Introduction to Probabilistic Methods in PDE
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Lecture 34
Solution to the Mixed Initial Boundary Value Problem

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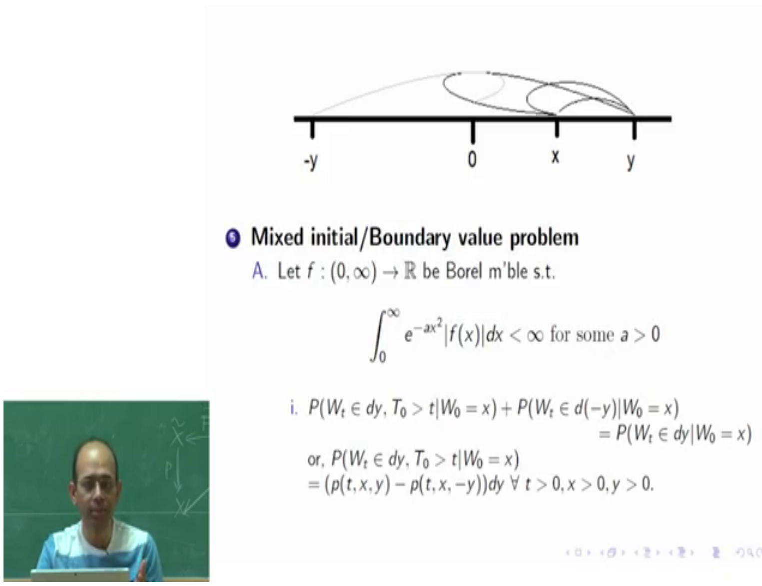

