Introduction to Probabilistic Methods in PDE
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Lecture 33
Widder's result and its extension on heat equation

Okay, in earlier lectures, we have seen heat equations, solution in terms of conditional expectation.

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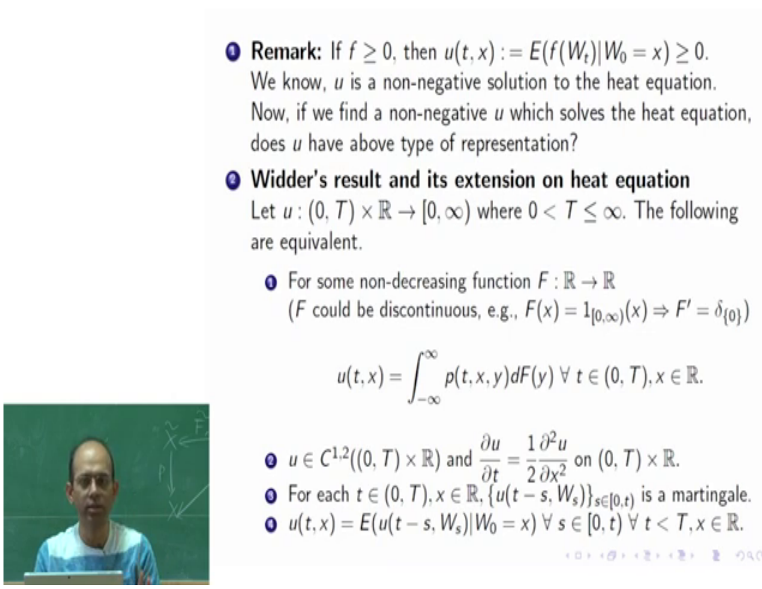

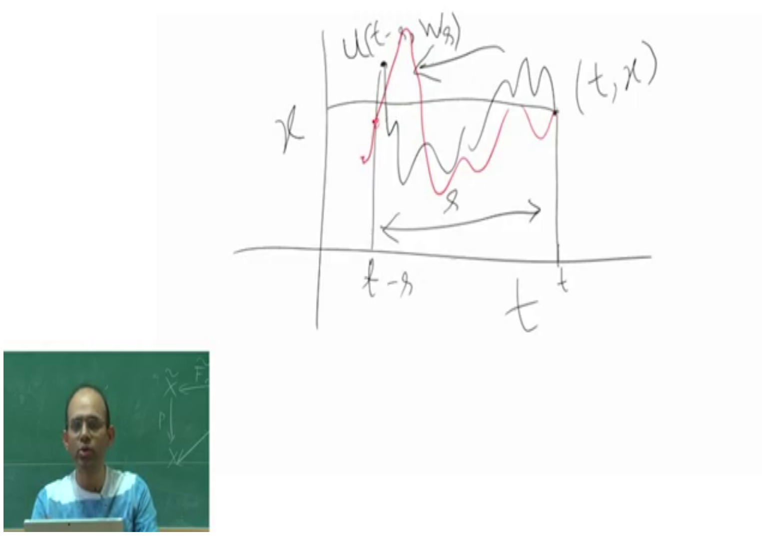

So, here we recall very quickly that if f is the initial data and then the expectation of f of Wt given W0 is equal to x, this conditional expectation gives a particular function of x, correct? Where W is a Brownian motion and that function of x if you write down u of t comma x, because this also is a function of t. So, u of t comma x, then this function solves the heat equation.

So, we have seen that and then what we are going to now ask a very straightforward question that, if my initial data is non negative that f is greater than or equal to 0, then from this representation it is evident that u should also be non negative, why? Because it is after all expectation of a non negative function of a random variable. So, whatever Wt value is positive or negative does not matter f of Wt, would always be non-negative and then you are taking exception, that is nothing but an average.

So, this is greater or equal to 0 and what we have seen in earlier lectures that given some kind of conditions on f. So, like growth condition, that this satisfy the heat equation however, the reverse thing we have not established, what is the reverse thing? That given the heat equation given a the heat equation, whether it has a solution I mean whether you can always write down this as in this form the solution can be written in this form.

So, we are going to ask this question, if we find a non negative u, which solves the heat equation, that is del u, del t is equal to half into second order derivative of u with respect to the space x. So, when we talk about multi-dimensional, we write down second order derivative their Laplacian, correct? Now, if we find a non negative u, which solves the heat equation, does u have above type of representation? So, this question we are asking here now, and we are going to answer a affirmatively.

So, this is the Widder's result. So, Widder's result is as follows, so let here u be a function on 2 variables, time and space. So, this domain is open interval 0 to capital T and space is whole R unbounded domain and this u is non-negative. So, its value is 0 or more than 0, so here the following are equivalent. So, this theorem is saying that the following are equivalent, what are these? That for some non decreasing function capital F, u of t comma x, this u, u of t comma x is equal to minus infinity to infinity p of t, x, y and then integrated with respect to capital F of y.

