Introduction to Probabilistic Methods in PDE
Professor Dr. Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture 30
Part - 2
Stochastic Representation of Bounded Solution to a Heat Equation

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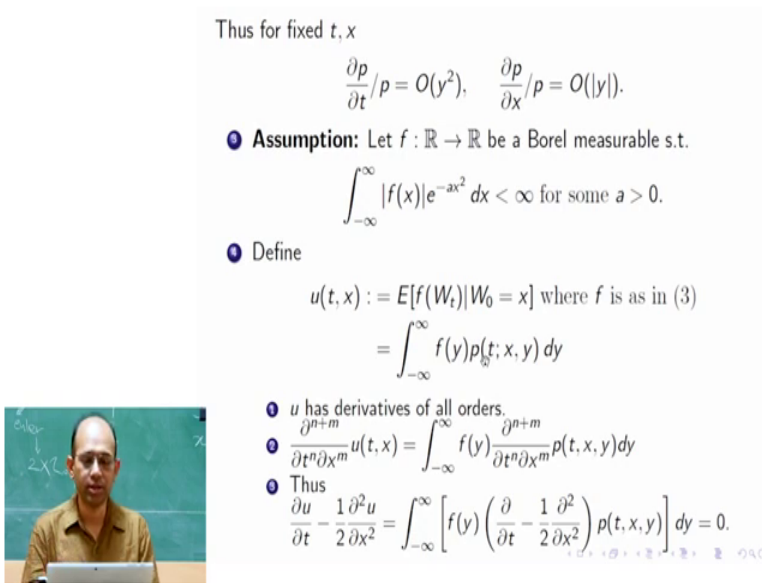

In the last lecture what we have seen is that the function u given as integration minus infinity to infinity f of y p t x y d y, where p is the heat kernel or in other words it is the probability density function of normal random variable with mean x and variance t. So, this value u t x is the solution of the heat equation del u del t minus half del 2 u del x two is equal to 0, this is a result which you have proved in the last lecture.

So, this basically says that u as a function on the upper a plane is upper a plane in the sense that, okay if you take your horizontal axis as your space variable like x, which is the full real number and a vertical axis you take is time, then u as a function defined on this upper half plane in the interior there it is C infinity function and it satisfies this heat equation. However, on the boundary of this upper half plane is basically the initial condition when times equal to 0, where u also satisfies the initial given condition f, but if your data initial data f itself is discontinuous, it is not possible to expect that u would be a continuous function on the closure of the domain.

But a natural question is that, if f is a continuous function, can we expect that it would be the solution would be continuous on the closure? This is a very natural question correct? So that we have not yet addressed, so first we are going to answer to this question affirmatively, we are going to see that okay, this is indeed true, we would continue assuming that f has this particular growth property, we have discussed quite a bit about this growth property the meaningful meaning of this growth property that this basically says that f is not having growth, like exponential kind of thing, e to the power of a x square type of things, but any polynomial growth is okay.

So f does not have, you know growth more than polynomial.

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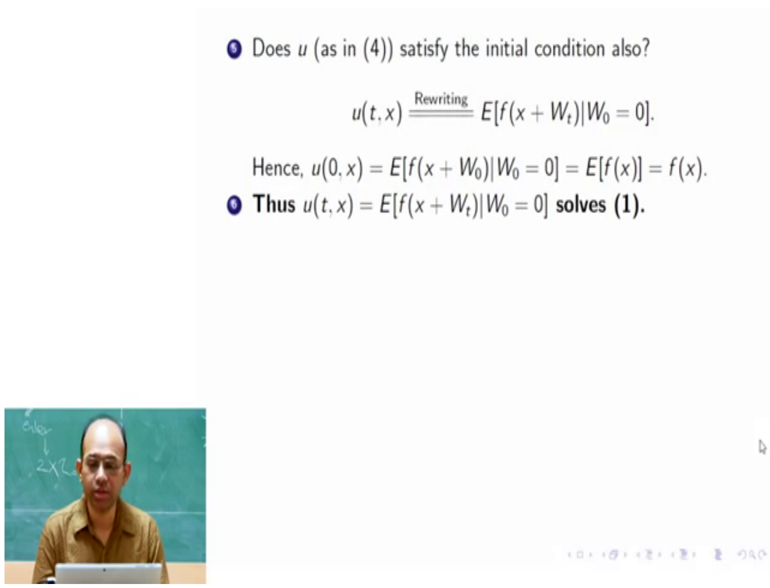

So, here we have seen that this part also we have seen earlier, that u t x when you write down this way, and at the time zero initial zero. If we plucked t equal to zero there in the formula, it becomes trivially f of x, so the initial condition is trivially satisfied. So, u satisfied that u is the solution of the heat equation, but then the question remains that whether it is continuous on the closure.