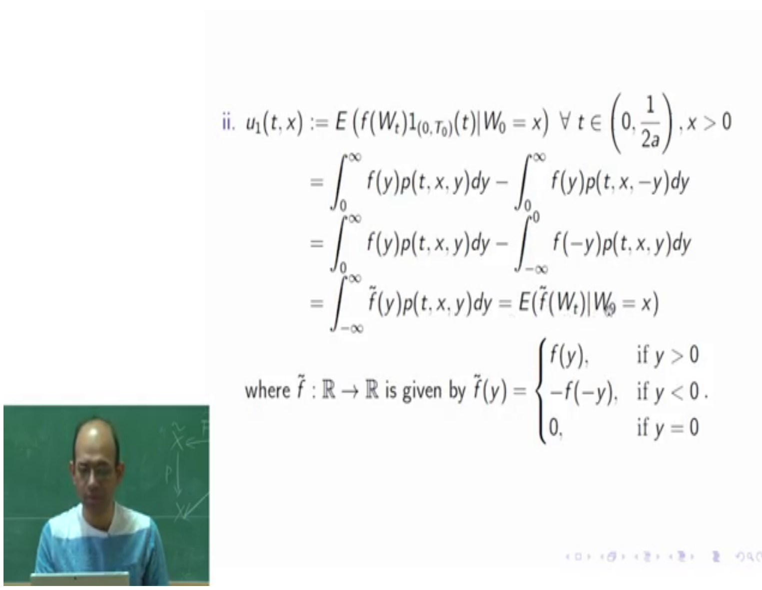

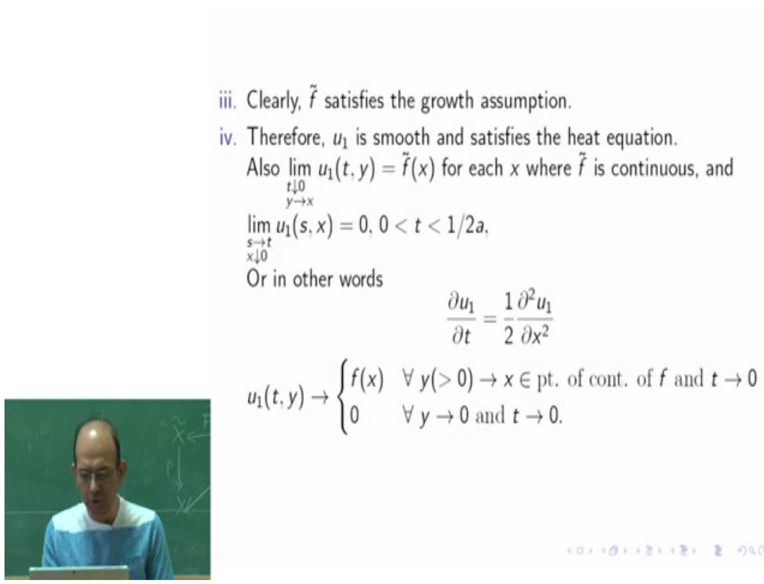

Now we look into the Boundary Value Problem, till now we were only talking about heat equation on whole real line, so the domain was unbounded, had no boundary and there only initial condition was given. But here we are going to address the question where some boundary condition is also given. For example, heat propagation on a finite rod, so earlier all the time we were talking about only infinite rod, infinite rod.

And imagine that two different end points has certain heat condition, possibly a sink or source is there. So, for those cases, this part is relevant, here let f be a function from 0 infinity to R and this is Borel measurable. So here, why am I taking 0 to infinity? Because you know I am not considering the whole real line, but all the positive real line, this is the simple way to introduce a boundary.

Because here, I mean it is lower boundary only, the simplest way to introduce boundary, so that would be my domain, so now we are talking about the initial condition on that rod, so which is you know left hand side it is bounded by 0, so there f is bounded Borel measurable is did not bounded, there is a Borel measurable function but has a growth property because you know on the right hand side, the domain is unbounded, f has plenty of chance to grow larger.

But we put a condition that we do not want very large growth of f, we say that it is 0 to infinity to the minus a x square mod of f of x dx is finite for some a positive. So now under this assumption of f we state the following, the probability that Wt would be found okay in a neighborhood of y okay, so this condition actually, yes so here I am not using f you know yet I would use it later.

But here it is very, you know this is very important result which is a consequence of reflection principle of Brownian motion, so let me elaborate. Consider this joint probability of this joint event, what is this? The W t that means if you start from point x here at time 0 and then, you know you just continue and then this one dimensional Brownian motion moves on these line. And then it goes towards left and then again comes to the right and imagine the case when it moves to the left but does not cross 0 like this type of thing that it goes left and then comes to y. So those events can be written as W t in the neighborhood of y, whereas T 0, T 0 is the heating time to 0 is greater than t.

So at time t, Wt is in a neighborhood of y, but small t is less than T 0, that means till time t the Brownian motion did not hit 0, is it clear? So this is joint event, so probability of this joint event given W 0 is equal to x precise that we starts from x. If you add this probability with Brownian motion at time t is in the neighborhood of minus y, and when given that you starting with 0 this two addition, addition of these two probabilities is equal to just probability that Brownian motion Wt would be in the neighborhood of y.

So this is the reflection principle, so let me give a sketch of the proof of this identity, now imagine that there are only two cases that either capital T0 is greater than t or less than or equals to small t, only two cases. So imagine that it crosses 0 and comes to y, when it does that, so then when it crosses 0 then we can reflect this path this way, this side this is just you know horizontal reflection.

When you reflect this path, then we get this grey colour path, so from here to here if you reflect along this you know this line, you are going to get this path and that will land up exactly at minus y because from here to here distance is y, so here to here distance minus y and for Brownian motion is concerned that, what does this reflection means? That at this point the Brownian motion when it took the left hand move, you just reverse it to the right hand.

And then you just follow, so that I mean, if Wt is a Brownian motion minus Wt is also Brownian motion, this negative of that is also Brownian motion, it is just the reflected part from this part to this part whatever the process is that if you just include the minus negative sign in front of that you are going to get this thing, form this point onward. So, therefore there exists one path like this, going to minus y.

