of W capital T minus small t into e to power of minus 0 to capital T minus t k of Ws ds, so this small k is a potential function, but this capital K is just another this is not same as small k, this is another constant, some positive constant.

So small k of Ws ds plus integration 0 to capital T minus t g of t plus theta W theta and this function is a function of theta and that is multiplied with this exponential function, e to the power minus 0 to theta k Ws ds and then this product is integrated from 0 to capital T minus t with respect to the value theta this variable theta, and this conditional expectation we are finding out given W0 is equal to x.

That means you know Brownian motion is starting from x. So this thing we can also rewrite it is easy to do that by some change of variable, so here that instead of 0 x I think the W t is equal to x. So since Brownian motion is time homogeneous Markov process we can do that, so I add t value to every time, wherever it is so then this becomes given W t is equal to x and capital W T minus t becomes capital WT, capital T.

So here I do not have minus t and here small 0 to capital T minus t becomes small t to capital T and then this g also becomes g theta W theta when theta runs from small t to capital T, so here theta was running from 0 to capital T minus t but T plus theta if we denote this whole thing as theta, then that is running from small t to capital T. So these are the adjustment one can do and one can get another alternate expression.

What is the benefit of this expression? That I mean this is precisely saying that f is evaluated at terminal time of w and f is a terminal data. So it is in that sense in more intuitive, and here this is like you know capital T to small t you know that you just when you want to find out v at time small t which is little past than capital T, then you just need to look at the interval small t to capital T.

Why? Because you know the data is given for capital T and now it is inside little past that capital T, so you need to integrate small k for small t to capital T, etc. So all this integration is from small t to capital T. So these expression is more intuitive than the earlier one.So what is the conclusion from this statement? The conclusion is that the solution is unique, because you know when I say that this v if satisfies the equation 1 and satisfies this growth condition then it has this expression.

It intrinsically says that this should be unique, there is no other thing because this right hand side expression is uniquely given. So, now we can recall we can compare this result with Tychonoffs uniqueness criteria for a special case, see the special case g was 0 k was 0 and there we just had these things that v was less than or equal to K times a into norm x square. So this is consistent with that idea.

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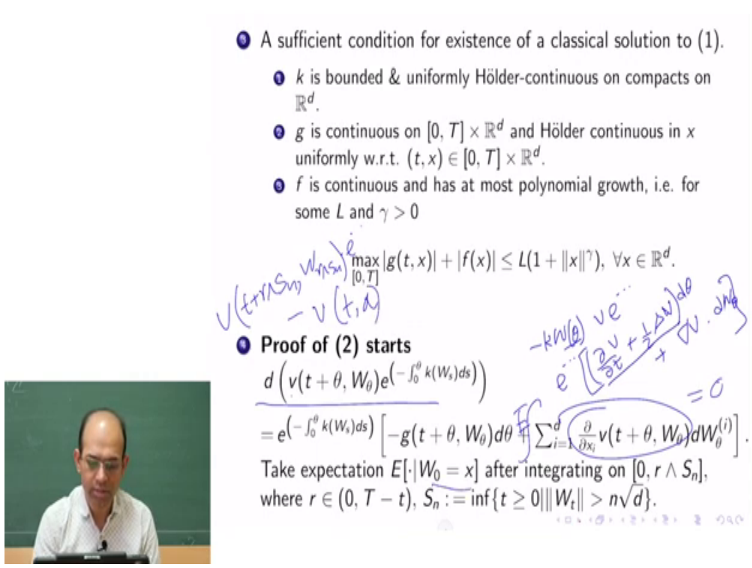