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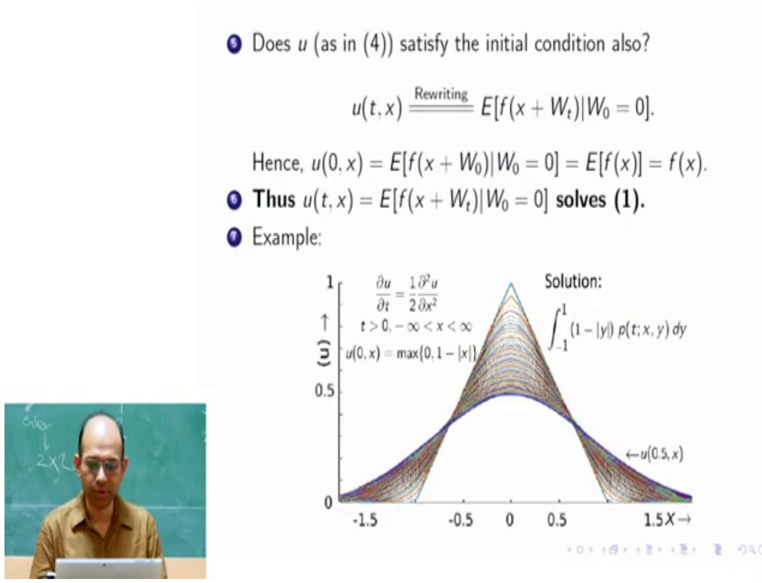

So, this was one example of what we have seen earlier lecture, that when initial condition is like one minus mode x then solution is given by this and then at time t is equal to 0.5 that is half, the solution changes and the value of this solution at t equal to half is smooth, we can see the smoothness is achieved immediately, immediately after initial point and it is C infinity function.

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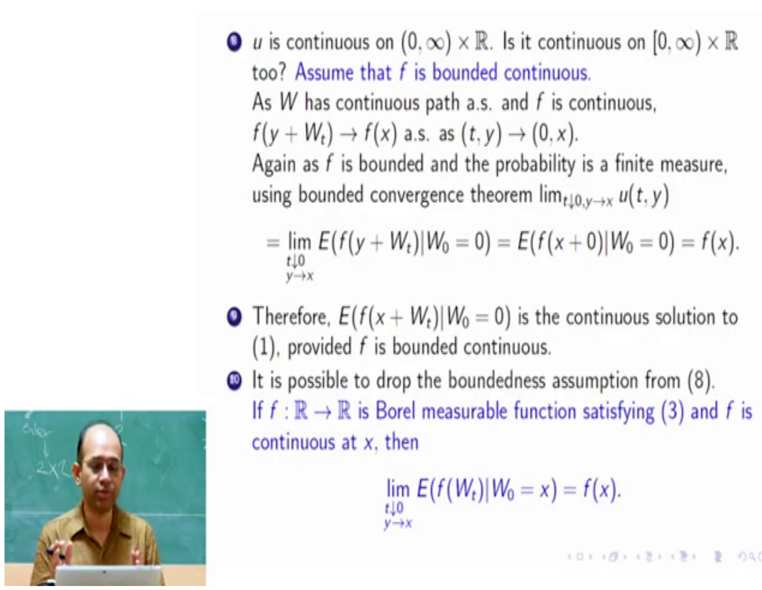

So, to answer this question, first we consider assume that u that f is a bounded continuous function, bounded means assumption is too straight, because our general theory that u satisfy the heat equation inside the domain there f is allowed to be unbounded and not only that, it can have any polynomial growth but here as a first step, we just see that, if f is bounded then this problem can be settled very easily.

And then, we would see the general statement and general proof, that would be little detailed but one should also know that okay, if f has boundedness property then one can conclude very easily. So, as w has continuous path almost surely and f is continuous, then f of y plus Wt converges to f of x if y converges to x and t converges to zero. Why is it so? As t converges to zero Wt converges to zero because W zero, and that is zero and y converges to x. And f is continuous, so this thing would go to f x.

So this is clear and y plus W t appears when we find out the solution, the solution this is u of t comma y, u of t comma y is exactly this. So inside we know that the limit I mean of f is f of the limit this can be conclude, but then how can you assure that limit can go inside expectation, that much only need to assure but that is also trivial, because f is bounded and expectation is integration with respect to finite measure, probability measure, so you can use bounded converges theorem. So, we can write down that we can put the limit inside and we get f of x.

So, this is very simple, therefore, this thing f of x plus Wt given W zero, so expectation of this is the continuous solution to the heat equation provided f is bounded continuous. Now, it is possible to drop the boundedness assumption from this, but then argument becomes longer. Of course, it is apparently not clear how can one manage if f is not bounded, okay.