So whenever it crosses, we can replace like this, so here that W t belongs to d minus and to reach minus y from x, you have no other way than this, I mean it must cross 0. So if it reaches minus y time t is fine if it does not it goes here then we just reflect, So probability that Wt is in d minus y, so region around minus y, given W0 is equal to x.

These two probabilities is same as just this happens because these are all possible cases, all possible way to reach y is written in terms of two different separate events, one is that reaching y before hitting a 0, another is that reaching minus y. So this is the sketch of this proof of reflection principle, we are going to crucially use this identity.

So we are writing down this, so equivalently the W t is in the neighborhood of y and T 0 is greater than t given W0 is equal to x just this term is equal to, now this I can take on the right hand side, so this probability is written in terms of the probability density function p of t x y starting from x reaching at y at time t, t x y but in a neighborhood yes, so I have dy here and from here I am going to get p of t comma x comma minus y.

Remember in the neighborhood of minus y dy. So this conditional probability of this joint event is written, I mean we know that it has this density, the difference of p t xy minus p t x minus y. So we are going to use this, let us define, u1, u1 is new function, function of t and x, u1 of t x is defined as conditional expectation of f of Wt where small t is less than T naught, so indicator function.

So when small t is more than T naught that means that Brownian motion hits 0 before time t, so T0 is smaller than small t, then this part will be 0 and it would not contribute anything to the expectation. So, this conditional expectation we now would be able to I mean that is written as u1 and we would now be able to compute this expectation, why? Because we know the density of that.

We know density of that, so this is f of y p t x y minus p t x minus y, from the last earlier slide we know that this is the density. So using that this is integration 0 to infinity f of y, why 0 to not minus infinity? Because my domain is now 0 to infinity, f of y pt x y dy minus integration 0 to infinity f of y pt x minus y dy. So now here this is rewritten so here just it is change of variable formula.

So minus y is rewritten as y, so y becomes minus y here, this change is this. Now if we introduce a new function f tilde, which is same as f on the positive line but it is minus of f of minus y when y is negative, then see I mean I can talk about it because if y is negative then minus y is positive, so f is defined on the positive line, so we can talk about it.

So then, we can use this f tilde then this integration is nothing but f tilde of y minus infinity to infinity p of t xy dy. And here we have minus infinity to infinity, full thing and we know that what does it represent, it is nothing but integration, this is nothing but the expectation, because this pdf is appearing here. So this expectation f tilde W t given W0 is equal to x, because this is the density of Wt. Okay so f tilde Wt given w0 is equal to x. So we got one representation of u1 that is using this.

So clearly f tilde satisfies the growth assumption, see here we had this growth assumption on f, so it basically says f cannot grow like e to the power a x square because then you know this would be infinity, it cannot be finite. So this growth condition also puts condition on f tilde on the both sides here, so here you get minus infinity to infinity, this is finite of f tilde.

So it satisfies the growth condition and we have already seen that when this is the case then this you know this conditional expectation when this subset of growth condition then this conditional expectation satisfies the heat equation, so u1 which is define in this manner satisfies the heat equation, and in addition to that if we find out the limit that what happens when t tends to 0 and y tends to etc.

We would see that it would also be f tilde x, why is it so? Because it inherits from the function f, we already had that. So continuous and this, and then limit u1 s tends to t, x tends to 0 because 0 is the point when f tilde you know, so there, that is 0 because here see I mean for y is equal to 0 we had 0, f tilde 0, so this is 0.

So this t, I mean from the theory what we have seen that we can define for t from 0 to 1 over 2a, so for that we have this solution. So this whatever I am written this fourth part that is also rewritten now again and exactly the same thing but in different language that u1 satisfy this heat equation and u1 ty almost is there, so obvious this is already I have stated, I mean this part is important that, I mean at the point of continuity of f because everywhere this limit will not converge here, only to for those point x where f tilde is continuous so there it would converge.