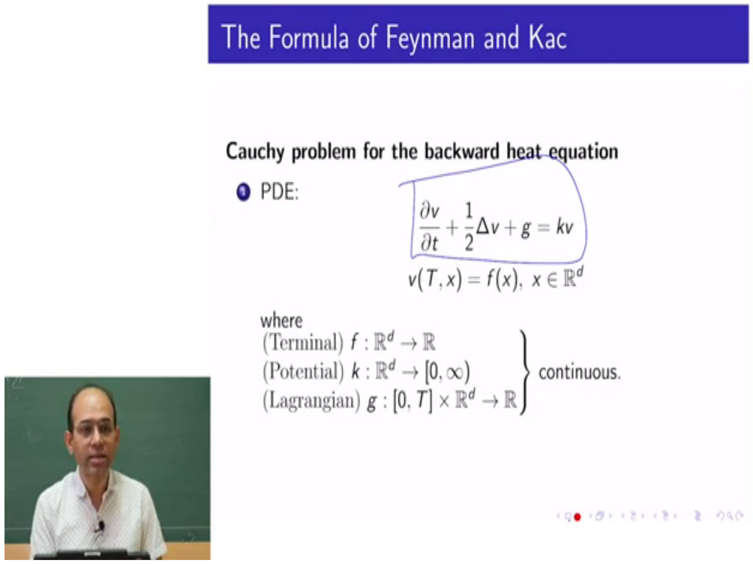

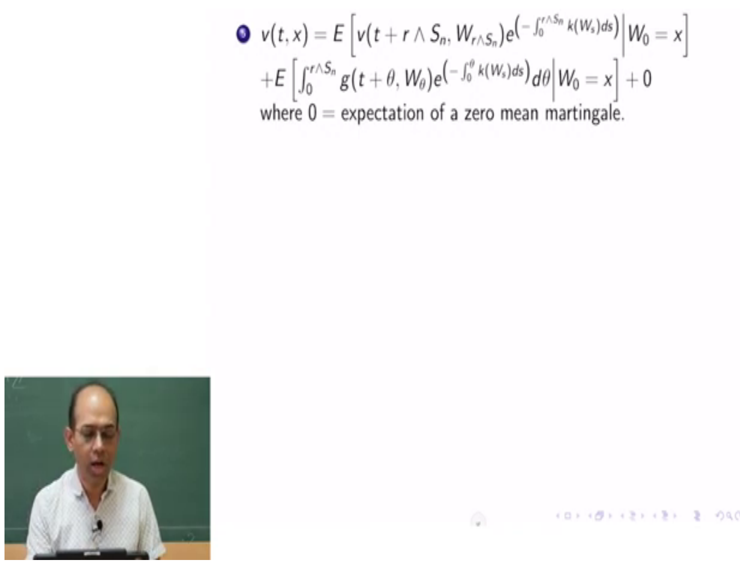

So a sufficient condition for existence of a classical solution to 1, so this is because earlier theorem does not say that when does it exist, it just says uniqueness, if it exist, it is unique but now one can ask very naturally that what is the sufficient condition for existence of solution to that equation. So this is one set of conditions that small k is bounded and uniformly Holder continuous on compacts on Rd and g is continuous on this domain and Holder continuous in x uniform with respect to t and x.

And small f, so basically we need to put conditions on all this given data k, g, f. So for f we assume that it is continuous and has at most polynomial growth, that is for some capital L and gamma you can write down that okay mod f is less than or equal to capital L times 1 plus norm x to the power gamma, so like polynomial kind of thing. So we start proof here.

In the theorem, we did not assert existence of solution, but rather we asserted that if solution exists, then how should that look like, and we have stated an expression and that dictates that the solution if exists should be unique. So in this proof, therefore we starts with the assumption that solution exists and then we derive, so here v is the solution so here assume that v is the solution of the equation.

So here we write down v of t plus theta comma W theta and we multiply with e to the power of minus 0 to theta k Ws ds, so this thing as a function of theta you think that theta is like a time like variable and you are writing differential of that. So we are considering the whole thing as a stochastic process with parameter theta, and then we are applying Ito's formula on this.

And then here since we have product, so for applying Ito's formula we need to first apply on exponential term and then v terms would remain, so if I do that, so the derivative of exponential term would give me minus k W s, and then we are going to get something like minus k W but here we are going to get theta minus k W theta and then we are going to get the exactly this term v term and e to the power of this term, from this side.

And then, if we differentiate the v on v and e we keep unchanged, so there we are going to get exponential of this term as it is and for v dvt, so there we can use the Ito's formula, so that Ito's formula looks like say there would be one del v del t term and there would be the half Laplacian term for this, half Laplacian of v term and that would be multiplied with d theta and then there would be the first order derivative with respect to the space variable and that would be multiply with the d W theta.

