Introduction to Probabilistic Methods in PDE
Doctor Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Prerequisite Measure Theory
Part 03

Next, what we are going to do is the? We are going to use some results to define integration of an arbitrary measurable function. So, for arbitrary measurable function one might not always define integration. So, you would figure out for what type of function we can define and the class of measurable function for which we can define, we will be would successful will be able to define successfully that class we are going to call as integrable functions.

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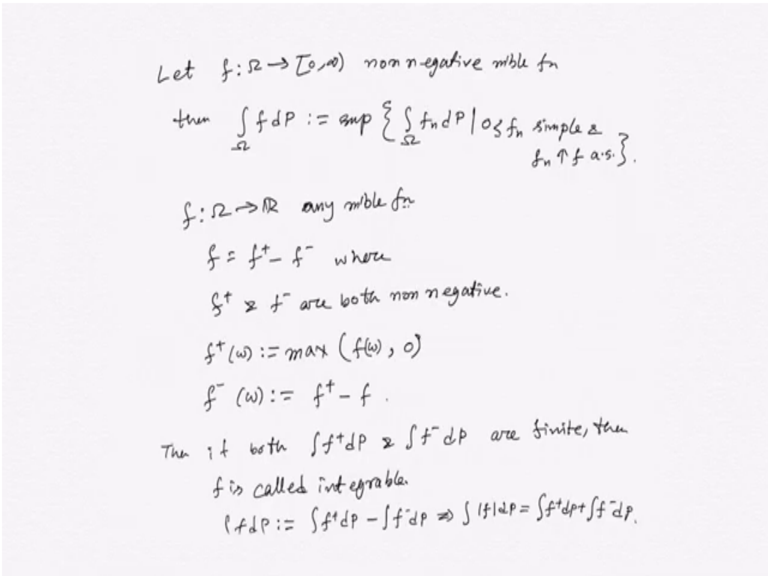

So, let we have f from omega to r plus that means, you know non-negative, if we have such function f then integration of f dp would be defined as supremum of integration of say fn of omega dp over omega where what is my fn? fn is simple functions and converging so it increasing to f, increasing to f.

So, fn is simple functions but they are increasing their the sequence increases to f. So, I mean almost sure convergence, if they increase to f, and then for each and every fn, I evaluate this integration I can evaluate using the earlier formula and then I would get a sequence of real numbers. Because you know when I integrate this I will get one particular number that will maybe real number or maybe plus infinity.

So, extended non-negative real number when and you get a sequence of that which is non-decreasing sequence because fn are increasing correct initial increase. So, non-decreasing and then we are taking supremum of that. So, that makes sense because it is a sequence of non-decreasing exterior valued sequence and we are going to get the supremum and whatever the supremum we are going to call that as integration of f.

Now, one can ask given a function f can one always find out a sequence of simple function which really converges to f and increases at constant df. The answer is yes, there are some constructions which I am not going into the details. So, this is true that given any function f measurable, I mean non-negative measurable function f one can get such fn. I also I would also require fn to be non-negative, so non-negative fn this is always true.

So then by this line we has given a notion of intuition of f when f is not negative, so what happens for general when f is a any real valued function, then we can write down f in two parts that is f plus minus f minus where f plus and f minus both are non-negative that means they could be 0 or more than 0, both are non-negative but I will be able to do that.

So, I mean a very simple also one can ask that why, what assures me of such decomposition. It is easy because given a function f you to the maximum f and 0 that would give you f plus and you subtract. So, f plus is you I mean the choice is easy it is you just take f of x and 0, so whatever the value you call that that is f plus of x. And then after become keeping that f plus you define f minus as f plus minus f, simple because you know this is when f is positive then f plus is also positive. They would have the same value they cancel and make it 0.

And if f is negative then minus f is positive, so this is given in positive but f plus is always non-negative, so I would again get f minus is non-negative. So f minus is non-negative so this is this is the way and then you know why should this be true because now you know if we keep put f on the left hand side and f minus on the right hand side I am going to get exactly same equation. So after decomposing f in these two parts, integration, or I mean and then define what is the integrability because as a told that for every measurable function, we might not be able to define integration of f successfully.