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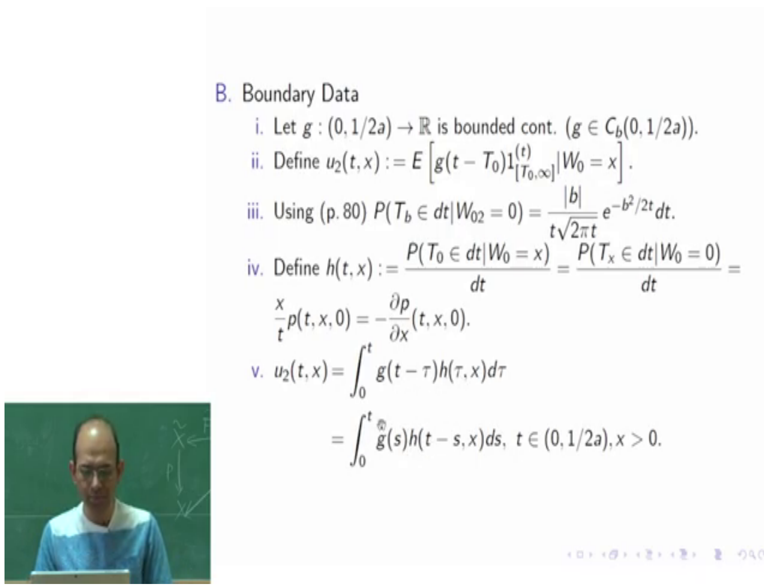

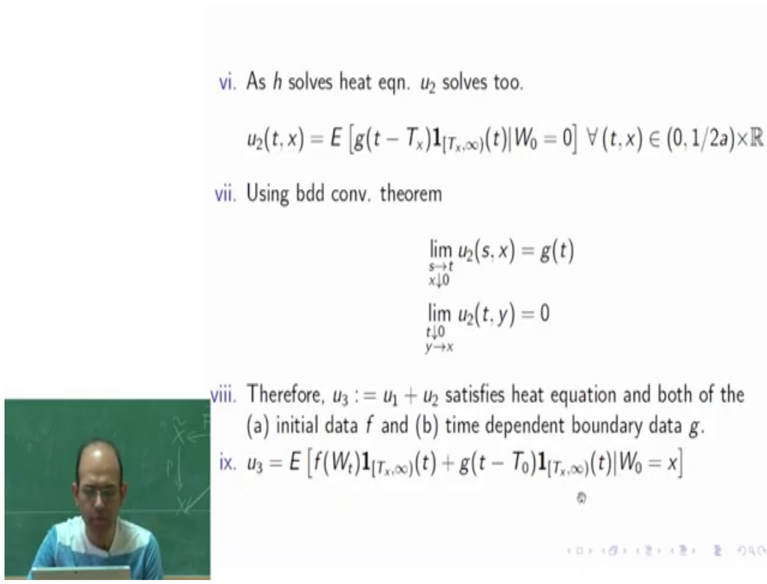

Now we talk about boundary data. So I mean, till now I was talking about f, f was distributional data my domain is bounded, so I should give some kind of boundary data here. So here g is so the boundary data should depend on time, because the boundary remains boundary for all time.

So how the boundary conditions are acting for each and every time that you need to clarify, so here time is between 0 to 1 over 2a, so g is a function of 0 to 1 over 2a to R is bounded continuous function that we assume here, so in short notation g belongs to Cb 0 comma 1 over 2a and you define u2 of t comma x, u2 of t comma x that is conditional expectation of g of t small t minus t capital T naught T0 into indicator function of capital T0 to infinity.

So here, this small t is more than or equals to capital T0, given W0 is equal to x. So there is a typo here it should be open infinity, so not closed to infinity, is open infinity. So ignore this reference, so what do we do is that, I mean this reference is actually the reference page number of the book of Karatzas and Shreve so there you would be able to find out this formula.