How can we do this? Here, already we have assumed F to be non-negative or non decreasing. So, non decreasing functions for that we can define the notion of Stieltjes integral. So, this is just stieltjes integral. And then, here capital F could be any non decreasing function it could be you know discontinuous function also like this, like this that indicator function of closed 0 to infinity x, basically heaviside function, that on the positive axis when x is positive or 0, then F is 1, if x is negative then F x is equal to 0. For such type of functions it is discontinuous you cannot talk about its derivative, it is not differentiable.

So, this stieltjes integral cannot be further reduced by dF y divided by dy, dy you cannot write on that way. So, it is little different than this type of expression, where we had u of t, x is equal to expectation of f of small f of Wt, where you can actually write down small f of y, dy and P t, x, y, because this expectation here looks exactly like this where the same density appears, where I mean like this same density appears, but the function here like small I mean, like you can think this is the anti derivative of small f.

But if small f is already a bounded measurable function, then its anti derivative would be continuous, but here we do not have that constraint. So, capital F is allowed to be discontinuous even. So, this is more general than this setting, so here the derivative of capital F can be using the distributional sense, the distribution sense derivative of capital F is Dirac mass at point 0.

So, now the following statements are equivalent, so here I have just written the statement number 1, so statement number 1, is saying that this non-negative function u is retained, I mean as if just this solves the PDE heat equation because it looks almost like same, but is more general actually the antiderivative of small f appears as F.

So, the second condition is that, that it solves the heat equation, is not surprising correct because of this particular nature, it is natural that, it would solve the heat equation I mean natural I mean it is not straight forward proof, because we have not proved this for this general generality but for the special case when capital F has a derivative, then this is exactly this. Then we have already proved that solves a heat equation.

So, this Widders result says that, okay even with this generality this u, would also solve the heat equation, because this is u is classical solutions. So, it is once differentiable with respective time variable and twice differentiable with respect to the space variable C12 and del u, del t is equal to half del 2 u, del x 2 on this domain.

There is also a third criteria, which is also equivalent to these two, what are these? This is for each t positive between 0 to capital T and x is real number, u of t minus s, Ws. So, this you can view that as a stochastic process as s runs from close 0 to open t. I cannot simply put s is equal to t, I cannot make it closed, why? Because for s is equal to t, this u of t minus t would be 0, u 0. And for and Wt.

So, now u 0, here if you put t is equal to 0 you do not what would happen here, because here for u 0, I can write down I know that W 0 is equal to x and it is expression f of W t and t is going to 0 say that okay, f of W 0 but W 0 is constant, so, you can write small f of x directly, but here you cannot do that simply, why? Because here you do not have exactly this type of expression, here it is because capital F derivative does not exist, it is this.

So, I may not have u 0 value at 0 I may not have a value, so I cannot have a closed interval t, I can have only open interval t, however above 0. So, in the term like t is equal to 0 is the boundary of the cartesian product domain is one part of the boundary, but it is not in the interior.

So, in the interior, we can of course define this function u, because and in the interior u is not only defined it is continuous, it is differentiable, it is you know it has all the regularity as required to solve these PDE. And this process as you know s is the time index, when t is fixed, this is a martingale, this is martingale u of t minus s, Ws is a martingale. So, how are you going to see this see, for example it is like a backward kind of thing, because here when s is equal to 0, then is it is actually u of t comma W 0.

So, let us see this, imagine that this is my time axis and this is the space, this is the space right down this way and at some particular time t your say you are at some point particular point x. So, this is this coordinate is t comma x and imagine that you are generating one Brownian motion starting from this point, but not as time forwarding but time backward.

So, this is a Brownian path and then you stop at some point, say here t minus s, so basically this distance is s, you go s, so here this direction and then where ever you stopped. So, here you find out this, this value u of t minus s Ws. So, here this coordinate is t minus s and then, what should be, this height because it started from here a Brownian motion and that Brownian motion here it took exactly s amount of time.

So, it is Ws here this position W0, Ws, so this point you evaluate the function u, so u of t minus s Ws. Now, you continue this Brownian motion s you continue and continue doing that for each and every s, starting from 0. So, here s is 0 starting with this and then, what are you going to get? You are going to get a stochastic process, because this function although this function is deterministic, but the path is random.

So, you are going to get a stochastic process and that stochastic process turns out to be a martingale. It is not clear otherwise because u I mean, W is of course a martingale, but a function of martingale need not be a martingale like W square is not a martingale. So, u is a function of this but since u satisfies the heat equation. So, these things becomes, so it is not true for all possible u. So, it is only special for some particular u and it is just saying that these three are equivalent.

So, am I clear? Could I emphasize that this third condition also clarify I mean, put some condition on u. So, here u is written in this way all function u cannot be written in this fashion, u satisfies this PDE, the all functions does not satisfy this PDE for all u, you may not get the martingale, but okay these three are equivalent. There is a another fourth statement that u of t comma x is equal to expectation of u, t minus s, Ws.

So, left hand side does not depend on s, right hand side depends on s, so this is true for all s, so if I fix one particular s, say s is not allowed to take value t, because if I put t, then u of 0 would appear, but I do not have that, I do not have from I mean, neither there, u is also define only on open interval 0 to capital T.

