Introduction to Probabilistic Methods in PDE
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Lecture No 31
Uniqueness of solution to the heat equation

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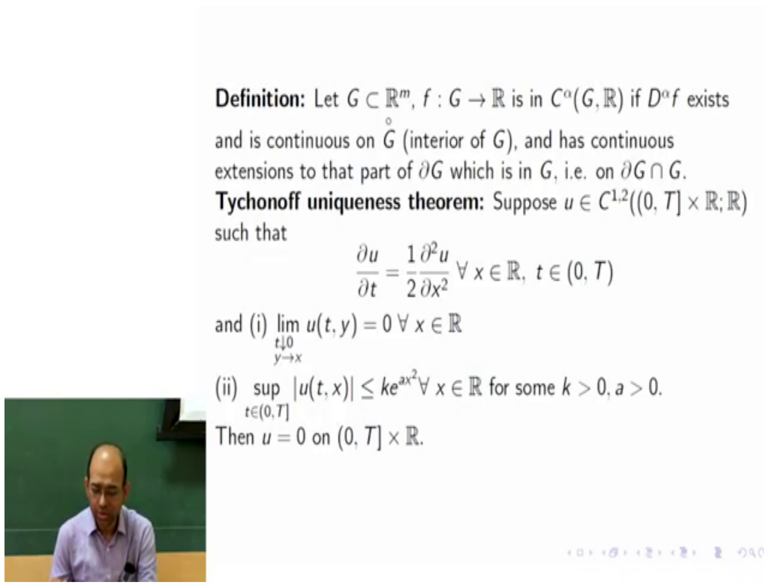

Today we are going to say Tychonoff theorem for uniqueness, this theorem says that, that heat equation, when the heat equation has the initial data 0. So, then the continuous solution would be 0 on D, I mean under certain mild condition okay. So, let us see this theorem, why do you call such theorem uniqueness? Because this basically says that, if I have one particular initial data whatever it is and then I cannot have two different solutions of the heat equation with a similar data, same data, why is it so? Because imagine for the time being okay.

So, let if possible that there are two different solution U and V, of the same heat equation with the same initial data, then U minus V would also satisfy the same equation, because this equation is linear. So, if U and V are both are solutions, then U minus V would also be solution. But in the initial data they would to nullify each other.

So, we are going to get 0 there, U minus V would be 0 in the initial line, what is my initial line? Initial line is actually time is equal to 0, and space is throughout whole. So, then this theorem would suggest that this U minus V, which is solution of the heat equation and which has 0 value in the initial line. So, that U minus V must be 0, although in the whole space. So, that would also in other ways say that U and V must be equal to each other.

So, this type of theorem actually asserts or assures uniqueness of the solution, there are certain you know specifications, which is very important this very important part to actually be careful. First, we start with what do we mean by a C12, like you know I said I told, I am looking for solution, which is continuous and which is smooth there.

So, the general definition goes this way and that I would say that if given function f is in C alpha to the power in the superscript, C alpha of G semicolon R, if f is a map from G to R such that the map this f is alpha times differentiable. Now, what is alpha time? Basically, alpha could be multi index correct, because this m need be 1, f need not be defined on 1 dimensional space. But it could be a subset of a multi dimensional space, there are actually m number of variables. So, this alpha is multi index correct.

So, the alpha could be actually 1 comma 2 if m is equal to 2, that means that, it is once continue continuously differentiable with this the first variable and twice conscious differentiable with respect to the second variable. If alpha is the multi index 1 comma 2 and m is equal to 2. So, in general, when alpha is just a multi index and D alpha f exists inside the interior of the domain G. So, G interior, this circle denotes interior and has continuous extension to the part of del G, the del G is basically boundary we denote boundary by del G.

So, continuous extension of del G, which is in G, that is that if the if the boundary is inside the concentration of G, where f is given, I mean if f is not defined on some point, we cannot talk about continuity of f at that point, we cannot talk about. However, if f is defined on some you know point of G, which is actually boundary of G, then we have the opportunity to cross verify whether f is continuous at that point or not.

So, how are we going to do that? We are going to say, that f is inside the interior it is defined and it is del, I mean alpha times differentiable and that is continuously extended. So, if you extend that you take limit and the limit would coincide with the value of f at that point on the boundary, then you are going to say that, f is continuously extended to the boundary. So, if that is the case, then we call that f is in C alpha G R, so that is a notation.

Now, if we look at this class of functions C 1 comma 2 here alpha is 1 comma 2, C 1 comma 2, open 0 close to T cross R, semicolon R. So, if you look at this and ask that what is it doing, so here boundary that capital T. So, this capital T cross R, so capital T singleton cross R would be the boundary of this, which is inside the domain, of course it has another boundary which is not inside the domain namely singleton 0 cross R, which is not inside the domain.

So, this was just a definition of this notation or what do you mean by these notations, C1, 2 function sometimes you know for such type of heat equation it is pretty clear that what is first variable, what is second variable, I mean generally it is an understood that the first variable is time and second variable is space. So, when you say C 1, 2 we mean there if it is once differential with respect to time and twice differentiable with this 2 space.

And this much a regular to we need for classical solution of this equation, because here equation involves the partial derivatives or the first order in t and second order in x. Now, we ask that, so what is the solution of this, which satisfies the following two conditions. First condition is that the limit of the solution, I mean if you extend to the another boundary which is not included in the domain.