Anyway, even if f is not bounded, but f has a growth property and Brownian motion this is for expectation when you find out expectation, that means we are going to multiply the p d f and that product the p d f the probability density function and f together, they can actually satisfy some bound some L 1 or something.

And then one there one can actually manipulate to allow limit to pass inside.

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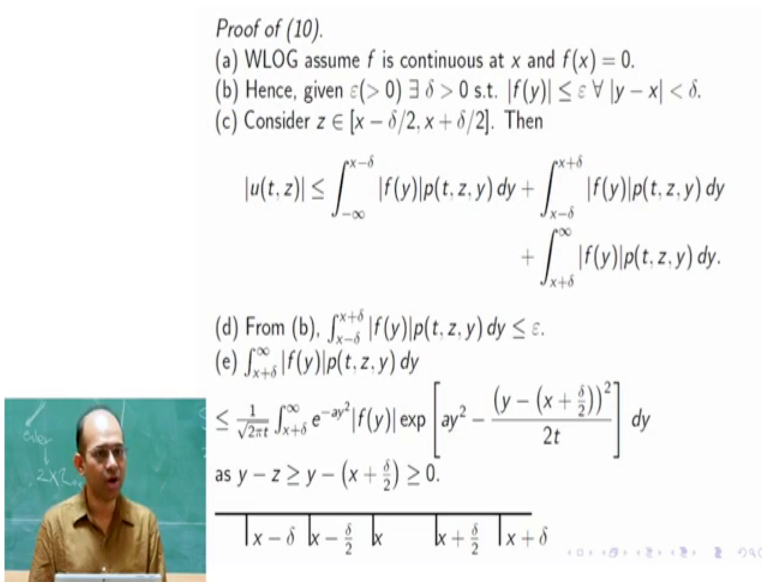

So, let us see that proof, so without loss of generality we assume that f is continuous at a, I mean f of x is equal to zero, we must assume f is continuous because otherwise there is no hope, there is no hope that u would be continuous on the closure because u agrees to f on the boundary, if f itself is not continuous u cannot be continuous, continuity is of course there but the value of f of x we assume just zero, why?

Because even if it is not zero, we can just whatever the value is I call that C, we can subtract u and C, and the new function whatever we are going to get would again solve the heat equation and that would have boundary data at the point x, it would have boundary data zero. So, we can always you know, reduce this problem to this problem. So, here since f of x is equal to zero, so and it is continuous at point x then, so at this stage we are not even assuming that f is continuous everywhere in the boundary.

So, just the information f is continuous at x, information, can you prove that the solution would also be continuous at x. So, since f is continuous at x and f x is equal to zero. So, given f some positive there exists a delta positive such that, mode of f of y is less than or equals to epsilon strictly less than actually epsilon, I mean the typo for all mode of y minus x is less than delta.

So we can do that, due to the continuity property. So, now, what we do? We consider the point z which is in the neighborhood of x, but a smaller neighborhood not the delta but delta by two neighborhood, it would be clear that why delta by two is taken, what is the importance of that.

Now, our goal is to show that u of t comma z as t and z, so t goes to zero and z goes to x, that mode of u t z goes to zero. So, that is our goal, correct? To show that that u is continuous because u of zero comma x is anyway f of x that is anyway zero. So, to show that u is continuous at x, our goal is to show that, if t is very small and z is also converging to x, then this whole thing would be very small and converges to zero.

So that is a thing, now we write down mode of u t z, so modulus of integration is less than or equals to integration of the modulus. So modulus I have taken inside so it is some of three different integrals minus infinity to x minus delta, x minus delta to x plus delta and x plus delta to infinity, integrands are all same. So, mode of f of y into p of t comma z comma y, d y mode of f y p t z y mode of f y p t z y.

Now what do, we look at each and every term. So, this middle term is the easiest to handle. Why is it so? Because already our assumption is that mode of f y I mean because given epsilon the delta was chosen, such that mode of fy is less than or equal to epsilon. So, this part is less than or equal to epsilon, basically it is less than epsilon. So, no need to put equality sign, epsilon and then this epsilon comes out of the integration then this integration is x minus delta to x plus delta of p t z y this p probability density function, correct?

Total integration minus infinity of this is one, and this is a sub interval of that. So therefore, so the total integration of pt z y, dy without this factor would be less than one, so we can write down that this is less than or equal to epsilon. Now, we talk about this term. I mean, whatever we are going to talk about this term, a similar argument would apply for this term also. Why is it so? It is because that p is a symmetric function but not symmetric around x but symmetrical arround y here, I mean symmetrical around z here, however, z is very close to x. So we would be able to manage that.