So this is a typo, there is no 2 here, W0 is equal to 0, so what is says is that, so when a boundary b and W0 is 0 then you are asking at this distribution T b, so like here T0 is the heating time to 0 T b would be the heating time till the point b for the first time, so that time is in a neighborhood of t d t, so that has a particular expression that is equal to mod of b divided by t times square root of 2 pi t e to the power minus b square by 2t dt.

So next we define one function h, so this h is basically a density of T naught, so probability the T naught belongs to dt given W0 is equal to x given and dt, so this is of our interest, so here this we can rewrite in the following manner instead of thinking that Brownian motion starting from x, you imagine the Brownian motion starting from 0 but then this would be t x because, I mean x is here and 0 is here so and Brownian motion starting from x and then heating to 0 is same as the corresponding distribution which is same as staring from 0 and heating to x.

So probability that Tx belongs to dt given W0 is equal to 0 divided by dt. So this is the place where we are going to use the above formula, so this is equal to from this place, b instead of b we are going to write down x, here it is non-negative so x here, so t and then this part would stand as p, t x, 0 because you know here we can see clearly that it is p t x0 if you replace b by x.

So this we have already recognize that this is nothing but the partial derivative of p with respect to x, so here what we have obtained is an expression of the density function of T0 which appears here. To find out this expectation is important to know that distribution of T0 and density functions, so that is having this expression.

Now, we are in the stage of computing this expectation, so u2 of t comma x, u2 of t comma x is expectation of g t minus T0 but we know the density of T0, so we write down that this function of T0, so write down g of t minus tau and then integrate with respect to the multiplied with the density of T0 that h of tau x d tau.

So that we do, and since here T0 to infinity, so I mean by small t I mean this T0 could be at most small t, T0 could be as slow as 0 but T0 cannot be more than t if T0 is more than t, this part is 0, so I would integrate from 0 to t, small t understand because I am finding expectation where capital T0 is a random variable.

So for that pdf I am using, but I cannot use T0, 0 to infinity here but because T0 can be at most small t. Now this is just a simple change of variable formula you write down t minus tau is s then you get here g of s h of t minus s x ds. And if t minus tau is equal to s, I mean and that and if t minus tau if tau is equal to t then t minus tau is equal to 0. So you would get 0 here, you would get t here but with a negative sign again you can change the limit upper and lower, you get this value.

We have already seen in earlier lectures that this del p del x solves the heat equation, so we are using that since ut is u2 solves heat equation 2, so u2 of t comma x is equal to expectation of g of t minus capital T x, so here you remember this T0 w0 is x and then we have argued that we can write down W0 starting from 0 and this is Tx, so evaluating that g of t minus T x, T x to infinity t, here it is corrected, open interval and W0 is equal to 0, for all t x.

So this is the expression of u2. So, what we have obtained that u2 which has this expression satisfies the heat equation, that we have proved here, from here because that heat, this h follows solves the heat equation, this integration solves the heat equation, u2. So u2 also solves the heat equation and then it has the expression.

Now what we want? We want to take limit of u2 t y, so here since g is already given a bounded continuous function, we can do so because here bounded continuous function is here bounded function and so you can do that. So limit s tends to t u2 of s x as x tends to 0 because 0 is the boundary, so x square whole right hand side here it is you know the space variable.

So that is equal to g t we are going to get it because of you can actually take the limit inside by taking limit inside we are going to get, we can take limit inside because g is bounded and also similarly that when t is coming to 0 the initial stage, so let us see what happens when t is closed to 0 then if this capital T x whatever it is when t is smaller, so it would cross that, so that means this part would become 0.

So for every omega you know that T x would be some positive this thing but small t equals to 0, so almost surely this goes to 0, so if you take limit inside, this whole thing will be 0, so u2 t y is 0. So here we can consider u2 solves of course the heat equation, and it solves this boundary condition because as x tends 0 it satisfies the boundary condition, on the boundary it is g of t and it satisfies 0 initial condition. So it satisfy the initial condition is 0, so we start with 0.