So, here t is going to 0 and y is going to some x, so this way actually this boundary point 0 comma x is the limit point of such points correct and we are asking that we putting this condition that limit of u is equal to 0. So, basically it is saying that although u is not defind on the boundary, because here the domain does not include the initial time correct, it does not include.

However, as we are looking at this equation, we are looking for the solution, which is defined on this and the solution if you now we approach to the initial data, initial value that means approx the t is goes to 0 and y is going to some x, then that value should be 0. So, this is the way to put initial condition for a problem where actually 0 is excluded from the domain of definition of the function.

And then, second point is that we also require that, u has some bound on its growth, that growth is e to the power a x square. So, it has some upper bound of the growth, which is independent of t. So, supremum over all t between 0 to capital T, if you take all these supremum, then this whole thing becomes a function of x and we are requiring that, this function is dominated by k times e to the power a x square. We understand that e to the power a x square grows very fast, because it is exponential growth.

But we are requiring that this is lower than that, we are going to see that why do we need this later, but now for the time being, we consider a problem which is like this, that we solve the heat equation, it has this initial condition and it has this growth property. In that case, if u satisfies all these case, that case we can assert that u is basically the 0 function, the trivial solution because we put 0 it solves trivial e, if it is equal to 0 its solves trivially.

So, it says the u is the only possibility is the trivial solution, u is 0 on the whole space. So, that is the Tychonoff uniqueness theorem and we call this theorem uniqueness, because this theorem helps us to establish uniqueness of heat equation with general initial condition.

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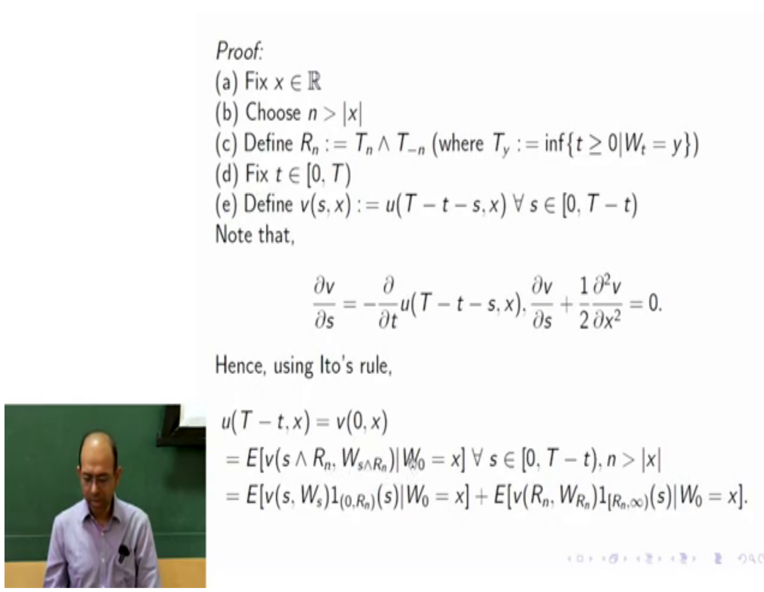

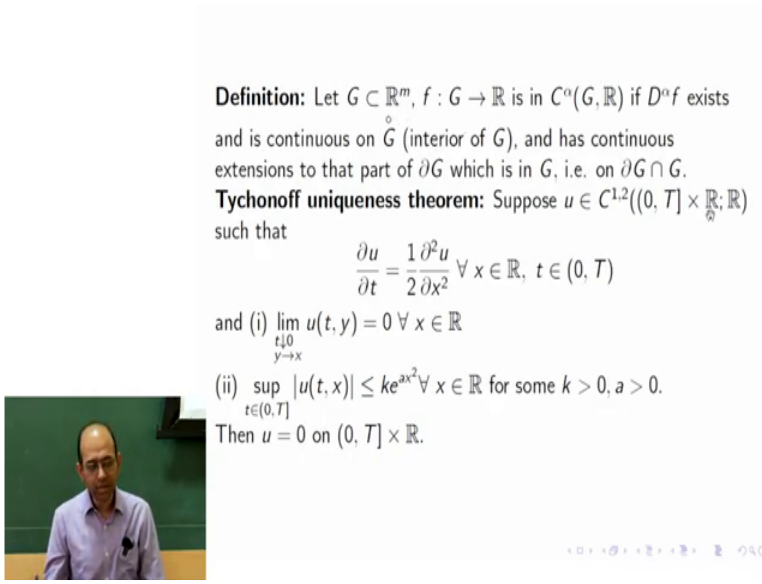

Next we see the proof, the proof is not too long, however it has many steps. So, we go through that slowly. So, what is the goal? The goal is to show that the solution is 0 at every point inside the domain also like. So, in the interior also, so to do that we have to pick a point there, that means a particular t comma x we have to pick up, all right?

So, we pick up a x some x we take some real number and then, so here for this exposure for this proof I am taking only one dimensional case. So, here we consider x is in R, although if you look at yes so here also here all these things I mean although this definition was for multi-dimensional case, but here we are taking space is only one dimensional.