So this is the more local martingale term, so here del v that the gradient of v actually, so the gradient of v DW theta, so that would be in the kind of terms. Now, at this stage we are going to use that the fact that v solves the pde, till now I have not used I have just the fact that v is smooth, since v is the classical solution of the pde so v is continuously differentiable with respect to time and twice continuously differentiable with respec to the space variable etc.

So we have just used that, so now we are going to use that, v is indeed a solution, so in the solution we already know that del v del t and then these things and minus k etc. we are going to get g from there, so let us quickly look at the PDE, so here we see that del v del t plus half Laplacian v in the first page Laplacian v that is half Laplacian v is equal to k v minus g.

If you have minus k v here, so g goes on the right hand side so minus g. So we understand that when we apply Ito's formula in this product, we would get minus g of t plus theta W theta d theta term and of course the term with integrated with respect to Brownian motion this local martingale term this, would remain, so this term is here. So this is dot product correct? This is dot product, so you are going to get a summation i is equal to 1 to d del xi v of t plus theta W theta dW theta, ith component of that.

So that is the expression for this part and now for are we going to do is that we first integrate because we have to get rid of this differential term and after integrating from 0 to r minimum Sn which I am going to define what is r, what Sn, then we are going to take expectation. So here, we mention I mean what is the range of theta on which I am integrating, so theta would be integrated from 0 to r Sn where r is the member between 0 and capital T minus small t.

So that t plus theta becomes a member between you know small t and capital T. Theta between 0 to capital T minus small t implies small t plus theta is in between small t and capital T. So that is my r and Sn is stopping time introduced to localize so that the Brownian motion remains bounded till time Sn.

So, here infimum over all possible t is such that the norm of Brownian motion is greater than n times square root of d, what does it mean? It means that the hitting time of Brownian motion or exit time of the Brownian motion from the ball of radius n times square root of d, so this Sn is the exit time.

Next we get the v of t comma x is equal to expectation of v of t plus r minimum Sn comma W r minimum Sn into e to the power of integration minus 0 to r minimum Sn k W s ds, given W0 is equal to x plus this another expectation.

So how are we getting these three terms? We are getting directly from integration because left hand term is going to give me two terms because the left hand side after integrating we are going to get v of t plus r minimum Sn and then instead of theta you are going to write down r minimum Sn, so that would be one term and then subtract it minus v of, so here we are going to get minus v of t comma W 0 e to the power minus 0 to 0.

So this integration 0 to 0 is going to give you 0, so e to power 0 is 1, so I would get v of t plus s minimum R n Ws minimum Rn times e to the power of this term minus v of t comma W0 only. But we are taking W0 is equal to x, so we are going to take conditional expectation, so on the left hand we are going to get v of minus v of t comma x that term, and on from the right hand side here this local martingale term that where Brownian motion is there this term you know after taking expectation it will be 0.

I mean here what is happening that since I am integrating only till time Sn there the Brownian motion is bounded, so the v of t plus theta W theta, so that function would be bounded function because the variable W theta is bounded by a ball, so v function is continuous function you know where that is now restricted on a compact set so it is a bounded function. So it is integration of a bounded function with respect to Brownian motion till the stopping time.

So we are going to get you know, a martingale term and that would have expectation 0, so from the right hand side only one term would survive, so two terms on the left and one term on the right, so you are going to get three terms. So here, we got three different terms, one is, two terms coming from left hand side and this part would become, I mean after integration and then taking expectation this part would become 0.

So from here, I mean the one term would be v of t plus r minimum Sn W of r minimum Sn times e to the power here also for theta you write down r minimum Sn and then subtract it with v of t plus, instead of theta you writting 0 that means t just V t comma x, why? Because W0 is equal to x.