On the other hand when we have seen this u1, in the earlier slide u1 that satisfies this heat equation and here the initial satisfies the initial condition f x but on the boundary, so when y goes to 0 then it is 0. So that means it satisfy zero boundary condition, so like you know here we have separated, so u1 and u2 are like you know capturing two different things separately, okay isolatetly.

u1 was capturing the initial condition, u2 is capturing the boundary condition and both solves the same heat equation inside the domain interior. So that is a good thing so we can just add these two, if you add these two u1 plus u2 then we get u3, u3 would also satisfy the heat equation because it is a linear pde the addition of two solution is also a solution and then the boundary and initial data both would be satisfied by u3.

So the new function u3 actually solves the initial boundary value problem, so this is the thing and so u3 now we want to write down u3, u3 is nothing but u 1 plus u2, so u1 is written here f of Wt 1 T x infinity t and u2 is written here g of small t minus T naught indicator function of T x to infinity. So, this expression combine this whole expression gives us the stochastic representation of initial boundary value problem for heat equation.

So next we are going to see the interpretation of this expression, I mean already we have seen when there is no boundary it is unbounded domain that how it means so for bounded domain we are going to see now.

Imagine that we have chosen a point t comma x and then so x is this and small t is here, so whole things is a function of t comma x and here we start a Brownian motion time 0 at point from the point x.

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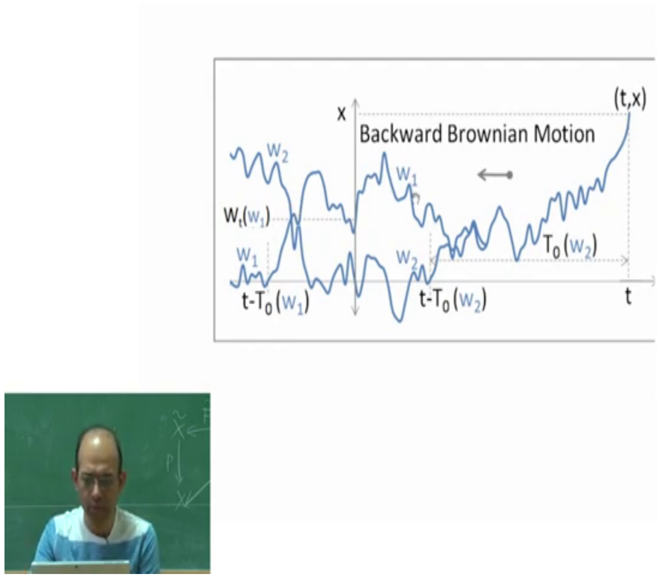