Now, we given that particular x, I need to show that u of t comma x is actually 0, for that so there are many steps, the first step what we do is that we choose one integer n large enough. So, that n is more than mod x, x could be negative also or positive whatever but mod x is positive number.

So, choose n is n such that it is more than mod x, then we define R n, R subscript n, which is defined as minimum of Tn and T minus n, where Tn is actually the heating time to the level of n, that means when Brownian motion which has started from 0, which become you know as large as n.

So, that is Tn and T minus n is that, when it first time hits you know minus n. So, this is definition Ty subscript y is infimum of t that is the first time, when Wt becomes y. Now then this

Rn you know that at time Rn, the Brownian motion valued be either n or minus n. And we also know that before Rn the brown motion value would be the mod of Wt would be less than n for sure for all the time.

So, that is going to give me some kind of you know control over the growth of the Brownian motion and Rn is surely a random time. Now, we fix some small t in close 0 to open capital T, I mean that is as expected as we have already decided, because we are going to do we are going to fix x and t. So, with this fixed x and t, then we define v of s comma x, so this function is defined as u of capital T minus small t minus s comma x, for all s in close 0 to capital T minus small t.

So, what we are doing is that, we have fixed t of course, we have fix x of course, but now we are defining a function v on the interval close 0 to open capital T minus small t. So, that when small s is 0 here, then I am going to get u of capital T minus t, when s is equal to the end point capital T approaching to capital T minus small t, then this would approach to 0, although u of 0 is not defined but this you know this variable argument would go to 0.

So, now given this is the definition of the function v, we can find out the partial derivative of v and write down the derivative in terms of the partial derivative with respect to u. So, del v, del s, here since the minus sign is there would be minus del del t of u, these two are equal of capital T minus small t minus s comma x.

Now, the other derivatives like space derivative of v, is same a space derivative of u. So, we are going to get del v, del s, plus half del 2 v, del x 2 is equal to 0, earlier what we had with respect to u, for u we had del u, del s is equal to del 2 half del 2 u del x 2. So, del u del t is equal to half del 2 u del x 2 we had. Now, we got sign changed, so we get del v, del s, plus half del 2 v del x 2 is equal to 0. This is the new equation for v when I write down del u del t, I do not mean that I am differentiate with this is this t I am saying I am differentiating u with respect to the first variable.

So, this is important thing to understand, that this notation del del t u or del u, del t does not matter what is my argument, it is first evaluated, that means when u as a function, you find out the partial derivative of u with respect ot the first variable and what are the function you obtain evaluate that function at the value T minus small t minus s comma x, its general.

So, it is like you know the del v, del s is equal to the first derivative of u with respect to first variable times, then you write down del t by del s. So, the first variable how it how this you know I mean this you know input is depending with respect to s, but it is depending negatively. So, you are going to get minus 1 sign that is here.

And you would also get other terms, like the second partial derivative of u with respect to this second variable and how second variable is depending on s variable it does not depend, so multiple is 0. So, only this would survive there is also another way to find I mean see, is just the chain rule?

So, hence using Ito's rule, we are going to so here there are some you know steps since I have spent lots of time in Ito's rule. So, I can expect that you would be able to follow, so let me explain v of s minimum Rn, Ws minimum R n given W0 is equals to x. So, this function, this function now v is evaluated at some random time and also with some stochastic process.

Now, we are going to use Ito's rule for this and if you do Ito's rule for this, we get that it would be v of 0 x plus integration of v you know partial derivative of v and the integration with respect to W with appropriate the time limit the integration the limit of the integration should be appropriately chosen, this s Rn would appear and then the second order derivative of v with half multiplied and the quality variation of W would also appear with the same limit of integrations.

And then when you do expectation, that stochastic intuition part goes to 0, I think that we have proved as a theorem in two three lectures before, and we have shown that, this thing can be written as expectation of v of s minimum Rn, Ws minimum Rn, given W0 is equal to x.

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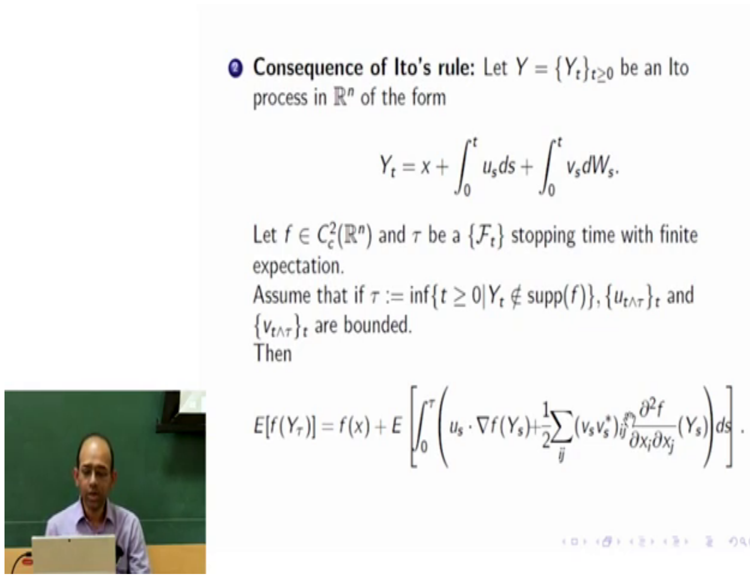