I mean, basically since z not equal to x, that is why you know, it shifts so we do not allow z to be more far from x, that is why we have taken z in, I mean in the interval x minus delta by two to x plus delta by two, so that we can manage the small change there. Now here, this third integration we are talking about here. So here we, I mean p of t z y we are writing down the full formula of p of t z y that is one over square root two pi t into e to the power of minus z minus y whole square by two t.

So, this term would appear. However, we can actually write down one upper bound of the of e to the power term, how are you going to take the upper bound? So let us see here, so this is the line drawing that you can see this is the real line and x is here, x plus delta by two, x minus delta by two, x minus delta and x plus delta is here. So z is inside the sub interval x minus delta by two to x plus delta by two, whereas y is here or here.

So, when we are talking about third integral, then y is here and z is somewhere here, so, z minus y, so this difference is greater than y minus x plus delta by two because this is closure. So, y minus x plus delta by two is smaller than y minus z. Now, y minus z whole square that would be greater or equal to y minus these things whole square. And the negative sign there we are going to get the opposite sign. So e to the power of minus y minus z square is less than or equals to e to the power minus y minus of this term whole square.

So that is the thing we are considering here now, that for after I mean writing down the full form of p, where we got this coefficient and e to the power of this thing. And then here we have multiplied and divided by e to the power ay square. So here we have e to the ay square, here we have e to the minus ay square, is just multiply and divided, but this is the term which we have this is the upper bound of the term of p t z.

And now I mean you can have intuition, why are we doing this multiplication and division because this is the term which actually appear in the assumption of the growth property of f, and we are going to use that, so for that reason. So now we need to control this, because this part we can manage, because we already have some assumption on this. The integration of over this is finite, only thing is that this part we need to manage that it should go to zero or something should be small.

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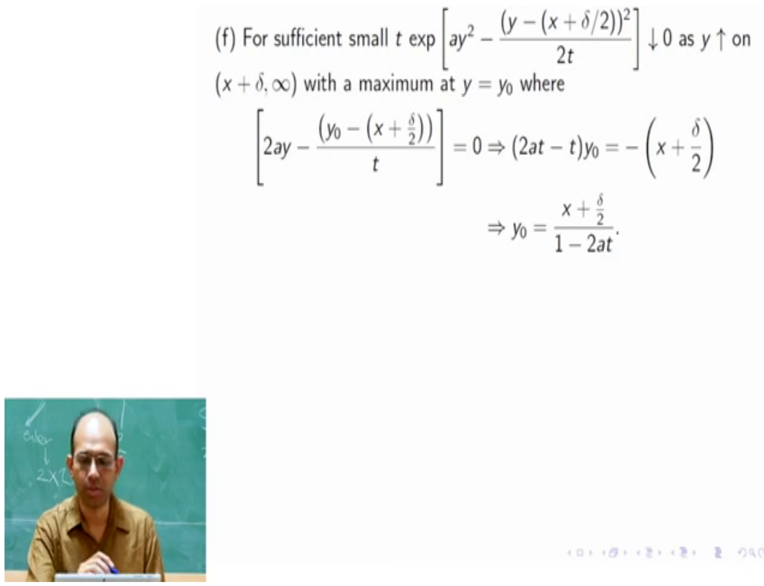

Now here for sufficiently small t, e to the power of a y square minus y minus x plus delta by two whole square by two t this whole thing decreases as y increases on x plus delta to infinity. So, it can be shown in many ways like you know just by analytically you can take a derivative with respect to y and check the derivative is becoming negative for y large.

So, that is going to give you that is decreasing and you can take limit and show that if t is small enough so, that means one over two t is large and then therefore, this term would start dominating over this if one over two t is much larger than a, then it will start dominating and then if y tends to infinity so, it would be minus infinity to the go to the numerator will become larger as if this would dominate over this.

So, it would be e to the power minus infinity so it will go to zero. So, that is one way of looking it, otherwise you know one can also view it geometrically like this is like a parabola upward and this is minus of this square is a parabola downward this has vertex origin but this has vertex different x plus delta by two which is away but, thing is that when we are talking about y on the other side, right hand side there this is monotonically increasing and this would also be monotonically decreasing.

So, on that side on the right hand side, where y is more than x plus delta and then it is just that one needs to find out appropriate choice of t such that this has higher extensity and then it dominates. So, I have described two different way to view this. Now this thing would I mean does not monotonically I mean, although have written that this goes to zero, that does not mean that I am saying that it is monotonic that this if you take derivative of this with respect to y you would not get say negative for every all y, but for large y you are going to get negative.