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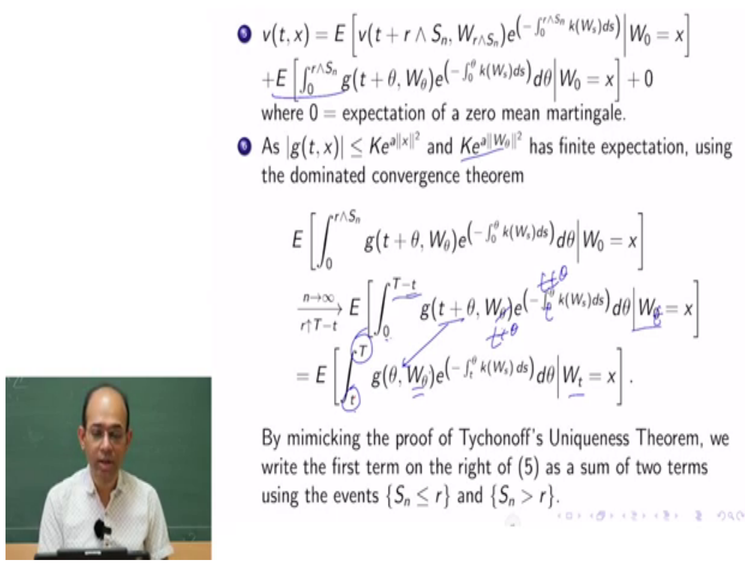

So, then rearranging of term would give you v of t x is equal to expectation of v of t plus r minimum Sn W r minimum Sn e to the power of minus 0 to r minimum Sn k W is ds given W 0 is equal to x plus expectation of integration 0 to r minimum Sn g of t plus theta W theta e to the power of minus 0 to theta k Ws d s. Integration with respect to theta given W0 is equal to x, this 0 is coming from the expectation of 0 mean martingale.

Next what we use? We use the growth property that g is less than or equal to K times e to power a norm x square and here if we replace x by the Brownian motion W theta then that random variable has finite expectation, so using this we can actually that means that dominate g of t comma W theta by one random variable with finite expectation.

Because here right hand side against this k times e to the, W theta whole square that does not depend on time t. So this would be used for applying the dominator convergence theorem. So here we use that, we take, so this expectation of integration 0 to r minimum Sn g t plus theta this expression was there earlier also.

Now we take n tends to infinity and r approaches to capital T minus small t. So when you do that we can take this limit inside of the expectation using this dominated convergence theorem, and then when we take it inside, we get 0 to capital T minus small t. So this r minimum Sn converges to capital T minus small t, why is it so? Because as n goes to infinity Sn goes to infinity almost surely, why?

Because Sn is nothing but the time to hit a ball of radius n times square root of d, as n goes to infinity the ball becomes larger and larger, so sn becomes also larger and larger, with probability 1 and on the other hand r I am also increasing to capital T minus small t, so I would get here capital T minus small t here. So g of t plus theta, I mean integrant is g of t plus theta W theta e to the power of minus 0 to theta k W s ds and then everything is integrated with respect to theta.

So here, this whole thing can also be written as expectation of integration small t to capital T g of theta W theta and this term. Here, why can you do that? We can do that because that we are just shifting time from 0 to T, whatever right hand side you see that condition was given in the earlier line was W0 is equal to x. Now if I declare that expect t so W t is equal to x then every time every other time point would be changed. That would be just shifted to small t unit away. So then 0 becomes small t, capital T minus small t becomes capital T. (done till this)

So therefore in this last line you got integration this lower limit is small t and upper limit is capital T and then in the integrant what you have g of t plus theta where theta was running from 0 to this then you can write down by change of variable that is g of theta itself where theta is changing from small t to capital T. So let us see, so here so Brownian motion I have changed so Brownian motion W I mean this W theta I mean that would change to I mean earlier whatever W theta was there, now it should t plus theta, but here also since we have changed t plus theta, is theta so I can keep the same notation W theta.

There is two change of you know notations, so g of W theta and then in the e to the power form so earlier W was from 0 to theta. Now since we have changed the time of the Brownian motion, so Ws s should run form t to t plus theta, so here first we have changed time index of Brownian motion so here W 0 to W t so then this thing should have become t plus theta W t plus theta and here this 0 would become so if this is t, then this integration e to the power minus t to t plus theta should appear here.

And then in this we do further another substitution that we change t plus theta as theta so this becomes d theta and W t plus theta become W theta and here since t plus theta is theta so we write down theta here and here lower limit of integration becomes instead of 0 we get small t and upper limit instead of capital T minus t it is capital T. So after obtaining this expression we now next mimic the procedure for proof of Tychonoffs theorem.