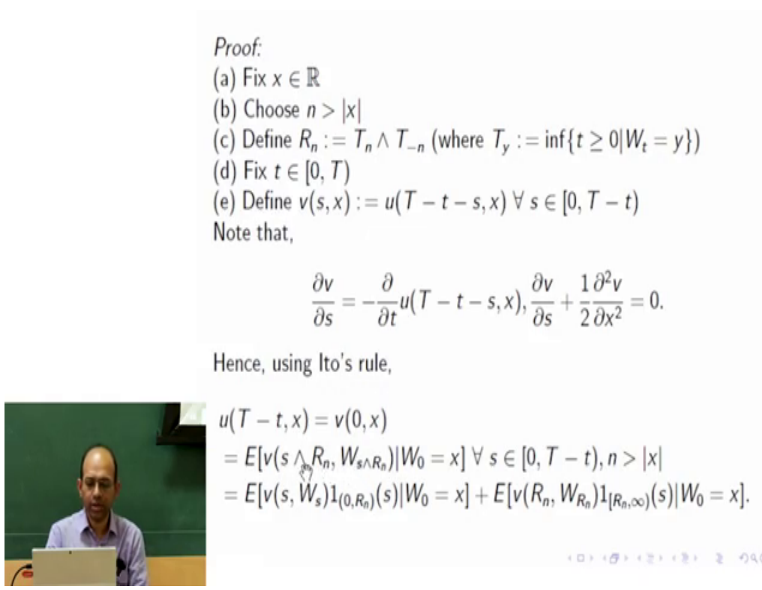

So, let us recollect that here what we were using here that expectation of v of this random variable, this thing is equal to v of 0 comma x there basically we are using one theorem which we have already proved in detail in earlier lectures that is expectation of f of W tau is equal to f of x plus this integration. And here you have the partial derivative of first order, here you have partial derivative of second order and for our case, f is not only function of the space but also function of the time.

So, that is not a big deal, so therefore, you have also a partial derivative of f with respect to time, and then partial derivative of this, and the way f is chosen in that case that is v such that, the partial derivative of v plus with respect to t, del v del t plus half del 2 v del x 2 is 0. So, this part would give you 0 there.

So, this would give you 0 there, so what would remain is that only that 0 x I mean only this term would remain. And this part would be 0, because of the choice of f here f is v here and for v we have already obtained that partial derivative of v with respect to s plus half, half also appears here, correct? So, half appears here and the second derivative appears here. And here this Yt, we are going to choose for our purpose the simplest one where u is 0 and v is 1.

So, all these just 1 etc. and is one dimensional, so only half multiplied with the partial derivative, second or partial derivative of f with respect to space x would appear here. So, we are directly using the theorem to conclude that, that v of 0 comma x is equal to expectation of v of s minimum Rn comma Ws minimum Rn given W0 is equal to x, this is true for all s between 0 to capital T minus small t and also we have I mean this is also not necessary to write down, but still you should keep it that n is more than mod x.

Because we have chosen one x and x we are not moving anywhere and we have chosen n is more than mod x, I mean that is important in the sense that, so that this Rn, etc the time. So, when you start time 0 your Rn is not 0, because if x is more than n then already your outside the domain. So, we chose that or you chose n such that it is more than mod x.

Now, what you do? We are just it is the matter of rewriting, so s minimum Rn, what does it mean that when s is less than Rn then this is s. So, when s is between 0 to Rn, then it is exactly s minimum Rn is s, Ws is minimum Rn is Ws, when s is more than Rn, this is indrcator function. So, then this part would be 1. And then, s may minimum Rn is nothing but Rn, it is just rewriting of the earlier expectation. Now, this is going to help us because this is the addition of 2 terms.

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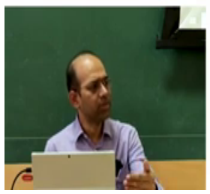
Now from (ii)

$$\left|v(s, W_s)1_{(0,R_n)}(s)\right| \leq Ke^{an^2},$$

and as $s \uparrow T - t$, $v(s, W_s) \to 0$ a.s. [from (i)]
Similarly, $|v(R_n, W_{R_n})1_{[R_n,\infty)(s)} \leq Ke^{an^2}$.
Using BCT and passing to the limit $s \uparrow T - t$
$u(T - t, x) = E[v(R_n, W_{R_n})1_{(R_n,\infty)}(T - t)|W_0 = x] \ \forall \ n \geq |x|$ since,

$$f(x) := \lim_{c \uparrow x} 1_{[0,b]}(c) = \begin{cases} 0, & \text{if } x < u \\ 1, & \text{if } x \in (a, b) \end{cases} = 1_{(a,b]}(x).$$