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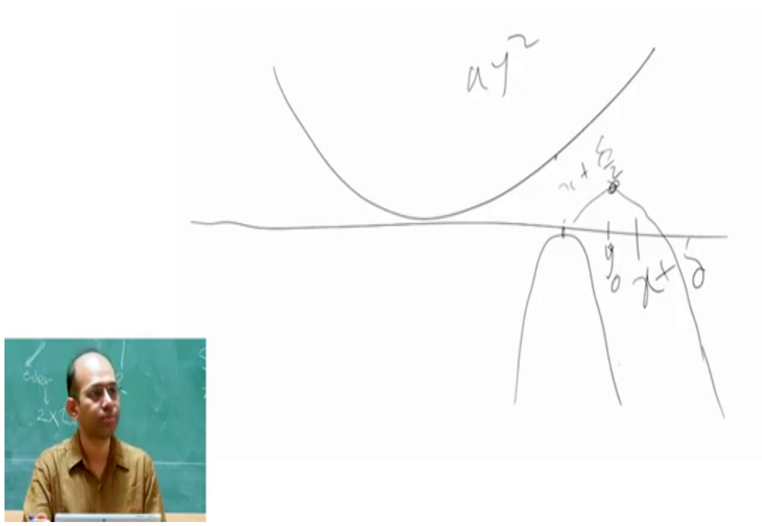

So, I mean I think if it draw some picture it would be better. So here, and imagine that so this is a y square and this is x plus delta by two vertex and this is that,other side parabola. Now if you add these two, I drawn outside because I mean below downwards, this is because it is negative correct? Negative sign is there, and then at this point, see this is increasing but this is not increasing, correct? It is like steady and then it decreases very fast.

So, if you add, so then addition would actually have some increment to the beginning here, so here x plus delta is there, but I cannot assure that whether here onward it would decrease but it would just have some you know it would increase for some time and then of course, this will dominate and then start decreasing.

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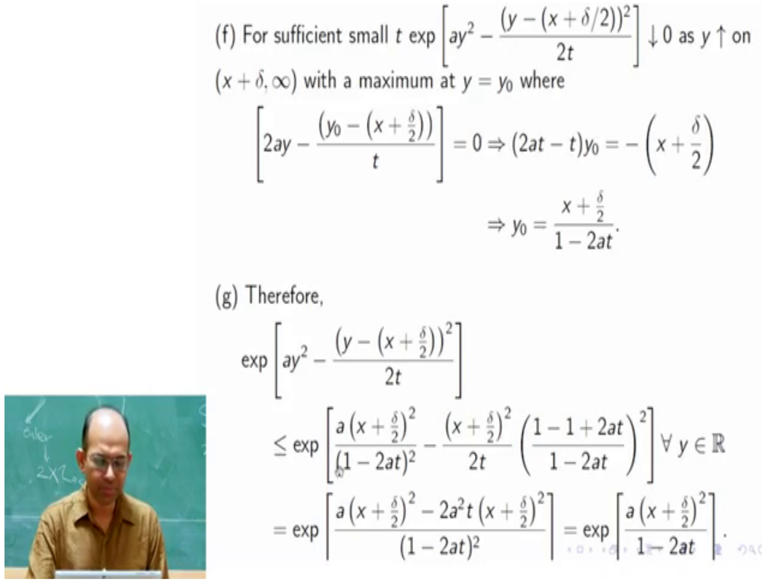

So, we can actually look at the maximum point here. So, maximum point here, so that is the thing which we are going to do now. So, with the maximum at y is equal to y naught, so here, so if you add these two things, say here this is y not, this is my y not. So, now, we take the derivative of this I mean how to find out the formula y not y not will be obtained if I take the derivative of this because I do need to look at exponential thing because it is a monotonic function, if I can find out the maximum this I would be done.

So, for finding maximum this I take partial derivative of this function with respect to y and evaluate y is equal to y not. So, this is a typo I should write y naught here y zero here. So, there is two a y not minus y naught, so here two is there square is there so two here these two and that two what would appear would cancel each other.

So, y not minus x plus del by two by t with this equals to zero and then you simplify, there is a typo it should be one here, so then two a t into y naught minus y naught then it take one common two at minus one times y naught is equal to the minus minus plus sign but this goes on the other side or the equality.

So, you get minus x plus del by two and then you divide everything by two a t minus one. So, y not is equal to x plus delta by two divide by one minus two a t, as it means that this is non zero. So, here for our case, since we are going to look at for sufficiently small t and a fixed, so, of course, in a neighborhood of zero I would be able to assure that this would be nonzero there. So, that is sufficient for us. So, this is the point y naught where we have that these you know this difference is maximum. Now, we observed that e to the power of this thing a y square minus y minus x plus delta by two whole square divided by two t, this expression, so we find an upper bound that is we write down its value at y naught because we know that I mean this thing is bounded above by this function at y naught because y naught is a maximal point.