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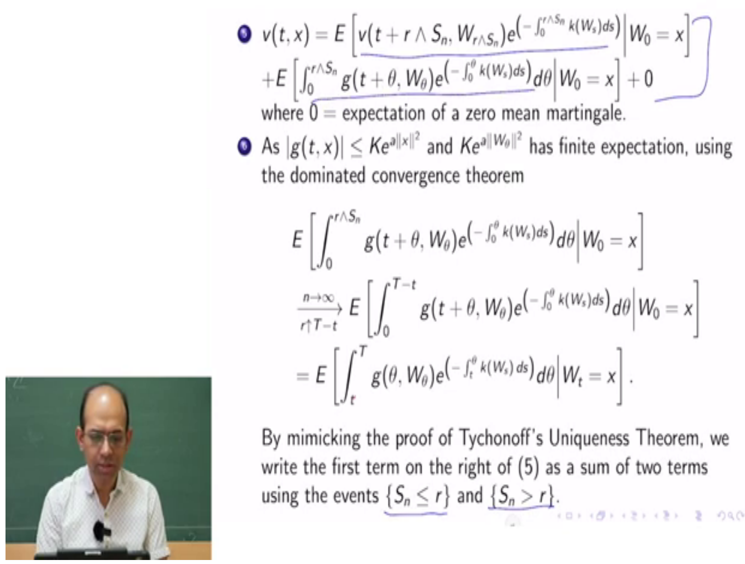

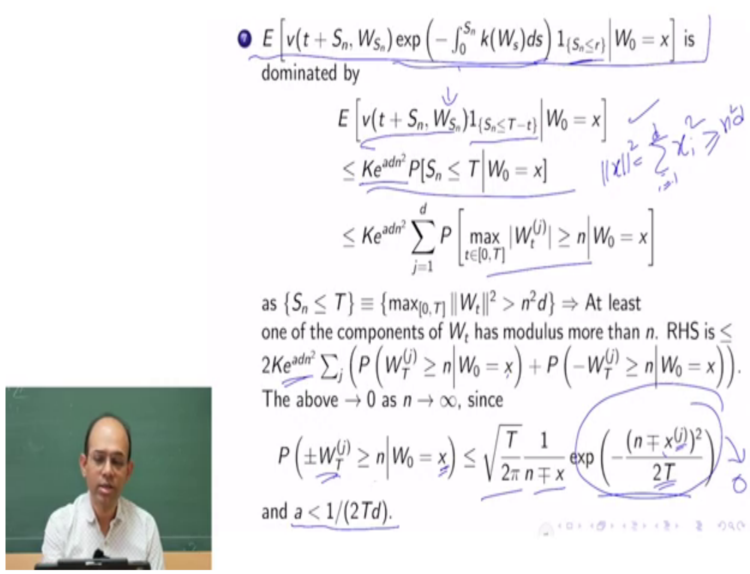

We use technique of Tychonoff's theorem, so what do we do is that we write down this event this you know this inside integrant as sum of two different terms, one term that we get by multiplying with indicator function Sn less than or equals to r another thing we obtain by multiplying indicator function of Sn greater than r, see indicator function of these two mutually exclusive and exhaustive events if you add you get 1, correct?

So, one can be written as sum of integrator function of these two mutually exclusive and exhaustive events, so that we do in the next slide, we consider only first the case when Sn is less than or equals to r, we have done exactly the same thing in the proof of Tychonoff's theorem also, when we do this our expression becomes simplified, so why is it so? Because earlier we had integration 0 to r minimum Sn.

Now if Sn is less than or equals to r, so r minimum Sn is Sn itself, so here look at equation number 5, in 5 you have r minimum Sn and then, so that thing we are writing down as v of t plus Sn comma W Sn into e to the power of minus integration 0 to Sn k W s ds indicator Sn less than equals to r, given W 0 is equal to x. So this is dominated by, so now we are going to give some you know domination of this.

But before that let us clarify that what are the other terms I am not writing here, so in this bullet 5, we are writing this term, v of t plus r minimum Sn so this term here and we are not writing the other term the g term, integration g term that we are not writing here, we are just writing the v of t plus Sn that term.