Therefore, for every fixed $t \in [0, T)$,



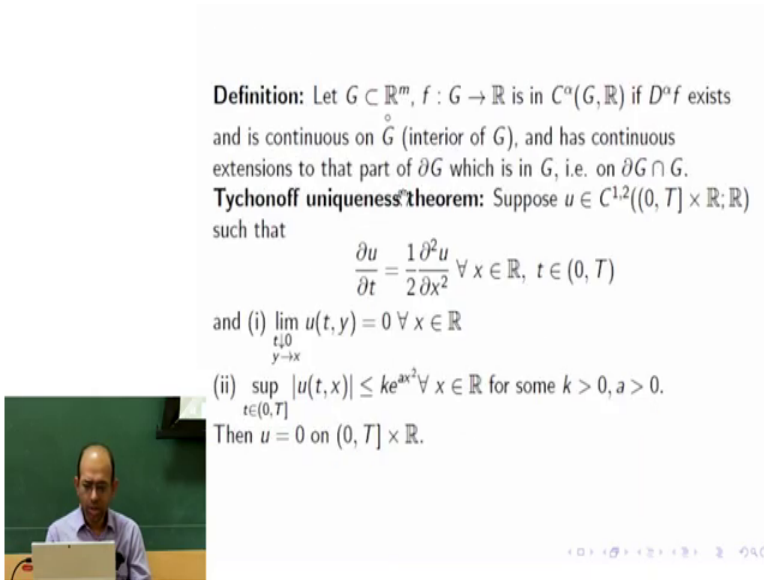
$$\begin{aligned}
&|u(T - t, x)| \\
&\leq Ke^{an^2} P[R_n < T - t|W_0 = x] \ \forall \ n > |x| \\
&= Ke^{an^2}(P(T_{n-x} < T - t|W_0 = 0) + P(T_{n+x} < T - t|W_0 = 0)) \\
&\leq Ke^{an^2}(P(T_{n-x} < T|W_0 = 0) + P(T_{n+x} < T|W_0 = 0)).
\end{aligned}$$

Proof:
(a) Fix $x \in \mathbb{R}$
(b) Choose $n > |x|$
(c) Define $R_n := T_n \wedge T_{-n}$ (where $T_y := \inf\{t \geq 0|W_t = y\}$)
(d) Fix $t \in [0, T)$
(e) Define $v(s, x) := u(T - t - s, x) \ \forall \ s \in [0, T - t)$
Note that,

$$\frac{\partial v}{\partial s} = -\frac{\partial}{\partial t}u(T - t - s, x), \frac{\partial v}{\partial s} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} = 0.$$

Hence, using Ito's rule,

$$\begin{aligned}
&u(T - t, x) = v(0, x) \\
&= E[v(s \wedge R_n, W_{s \wedge R_n})|W_0 = x] \ \forall \ s \in [0, T - t), n > |x| \\
&= E[v(s, W_s)1_{(0,R_n)}(s)|W_0 = x] + E[v(R_n, W_{R_n})1_{[R_n,\infty)}(s)|W_0 = x].
\end{aligned}$$



So, here what we use that this v function from above because v is actually obtained using u and u is a solution satisfying the condition 3, condition 2, which is saying that which put some growth condition on u. So, v are also inharites that growth condition, so mod v is less than or equal to K times e to the power of a n square, why n square? Because does not matter even it Ws is very large time, but W even if W exceeds the n minus n but this would become 0.

And this would become remain non zero only when Ws is W is between minus n to n and then I can use this growth property, that means this is less than or equals to K times e to the power a, W square, W Square should have written, but Ws when it is less than n then only this part is non-trivial, for that I can replace that by n square, this is even larger is also upper bound.

So, we got some upper bound, which is deterministic not random, that is good part of it, so I can say okay this is like a bounded random variable. Now, as s is now approaching to capital T minus small t. So, here we are not changing n, n is fixed, so it is not n tends to infinity we are doing, what we are taking s tends to capital T minus small t. Why are we doing that? Because if s tends to capital T minus small t, then this thing argument goes to 0 and that is our goal, why? Because we have information, when u of this variable goes to 0.

So, that information we want to use now. And we can do only if s goes to take capital T minus small t. So, I mean till now we have used condition 2 here but condition 1, we are now going to use. So, as s approach is to capital T minus small t, then v approaches to 0, from condition 1. So, this is converges to 0 almost surely.

So, now this modulus, here modulus is missing, so v of Rn comma W Rn instead of s, Ws etc, I can talk about this, this s should be here. So, what does it say? It is saying that if s is more than Rn, then only this is non zero, but for those occasions here this Brownian motion is not moving anyway.

So, it is at W Rn, it is not depending on s, here it was depending on s, but here it is not, so this thing is v of Rn comma W Rn, but what is the value of W Rn? Is precisely either n or minus n. So, I can again use the growth property that this whole modulus is less than or equal to here. So, there is a ambiguity about notation, here I have started using capital K, whereas I think I was using, I was using small k here, actually both capital K and small k are same for this like okay.

Now, so as I was telling that, we have now bounded and this thing, so bounded random variable here. So, since we have bounded random variable, we can use the bounded converges theorem. So, how are we using here? Because we are now changing s here, s going to capital T minus small t and we know that v converges to 0 there.

So, u of capital T minus small t comma x, which was v of 0 comma x is u capital T minus small t comma x, correct? So already we have that so left hand side I am writing just here and right hand side I have this thing, but s appears here and now I am letting s tends to 0 capital T minus small t. So, we are going to get this limit would come inside.

So, here v Rn, W Rn one indicator, so earlier I had closed Rn comma infinity but when I am going to take s tends to capital T minus small t, that means from below to up, then the limit of that would be an indicator function of open Rn to infinity of T minus small t. So, I have explained it here, so I would explain this why do we get it, but first let me finish this reading this line.