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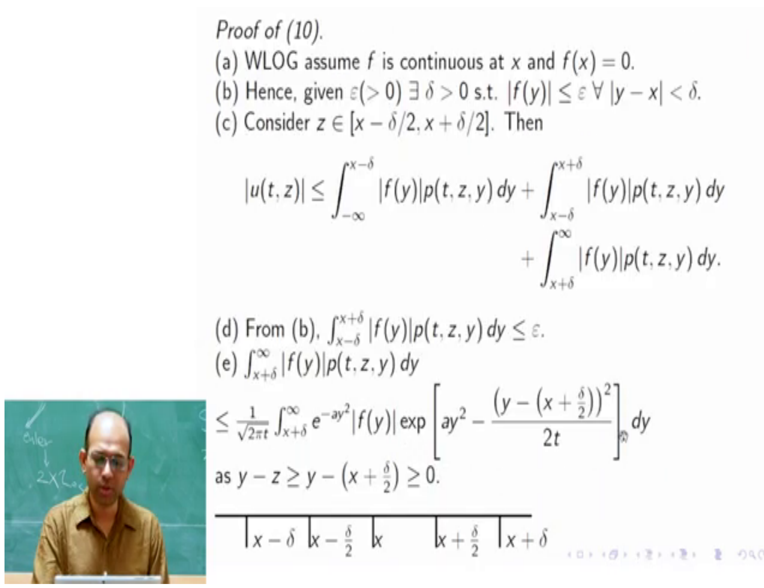

So, we evaluate this function at value y naught, so that would be an upper bound of these expression and that was our goal, our goal was to manage this term, actually to show that this actually goes to zero to show that actually we have first figured out what is the maximum of this function.

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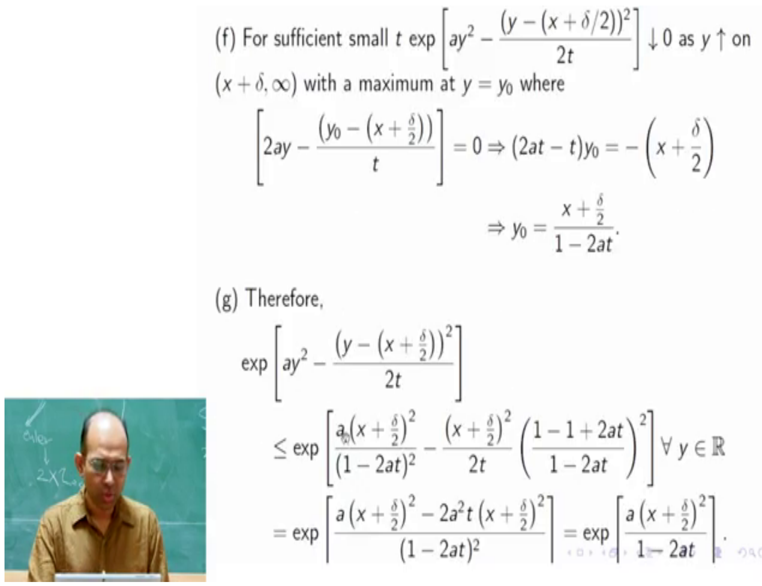

So, here a times now, y not is like this x plus delta by two whole square and one minus two a t whole square is this one it is corrected here but there is a typo here. 1 minus two a t whole square minus, so this y also write down here and this y also has x plus delta by two here also we have x plus delta by two, we can take common x plus delta by two.

So x plus delta by two whole square is taken out of this square and then we would have one over one minus two a t there. So one over one minus two a t minus one so, that is one minus minus one plus two a t, so this whole square will remain and then two by two t was there and keep it here.

So this numerator we are writing this way, one minus one cancels and two a t and two t, so, two t the whole square so, I would get two a square t yes I will get two a square t minus two a square t here. And one minus two a t whole square is minus one minus two a t equals square and x plus delta by two whole square is remaining here. And this thing is also remaining, so I have one minus two a t whole square and this thing.

So this is the expression, it can all further be simplified because here x plus delta by two whole square is common. So then a can also be taking common, so one minus two at again. So that would cancel with this square term. So I would get a times x plus delta by whole square divided by one minus two a t. So this is just you know, manipulation algebraic manipulations, but these are very important because otherwise one does not get that particular estimate.