So for that term we are now getting you know one estimate upper bound for upper bound, what are the things we are doing? We are replacing these e to the power of minus of this term by 1 because k is non-negative, so this e to the power of minus 0 to Sn k Ws ds is less than or equal to 1, so it is dominated by this term that when this expression is replaced by 1 and then indicator function of Sn less than or equals to r is smaller than integrator function Sn less than capital T minus small t.

Why is it so? Because r is smaller than capital T minus small t, so Sn less than or equals to capital T minus small t is a larger event so indicator function of this is a larger function, so I mean not strictly larger I mean not smaller than this, so we are going to get this estimate. Now here we use the growth property of v, growth property of v is saying that v of t plus Sn W Sn is less than or equal to k times e to the power of norm of mod x square.

But x is here like W Sn and W Sn is what? The position of the Brownian motion at time Sn at Sn Brownian motion lie on the surface of the ball, on the sphere of radius n times square root of d, so the square of that will be n square into d so K times e to the power a d n square appears here to estimate that v function.

Now what remains is the indicator function, expectation of indicator function is a probability of the event, so we write down that is probability Sn less than or equals to capital T, here again further we have removed small t capital T minus small t I have removed to get even a larger probability perhaps, because capital T is greater or equals to capital T minus small t.

So here, this estimate is independent of small t, so we could dominate this expectation by some quantity which is independent of small t. This we can again further get a better estimate here Sn less than or equals to capital T, that means that Brownian motion touches the boundary of the ball before capital time T that means, that if we look at the component of the Brownian motion so the Brownian motion modulus is more than or equal to n times square root of d.

So that means its squared of the norm is more than n square times d, and then that square of the norm, so norm of x square inverse summation xi square i is equal to 1 to d, so there d number of sums if the whole sum is greater than or equals to n square times d greater or equals to n square times d that means at least one of the term must be more than n square, sum of d number of positive terms is more than n square times d.

So at least one should be more than n square, so that means at least one of the component xi should be no more than n, so at least one of these components will be Brownian motion is more than n and then that probability would be upper bounded by some of the probabilities of the terms more than or equal to n, so here we get because at least is like union.

So probability of union of event is less than or equal to sum of the probabilities so we are using that so sum of j is equal to 1 to d probability of this maximum of Brownian motion in between 0 to capital T that is more than equals to n, so this is the justification. So now we get that right hand side is less than or equal to so here what do we use? We use, reflection principle, right hand side is less than or equal to two times

So here I did not have two, I had just K times e to the ad n square earlier but now I am getting 2, why? Because I am replacing maximum 0 to capital T by capital T itself, the probability that w capital T at the component j is greater or equals to n plus probability minus W t greater or equals to n given W0 is equal to x.

So this is coming for reflection principle what we have already seen in earlier class, so now here from this we consider n tends to infinity, when you take n tends to infinity I mean we would like to see whether this whole thing goes to 0 or not, and here we see e to the power ad n square as n tends to infinity this term grows in large however the other terms where Brownian motion is greater than equals to n that goes to 0.

And then what would survive, so we see that probability W t greater than or equals to n that probability is less than or equal to square root of T by 2 pi 1 over n, so minus or plus depends on you know plus or minus here we are talking about, e to the power of minus n square by 2 T, n minus plus x j whole square by 2 T.

So this x j is a jth component of initial position x, so that does the value of x j does not matter much as you know we are talking about large n, so what matters is actually this denominator 2 T so here if you know we choose a is less than 1 over 2 T d, so then this a d n square the way it becomes larger would be dominated by the way the e to the power of minus n square by 2 T goes to 0.

As n tends to infinity this whole term e to the power exponential term goes to 0, so this converges to you know 0, this would dominate this growth if a is smaller the coefficient a d is smaller than 1 over 2 T, so that is the idea and since we have assumed that a is less than 1 over 2 T d, so this is true and we can get this you know above you know this upper bound goes to 0 as n tends to infinity.

So what does it means? It means that the expression in the bullet 7, this expectation that goes to 0 as n tends to infinity.