So, v of Rn comma W Rn, this does not depend on s, so this rest does not change, however this function surely changes because s becomes capital T minus small t, only things of this domain also changes here and given W 0 is equal to x. So, for all now this also we want to remind ourselves that our n was chosen which is more than you know our n is actually strictly greater greater than mod x.

So, now here the explanation comes, if we have indicator function of say there is also so my a and 0 is the same. So, basically I should have written a here, indeed a function of closed interval a, b evaluated at point c and then you are taking limit c tends to x. So, what should be its limit? It should be a, if x is outside of the closed interval a b, then x is less than a then c is also less than a.

So, indicator function of a b at c would be 0 and it is 0 for all c, which is less than x. So, when you take limit c tends to x this would be 0, so you are going to get 0 here. However, if x is inside the interval a comma b, then I mean of course you would get that sum, so then when c is less than x, that means c is less than a, x is between a and b, open a, b. So, x is an interior point, so x has a neighborhood which is strictly inside a, b.

So, when you approach x from the left, then you would enter the interval a, b before and this would become 1 and the limit would be 1. So, here only thing is that, when x is equal to a, so that I have not written. So, here I should have written less than or equal to a, because whenever x is equal to a, then also from the left if you approach, all the values would be 0. So, the limit would also be 0.

So, there is a typo here, so this part is okay. So, then that would give me that this limit function is indicator function of open a closed b. So, apply the same rule here, so we are going to get the limit here would be open Rn infinity. Now, for every fixed t, so we are now considering u of capital T minus small t comma x actually we have started proceeding from earlier slide, but now we have the details about this expression. So, we are using this expression, now this is upper bounded by this K times e to the power a n square.

So, we are going to use that, so we are using this, that this is less than or equal to K times e to the power a n square times the probability, so because replacing the function by upper bound, then from that expectation value we are going to get the probability that these Rn is less than capital T minus small t and I am just keeping write, I keep on writing n is more than mod x. So, this K times e to the power a n square probability that so in other words so just it is just writing the same thing in explicit manner.

So, when the Brownian motion starts at point x, then you are constructing this Rn as T and minimum at t minus n, but if you start from 0, then it you have to translate it. So, then it is the T n minus x is less than capital T minus small t and probability that T n plus x is less than capital T minus small t. So, both the sides correct. So, either n minus x, or n plus x. So, you are going to get these two addition, addition of these two probabilities.

So, this is less than or equal to K times e to the a n square times probability Tn minus x is less than capital T given W 0 is equal to 0, how do you getting this? Because this part is less than capital T minus small t. So, this is smaller than this, so I have replaced this number by is bigger larger number, so this has larger scope. So, the probability is not smaller, so I can get this inequality, and probability Tn plus x is less than capital T here also for the same reason.

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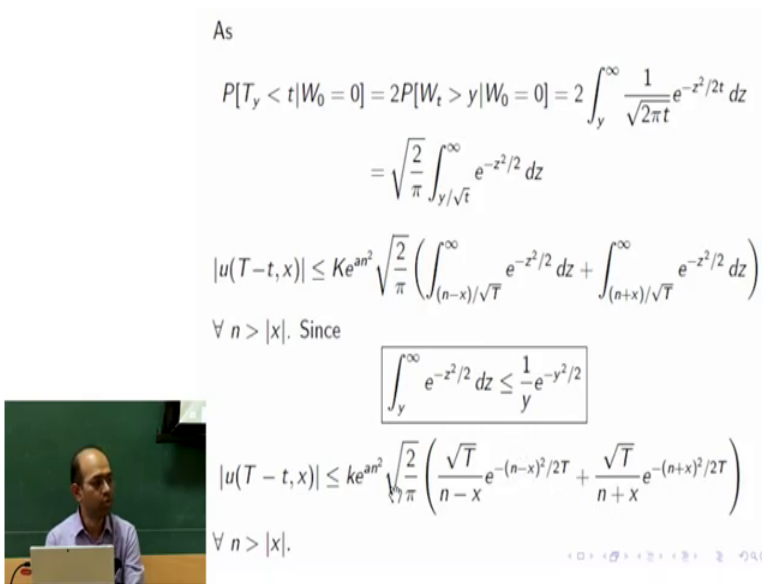