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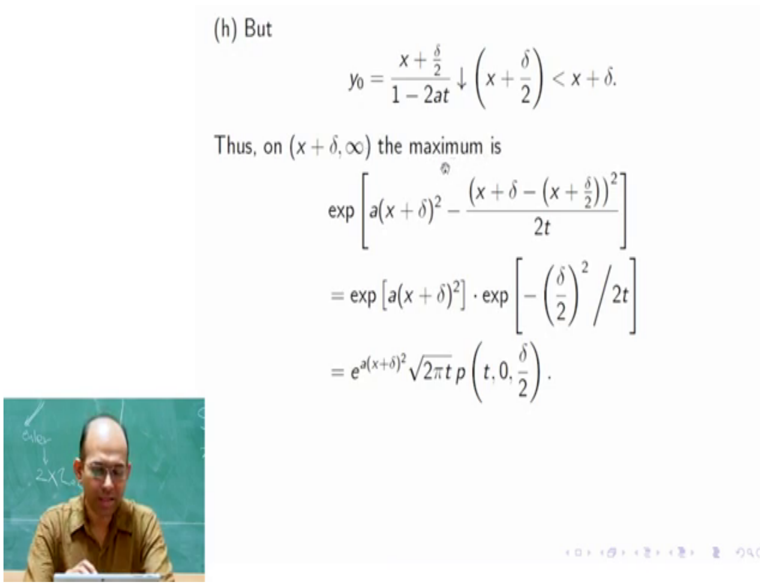

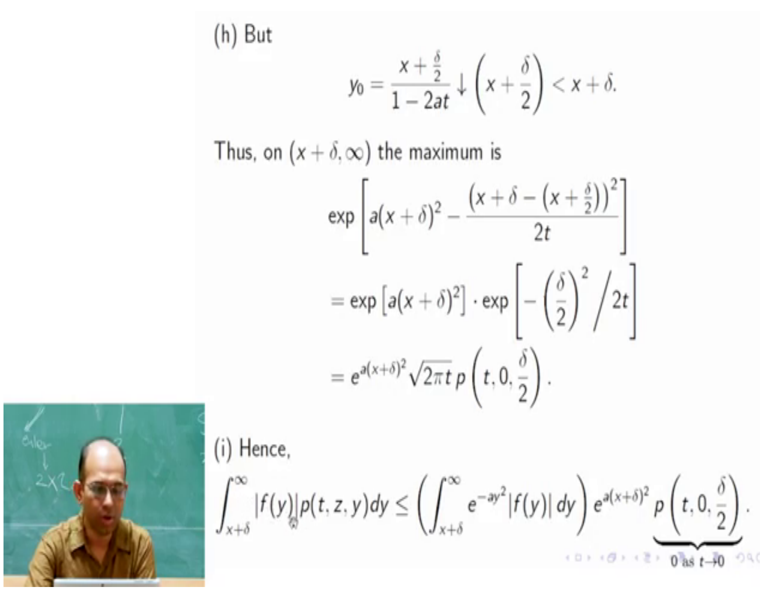

Now we look at this expression x plus delta by two one minus two a t as t is small, t tends to zero. So, here let us see, what does it do? It is little smaller than one, correct? So if t is small, then one minus 2 a t is still positive, so but this is little less than one so little less than one is the denominator, so this whole thing is little more than x plus delta by two, but as t turns to zero, this is converge to x plus delta by two.

So it will converge on the upward. So, it is converging from upward, so it is decreasing to x plus delta by two. So, it decreases x plus delta by two so where x plus delta by two is some less than delta, so that means after certain t small sufficiently small this quantity would be inside within x plus delta.

So, now on x plus delta to infinity that positive line, the maximum is e to the power of so I am writing from here. So, here e to the power of a times x plus delta by two whole square into divided by one minus two a t, correct? This was the maximum it was maximum on this whole x plus delta two delta by two to infinity. However, if that maximum below is x plus delta, so, the maximum which was achieved which is below this, so that means here it would strictly decrease. So, for x plus delta by two to infinity I have the upper bound, but that upper bound is here then the function would decay.

So, that means at the point is x plus delta it would be little lesser possibly whatever the value is, and then that should become the new upper bound for the function over this interval. So, if I look only the subset x plus delta by x plus delta to infinity, instead of x plus delta by two to infinity. So, then they are the upper bound is actually the value of the function at x plus delta, why? Because the value of the, of the functions maximum achieved at some point which is below x plus delta and the function is decreases, hence onward.

So, the functions maximum value from this point to infinity should be on the boundary. So, here when we consider instead delta by two we consider x plus delta to infinity then the maximum for finding out maximum, we can now safely evaluate the function at the point x plus delta itself. The all these analysis actually earlier help me to do this confidently, but this we cannot do for any arbitrary t, for sufficiently small t, for sufficiently small t, when this is below is x plus delta. So, then only for sufficiently small t we can sufficiently small t onward, all the time, this would be function of t. So, what is the benefit here see earlier that upper bound had was quite complicated.

So like here many t, so a plus, a into x plus delta by two whole square divided by one minus two a t, this way. So here it does not have the expression of the kernel heat kernel with one minus two a t is there, but here it appears a times x plus because my function is actually a y square minus y minus this whole square by two t. So, y is replaced by x plus delta a x plus delta whole square minus is to y I am writing x plus delta by two whole square by two t.