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- Again $E\left[v(t+r, W_r)e^{\left(-\int_0^r k(W_s)ds\right)}1_{\{S_n>r\}}\middle| W_0 = x\right]$
  $\xrightarrow[r\uparrow T-t]{n\to\infty} E\left[v(T, W_{T-t})e^{\left(-\int_0^{T-t} k(W_s)ds\right)}\middle| W_0 = x\right]$ using DCT.
  $= E\left[f(W_T)e^{\left(-\int_t^T k(W_s)ds\right)}\middle| W_t = x\right]$
- Hence, by taking limits in (5), (2) is proved.
- **Corollary:** $\frac{\partial u}{\partial t} + k(t,x)u = \frac{1}{2}\Delta u + g(t,x)$, $u(0,x) = f(x)$ and for every $T > 0$, $\exists\ K(T) > 0$, $a(T) < \frac{1}{2Td}$ s.t.

  $$\max_{t\in[0,T]} |u(t,x)| + \max_{t\in[0,T]} |g(t,x)| \leq K(T)e^{a(T)\|x\|^2}\ \forall\ T > 0.$$

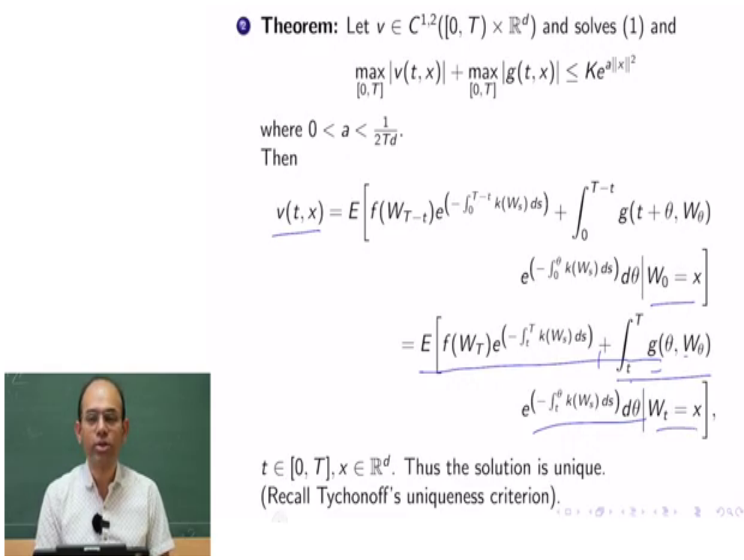
  (Typically $a(T) \downarrow 0$ as $T \uparrow \infty$ and $K(T) \uparrow$ as $T \uparrow \infty$) Then

  $$u(t,x) = E\left[f(W_t)e^{\left(-\int_0^t k(W_s)ds\right)} + \int_0^t g(t-\theta, W_\theta)e^{\left(-\int_0^\theta k(W_s)ds\right)}d\theta \middle| W_0 = x\right].$$

- $v(t,x) = E\left[v(t + r\wedge S_n, W_{r\wedge S_n})e^{\left(-\int_0^{r\wedge S_n} k(W_s)ds\right)}\middle| W_0 = x\right]$
  $+E\left[\int_0^{r\wedge S_n} g(t+\theta, W_\theta)e^{\left(-\int_0^\theta k(W_s)ds\right)}d\theta\middle| W_0 = x\right] + 0$
  where $0 =$ expectation of a zero mean martingale.
- As $|g(t,x)| \leq Ke^{a\|x\|^2}$ and $Ke^{a\|W_\theta\|^2}$ has finite expectation, using the dominated convergence theorem

  $$E\left[\int_0^{r\wedge S_n} g(t+\theta, W_\theta)e^{\left(-\int_0^\theta k(W_s)ds\right)}d\theta\middle| W_0 = x\right]$$
  $$\xrightarrow[r\uparrow T-t]{n\to\infty} E\left[\int_0^{T-t} g(t+\theta, W_\theta)e^{\left(-\int_0^\theta k(W_s)ds\right)}d\theta\middle| W_0 = x\right]$$
  $$= E\left[\int_t^T g(\theta, W_\theta)e^{\left(-\int_t^\theta k(W_s)ds\right)}d\theta\middle| W_t = x\right].$$

By mimicking the proof of Tychonoff's Uniqueness Theorem, we write the first term on the right of (5) as a sum of two terms using the events $\{S_n \leq r\}$ and $\{S_n > r\}$.