But say u of, if I put s is equal to 0, then I can check what does it mean, what does it mean? That here the Brownian motion is starting from point x as I should have shown in the picture and then for s is equal to 0 the W 0 would be not a random variable but a deterministic value that x and u of t minus 0 is u t. So, here expectation of that would be just u t, because this is deterministic. So, u t x you are going to get that.

So, left hand side and right hand sides are equal when s is equal to 0 that we can see very easily, but in general I mean, s is equal to 0 is trivial I mean, that is true for any function u, but in general for any arbitrary function u you would not get this to be true, because if s is non 0 then it is not obvious.

So, here this part also we can visualize in this that you fix one particular s and then you simulate Brownian motion backward direction may be you know a million time. So, another path possibly here and then evaluate u function at this point, u function is defined on the Cartesian product t comma s.

So, here so we have drawn this black and red color, these two paths of Brownian motion and then well there we see that it stopped somewhere I mean, at the point time point s and then if we generate such type of millions of paths, and then at each and every for each and every path when it stop somewhere at some points t minus s comma Ws.

So, there we evaluate the function u and we are going to get some particular number, we are going to get some particular number and we take average of that. And here since our function u is non negatives we are going to get millions of such non negative numbers and we going to take average and that average would be an estimator of expectation of u of t minus s comma Ws. And that value would match with the function u at the point t comma x, which was a starting point of the Brownian motion.

So, this is the fourth statement of fourth statement that u of t comma x is equal to expectation of u, t minus s, Ws expectation. So, one can find an estimator of this expectation by taking you know by simulating W say million times and then finding out that point and then taking average of those values.

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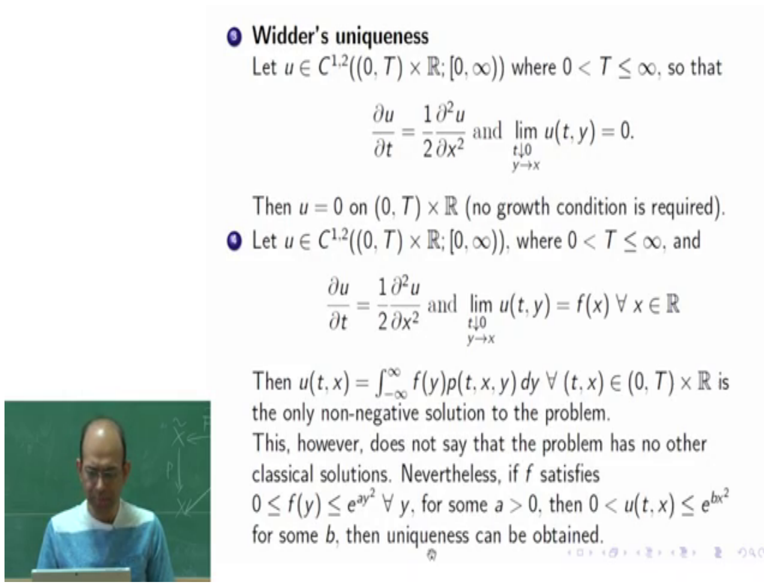

So, next we come to Widder's uniqueness theorem, so we are not going to prove this theorem, we are going to just state this theorem and while stating we are going to you know clarify that where this is you know different from other earlier theorem. So, here let u be a smooth function that C 1, 2 function must continuously differentiable with respect to time and twice conscious differentiability of space, and it is non-negative function and u satisfies the heat equation del u del t is equal to half del 2 u del x 2.

And here, furthermore that at the initial time if you come close that t tends to 0 and y tends to x u t of y if you find out and if that value is equal to 0 for, I mean each and every choice of x. Then the function u should be 0 inside the interior, should be 0 identically. So, in this statement no growth condition is required on u, remember that Tychonoff theorem we had also another uniqueness theorem with similar conclusion.

But there we had to assume that u satisfy certain growth condition, that you know of t comma x into I mean is less than or equal to e to the power of a x square for some positive a. So, here we do not need such kind of condition, but however we of course require u to be non-negative you understand that.

So, earlier u was allowed to be positive or negative but should satisfy some growth condition here, we do not need the growth condition but we need u to be non-negative. So, you understand that these two theorems are not actually a special case of either of these, but they have different applicability.

Next let u be again you know the C1, 2 function and del u del t is equal to half del t u, del x 2 and here instead of 0, I am writing f of x, that u of t y as t tends to 0, y tends to x, u of t, y goes to f of x, for all x in R. So, if that is the case, then u is equal to minus infinity to infinity f of y, this is the representation. Then u the solution u has this representation it can be written in terms of integration of f and integrated using that probability density function p.

Is the only non-negative solution to the problem. So, here we have got the uniqueness that the solution, it has only one single solution and that solution is written in terms of this. So, this is in other words the conditional expectation of f of Wt given W0 is equal to x. So, this representation is there

However, this does not say that the problem has no other classical solution, so here so far non negative solution we are talking about, but it can have positive, negative like the classical solution. So, then as I told that, this is not a special case of Tychonoff's theorem these are two different applicability. So, if you want, I mean if you have these you know bounded the condition in addition then you can talk about unique solution.