Now here is x x cancels, so you get delta minus delta by two that is delta by two whole square so delta by two whole square divided by two t and here I have as it is e the power of a times a x plus delta whole square into e to the minus delta square by two by two t. Now, this quantity looks like you know the heat kernel the probability density function for normal.

Only that we do not have appropriate coefficient and multiplier. So we multiply and divided by this square to by t, and then divide on the square, square two by t, then this looks like you know that with mean zero and variance t and evaluated at the point of delta by two. So p of t comma zero delta by two. So this term is equal to square root two pi t times p of t comma zero comma delta by two, I mean this is kept as it is.

So this is the new upper bound for that function on x plus delta to infinity. And that is a thing what is of our interest because we would like to upper bound all the integral third integration, where the domain of y is x plus delta to infinity. So x plus delta to infinity mod of f of y p of t comma z y dy. So here we have written that this is less than or equal to x plus delta to infinity e to the power minus a y square, mode of f of y dy into this e to the power x plus delta whole square and then this thing, correct? This square root two pi t we do not need anymore because that got cancelled from the earlier expression, correct?

So, here we had already one over square root two pi t we had this term and this term, so this term we have you know estimate upper bounded and this term was there. So, we got this term, so, we do not have to worry about this term.. See, I mean, how important was this analysis because we were anyway going to take t tends to zero and then this explodes, correct? So this whole thing I mean, this manages, correct? So this things manages these balances so that balancing is now appropriately illustrated and captured here.

So now there is no t term everything is here. Now here in this as t turns to zero, we are going to see. So, we know that as t turns to zero this is the probability density for normal random variable with mean zero and variance t at value delta by two and this does not depend on t. If variance decreases, then the probability decreases.

And then this kernel goes to zero, you can also see analytically instead of this probabilistic justification explanation, with this goes to zero as t tends to infinity. And now this term is finite, I know this and this goes to zero. So, we can actually have sufficiently small t equal delta prime, below that this whole thing would be bounded by epsilon.

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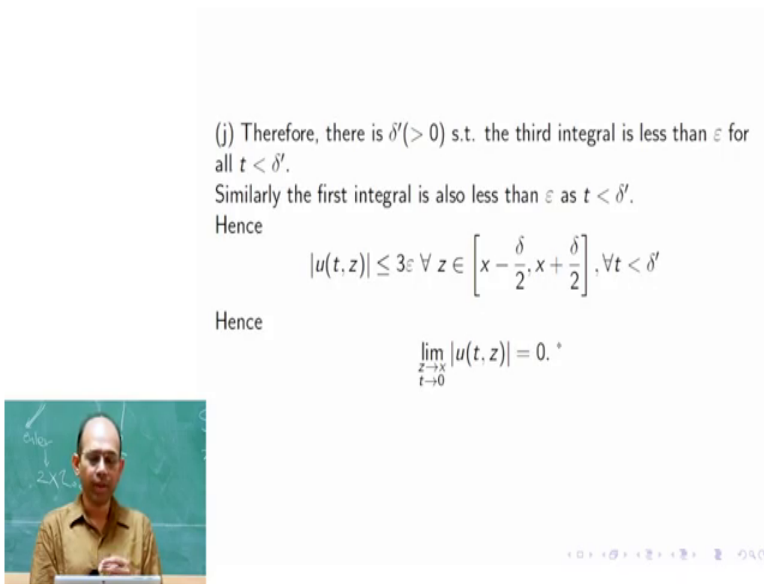

So therefore, there is a delta prime positive such that third integral is less than epsilon for all t less than delta prime because here, this whole thing does not depend on t anything, this is fixed number and this number is only decreasing as we are decreasing t. So, we can make this therefore, there is a delta prime positive such that the third integral is less than epsilon for all t less than delta prime.

Similarly, the first integral is also less than epsilon as t is less than some, I mean I have written the same delta prime one can take some other delta prime also was because due to the same argument because there also one would have exactly same way of arguing. So hence, from these three terms, we are going to be three epsilons mod of u of t comma z is less than or equal to three times epsilon.

So this is true, no matter what z is, z is between, for any z between x minus delta by two to x plus delta by two and for any t less then delta prime. So, even if it is difference of delta double prime you take minimum between this and then you will you can use that here. So that is basically saying that there is a double limit because given epsilon you got a upper bound, I mean the neighborhood for t and z both. So z limit z tends to x and t tends to zero then more modulus of u of t comma z is equal to zero. So this proves that the solution of the heat equation which is written as conditional expectation, that is continuous at the initial point, that is boundary of the domain now, because domain is t and z two different variables.

So, it is continuous at that initial point data if the data is continuous there. So, in other words, if your initial data is continuous function on the whole domain then the solution of the heat equation will also be continuous on the whole domain. And I mean interior and the closure also it is continuous to the closure also. So, that completes the proof of this theorem, thank you