So expectation of v of t plus r Wr e to the power of minus 0 to r k W s d s 1 Sn greater than r. So here I am considering the other part of the event Sn greater than r. So you remember than we promised that we are going to have two different terms only we have talked about one term there, when Sn is less than or equals to r.

We are considering Sn greater than r, so in that case r minimum Sn becomes r itself so we are getting r v of t plus r here W r here and e to the minus 0 to r here, and then here this thing again we are going to consider the limit, so as n tends to infinity and r tends to capital T minus small t then v of t plus r would become capital T and e to the power of minus 0 to r that means 0 to the minus 0 capital T minus small t.

And here Sn, so as n tends to infinity what happens that Sn goes to infinity with probability 1, so r is on the other hand a small number between capital T minus small t, so the probability that Sn would be more than r goes to 1, so this indicator function grows to 1, with probability 1, so I am going to get only I mean 1 here or nothing we do not need to write down this again.

So here this limit is easily obtained as n tends to infinity, what happens? So using dominating converges theorem, we can put the limit inside because you know all these things are bounded here. So now we used dominated convergence theorem not because v is bounded because here not because of that but we can use it because v has that growth property for which we take the limit inside. Which we have already discussed earlier. So now this whole thing, we know that v is the solution or the PDE, so at time capital T v satisfies the terminal condition, the terminal data is small f, so v of capital T of W T minus small t is nothing but f of W capital T minus small t. So this is same as small f of W of capital T minus small t.

And then there what we do is that we change variable we do say change of variable here, so we just shift the time of Brownian motion from 0 to t, so earlier it was W0 is equals to x, so I am writing Wt is equal to x here assuming that starting from time t then everywhere wherever capital T minus small t minus there it would be capital T itself and integration minus 0 to capital T minus t is becoming now integration minus small t to capital T.

So we get this expression now, hence by taking limits in the bullet 5, so what is bullet 5? This one, this whole thing that I have written as you know the first term was written as sum of two different terms, one term that goes to 0 that we have proved and another term is going to another limit, and then this g term remains there.

So when we take that then we get that the formula here, that v of t plus x, so we go to the slide number 2 here that in the statement of the theorem v of t comma x is equal to expectation of this thing f of W t e to the minus small t to capital T, k Ws d s that we have just obtained now. And that integration g things that is as it is here, integration g theta W theta this thing.

So here, bullet number 5, we had r minimum Sn only but as n tends to infinity this Sn would also go to infinity and we do take r tends to capital T minus small t, so expectation of integration of g function would converge to the term which is appearing in the statement of the theorem that integration small t to capital T g of theta W theta e to the power minus this thing.

So actually you know here, the statement is written for with two different versions where conditional expectation was given W0 is equal to x another was W t is equal to x, so both the versions were written here. So as a corollary of the earlier result, I mean earlier this theorem what we have proved now, we can write down that look at the equation del u del t plus k t xu is equal to half Laplacian u plus g t x, u 0 comma x is equal to f x. So here it is initial value problem, earlier we talked about terminal value problem.

Now if you consider an initial value problem, initial value problem what happen? That Laplacian is on the other side of the equality, terminal problem that del u del t and Laplacian was on the same side of the equality, but it is on the other side of the equality but you get just you know by changing the variable T minus t as t, so did not get the negative sign so it should appear.

So for this we can since you know, this I mean the terminal value problem and initial value problem can be related or can be connected with only this small change of you know change of variable, so the same result apply there only suitable change of variable there. So we write down here, so for every capital T greater than 0 there exists one k positive and one a less than on over 2 T d such that.

If that mod of u plus maximum of you know supremum norm of g is less than or equals to these growth condition k times e to the power of a norm x square then as a goes to 0 as t tends to infinity I mean then u of t comma x is written as expectation f of W t e to the power of minus 0 to t k W s ds and integration 0 to t g of t minus theta W theta see earlier we had d plus theta but here we will going to get t minus theta W theta e to the power minus 0 to theta k W s ds, given W 0 is equal to x..

So earlier our integrant expectation that random variable was like you know of the form f of W capital T minus small t. So small t that running variable always has a negative sign there. Now here it is not, it is just this is time change happens so f of W t is appearing here. So that is the result, so we stop here, thank you.