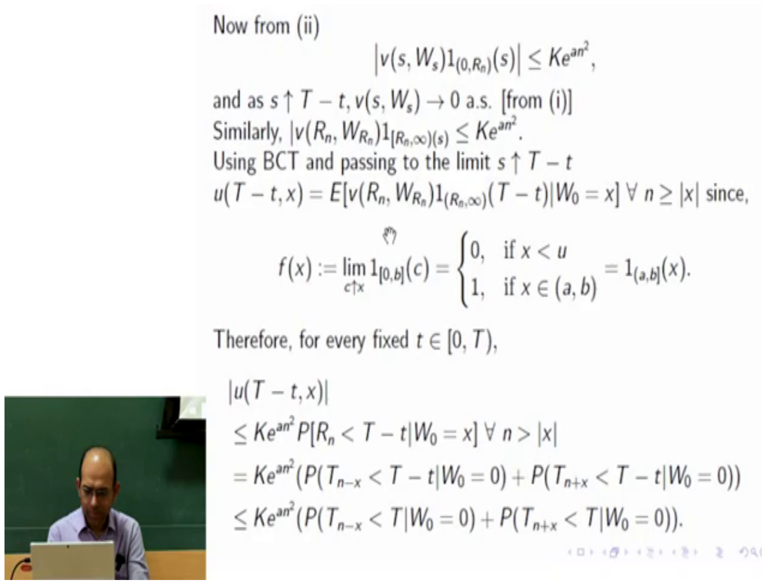

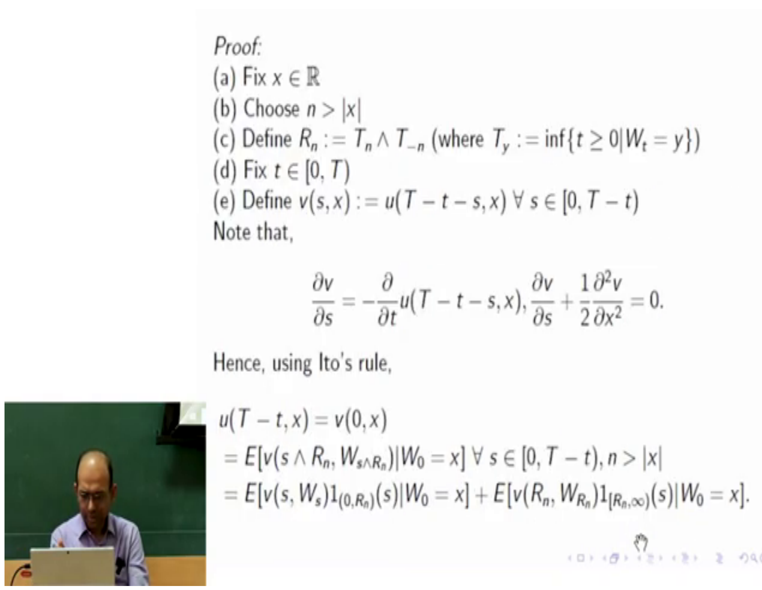

Now, here we explain that, this is the, I mean this inverse some calculation some estimates as T y is less than small t, that this probability is equal to 2 times. So, this is this result is obtained from reflection principle, that T y is less than small t, what does it mean? That before time t or the function hits y. So, that probability is same as that 2 times the probability the Wt is more than y.

So, at time t, because you know it does not say that, before time t by motion hits y, that does not say precisely there at time t it would be more than y or not, it will be lower, but it will above or lower with equal, equal probabilities. So, from that consideration, there is a actually theorem which is called reflection principle, that has this you know formula the calculation based on this very simple argument, what I told that, okay it has half of probability going up and down, that to show that okay it is equal to 2 times probably the Wt is greater than y.

And now, this probability is very easy to compute, why? Because Wt is nothing but normal random variable with mean 0 and variance t, I am just trying to find out this value, 1 minus the CDF, correct? So, this is 2 times y to infinity 1 over square 2 pi t, e to the power minus z square by 2t dz. Now, this whole thing you know that I can write down in a more simpler manner using some change of variable.

So, here we get y by square root of t infinity e minus z square by 2 dz. So, this 1 over square t you can get rid of. And then, what you do? We actually take modulus of you I mean again we go back to that same quantity u capital T minus small t, x because that is the thing we want to show that it is basically 0. So, u of capital T minus small t, x mod, so that is less than or equals to K times e to the power a n square times. So, here this probabilities, we are now writing down, we are now using this, that this probability is where at in square root of 2 pi, 2 by pi. So, here we are writing instead of y, because n minus x we should consider now because this Tn minus x correct n minus x. So, n minus x divided by Square root t, and here n plus x, square root of t. So, those two probabilities are now having you know these type of expressions.

And then, we are using this estimate also, we are going to use it, because now it is in terms of integration, but this integration do not have any closed form expression, this is basically error function. It can be written in terms of error function, but does not mean closed form expression. However, we have this estimate that e to the power minus z square by 2 dz, integration from y to infinity is less than or equal to 1 over y e to the minus y square by 2.

So, one can understand cross verify in many different ways one can actually, you know so say like y tends to infinity, this part would be 0 here also this part is 0 and one can actually, you know do these calculations, it would involve some gamma integral but one can get this expression. So, we are just using this estimate for replacing these integral term in terms of just exponential term.

So, we have obtained that mod of u of capital T minus small t comma x, this left hand side is less than or equal to K times e to the a n square, square root 2 by pi and then this integration is replaced by 1 over of this thing. So, 1 over of these things means the square root of T by n minus x, square root T by n minus x, e to the power of square of this term. So, n minus x square by 2 T and similarly here we have square instead of minus x, we have plus x here. So, have this term.

And we should not forget that n is chosen, such that n is more than mod x, n could be chosen any other number, any number more than that, any number more than mod x is fine. Now, when we have obtained this expression, we can now check what happens if n becomes very large, what can you say? We can say something, that here denominator goes to 0 here also denominator goes to 0. And here numerator also becomes very large, very large.

And then that means also the negative sign, this will also be small number, however n also appears here n square. So, these terms, actually give you confidence that okay, this part would go to 0, this part would go to 0, but here also I have a n square. So, here we can observe that, here also I have n square term, but there are some coefficient 2 T here a, if I can say that this 1 over 2 T is more than a, or a is less than 1 over 2 T, then this part would dominate. So, this decay of this numerator to 0 would actually lead everything go to 0.