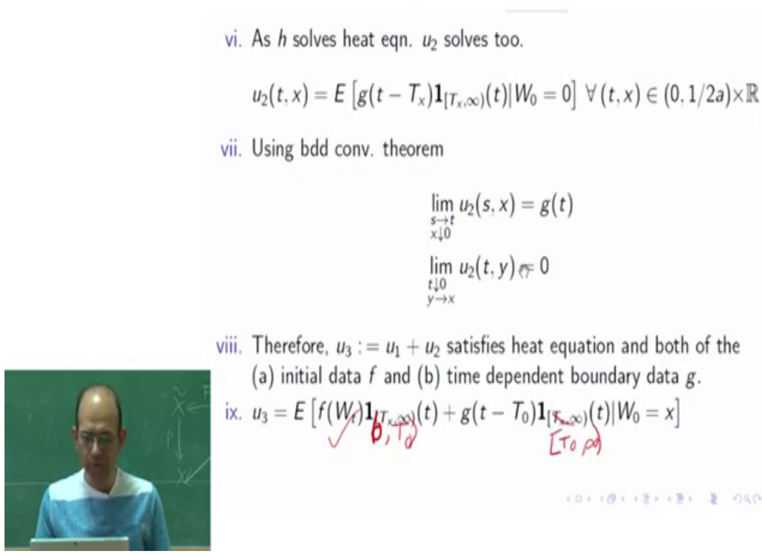

So we start a Brownian motion from here and imagine the Brownian motion so this is all path of Brownian motion, which is moving and then at some point we see that possibly there actually for every point you can go many different place spaces, so we just see only two paths, so one path is moving like this and coming here and the another path is coming like this and then going here.

We call this above path is W1 and below path is W2, and then see that when I mean this W1, W2 are two different paths which has I mean which I have chosen for a particular reason because they do things differently, W2 hits the this you know time axis first, whereas W1 heats the x axis first and then time axis.

So, whatever the case is, so what we do is that we find out this point, so this point is for the path W2, so this distance, this time distance is T naught W2 because this is the first time it reaches 0 because this is the x axis, so this time axis you know hitting time axis means that space becomes 0, for the first time that the motion reached 0, so that is T0 W2 and this time point therefore small t minus T0 W2, so this time t minus T0 W2.

On the other hand, if for W1 when it hits this time axis then we are going to say that okay this whole thing is T naught of W1 and then this time point is t minus capital T naught W1, clear? And when this W1 hits this x axis, then what happens? That means that it is like initial time because t is equal to 0, here t is equal to 0.

So initial hits the initial time point, so there this point the locus of this point is W t of W1, why W t of W1? BecauseWw1 is the path and this your Brownian motion started here so it is the time 0 and then your clock started, since this distance exactly time t you need exactly time t to hit here, does not matter whether you go up or down you need to exact time to hit here.

So when you heat here, then in you clock you have t time Brownian motion, so Brownian W t of W1 is this point, of course this is a random variable, so now we see this slide here f of Wt when small t is more than T x, here it should be x there is a typo, g of small t minus T x, so if I write down W0 is equal to x then it should be 0, it should be 0, it should be 0, so there is a typo that I should have 0, 0 here.

If I write down 0 here, then I should write x, so it should be 0. Now so these are the two different cases, so now for W1, here T0 W1 is more than t small t, T0 W1 is more than small t, so this part would be 0 because T0 is more than a small t, so small t is or in other words small t is less than so this part would be 0, so this part would contribute only.

Here it should be open 0 comma T0, there are many typos here, so let me write down on this, T0 comma infinity, and other things are all correct, so this part is wrong, open 0 comma T0, so here, when let us go back to the picture here when we talk about W1, so in W1 capital T0 W1 takes more time from here to here, is ofcourse longer than small t over small t is from here to here.

So here we would so capital T0 is longer so therefore, small t of this is 1, so this part would survive here, so it would pick up the f of W t, however for W1 since T0 is longer, so this part will be 0, so g function would not be appear here, so for W1 we are going to consider f, f would only appear, so f is initial condition. So f appears here, I mean f is given on this line because this time t is equal to 0 line.

On the other hand, for W2 when T0 is smaller than t small t because T0 is smaller than small t for W2, then this part would not contribute because small t I mean is more than T0 this indicator function would give you 0, whereas here you are going to get a non-zero value it would be 1 and then g of t minus T naught would be there.

So and g is defined on the boundary, so this t axis is the boundary of x, so it is clear, correct? So it is now clear that what you do if you want to find out this you know this u3 using stimulation, for example so you just generate the Brownian motion if it and check which part does it hit first.

This line or that line, whatever so if hits this line then at this point you compute g function, if hits this line, then at this point you compute f function, and all this values you just take you know at an average, you just average all these values that is it, nothing other than that. So here I mean at this point you have to evaluate the function of g of t minus T0 the amount of time it is required, and then you average.

And then that would be an estimator of u3, so that much just I wanted to say, thank you very much.