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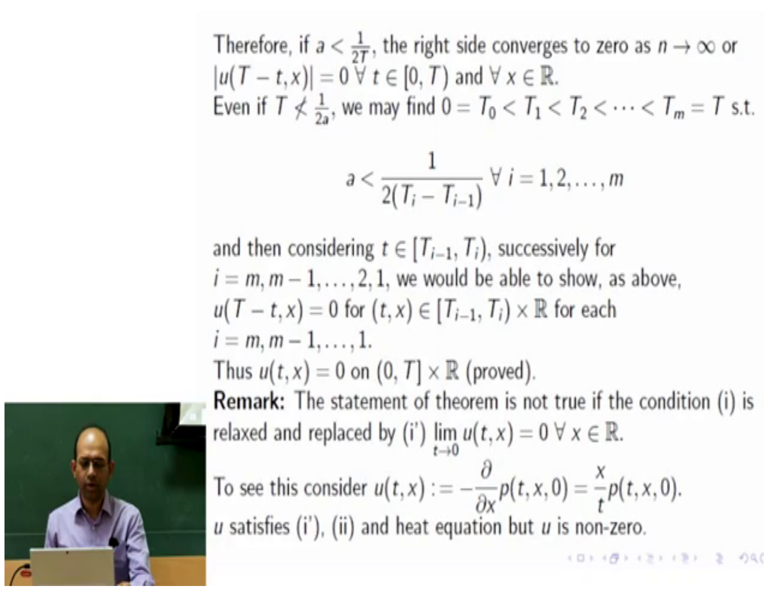

As

$$P[T_y < t | W_0 = 0] = 2P[W_t > y | W_0 = 0] = 2\int_y^\infty \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}\, dz$$

$$= \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{t}}^\infty e^{-z^2/2}\, dz$$

$$|u(T-t,x)| \le Ke^{an^2} \sqrt{\frac{2}{\pi}} \left(\int_{(n-x)/\sqrt{T}}^\infty e^{-z^2/2}\, dz + \int_{(n+x)/\sqrt{T}}^\infty e^{-z^2/2}\, dz \right)$$

$\forall\, n > |x|$. Since

$$\boxed{\int_y^\infty e^{-z^2/2}\, dz \le \frac{1}{y} e^{-y^2/2}}$$

$$|u(T-t,x)| \le ke^{an^2} \sqrt{\frac{2}{\pi}} \left(\frac{\sqrt{T}}{n-x} e^{-(n-x)^2/2T} + \frac{\sqrt{T}}{n+x} e^{-(n+x)^2/2T} \right)$$

$\forall\, n > |x|$.

So, this is the argument we are now going to put. If a is smaller than 1 over 2 T, then the right hand side I mean this is a some condition which we have not introduced earlier, remember. So, because we have started with any particular capital T and a was just parameter which appears in the assumption. But here we are requiring one relationship between a and 1 over 2 T, but we would show that we can avoid I mean, for this step we actually required it, but for general case also we can get the same conclusion but that would come later, first we finish this part.

So, a is less than 1 over 2T, then right hand side converges to 0 as in just infinity, therefore this left hand side who is equal to 0, why? Because the left hand side does not depend on n, this is true for all possible n large. So, you just take n tends to infinity and then this goes to 0 for the special case, where a is smaller than 1 over 2 T. So, that proves that this would be 0, but then it is yet incomplete proof, this is a proof only for a special case.

Now, let us talk about the general case, so if capital T is not less than 1 over 2 a, then what we do that, because we do not have any control on a. Why? Because a is coming from the class of functions, we are looking for class of functions and solutions, such that the growth property is you know like K times less than or equals to K times e to the power a n square x square, a x square, but T is a given problem, because 0 to T we need to solve the equation for the interval time 0 to capital T.

Now, if capital T is too large, then we can actually subdivide capital T in small, small parts, T 0, T 1, T 2, T 3, etc. Tm where Tm is equal to capital T. So, when we subdivide capital T in the small parts, we divide it such a way that this every sub interval width is small enough, such that that is less than 1 over 2a, a is less than 1 over 2 T, then T is less than 1 over 2a, but if we do not have we actually have the sub interval. And for heat equation, we are now going to take this sub intervals, each one at a time.

So, now this is 0, so now we are going to instead of capital T we are going to take only T 1 as our last final time interval. And then we are going to prove that, this is 0 every here till T1. And since we know that till T 1 it is 0, we take that as my initial condition again and then I have tried to write down that here, so and then considering small t between capital Ti minus 1 to capital T open i successively for i is equal to m to m minus 1.

So, this is like you know, either you start from 1 or m, so here if you do for m, that means you assume that it is 0 at, I mean like you know for induction. So, you have done it for till m minus 1, then you do for m, otherwise you start from i is equal to 1, like 1 at a time. So, for each this thing i you do separately.

So thus, we can prove that for any capital T positive real number, so this would be you know 0. So, what we have obtained is that? For a heat equation on infinite horizon. So, like 0 to capital T and initial condition if that is 0, I mean initial condition is given in a way that, as limit distances 0 than that is 0. So, there then the solution must be 0, solution must be the trivial solution, there is no non trivial solution for that problem.