So, we introduce a subclass the integrable functions. So, if both integration of f plus dp and integration of f minus dp are finite, remember earlier as we told that given a function f which is maybe unbounded some function I can have fn simple, simple sequence of simple functions whose its own integration might be infinity say and then there is no hope that intuition f would be infinite.

Otherwise sometimes you know these are all finite but intuition f and all finite but they grow to infinity. So, I can easily have infinite value for this for this integration. Now, we look at only for those functions such that both this things are finite, then we call that if then f is called integrable, integrable then f is called integrable if it is going to be.

And then for an integrable function f integration of f dp is defined to be integration of f plus dp minus integration of f minus dp. Now, you understand that why do we need this finite thing. Consider an extreme situation when both integration f plus an integration f minus are infinity then we are into the situation of infinity minus infinity that makes no sense correct. That makes no sense.

So, to avoid that thing, we consider the cases where both are finite one can actually also either of these is finite one can put that is there is a (())(7:55) relaxed situation which we are not going to consider. We are going to consider only the situation both are finite. And then this is the thing and then it implies that integration of mod f dp is equal to integration of f plus dp plus integration of f minus dp, good. Now, we discuss what do we mean by convergence of integrals.

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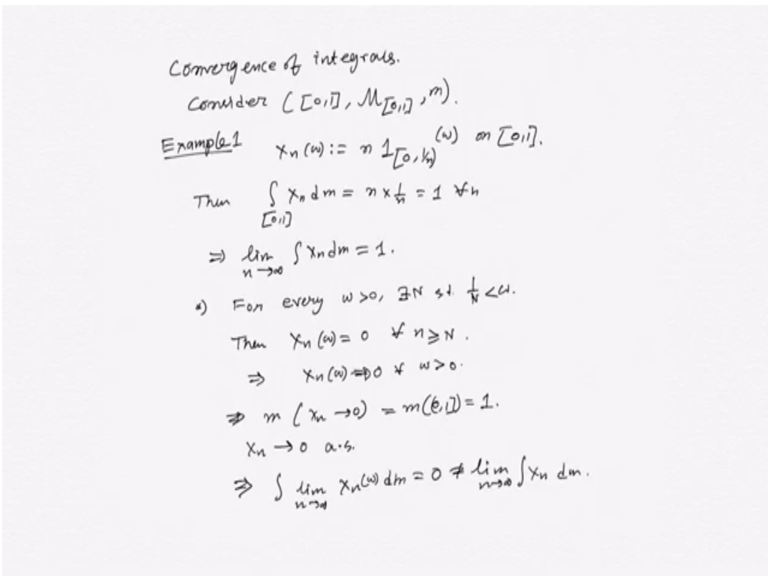

Convergence of integrals, imagine so, we start with some examples. So start with some example, consider a sequence X n of omega, which looks like n times indicator function of closed 0 to 1 over n open. So, this is function of omega and that is defined on close 0 to 1. So, in on the indeed interval we have this function X n and the measure space we consider here say close 0 1 and lebesgue sigma algebra and the lebesgue measure that is the measure space.

And then we find out the integration of X n on these intervals 0 1 what would be its value? Its value is the just the area under the curve, the height is n, the base size is 1 over n. So the area is n times measure of 0 of 1 over n that is 1 over n, this is 1 that is for all n.

Now, we ask that what happens if we put limit so, limit intense to infinity, integration of X n dm. So, that is one because it is one for all in this is one. If we now ask I mean what happens if I just consider a limit of X n and then integrate.

So I am talking about can we interchange the limit and integration these two operations. So, let us see what happens? So far, so this is the next point for every omega, which is more than 0. So, I write which is more than 0, which is more than 0.

I would always be able to find out 1 in such that 1 over n is less than omega. And then consider that type of n. So, n so n is such that 1 over n is less than omega. Then X n of omega is 0 for all in great or equals to capital N. That implies that X n of omega converges to 0 for all omega greater than 0. Measure of the set on which X n converges this to 0 is equal to measure of open 0 to 1 closed but that measure is 1. So that means X n converges to 0 almost surely.

However, integration of the limit so, 0 is a limit of X n almost sure limit almost sure limit of X n. So integration of limit of X n is 0 because integration of 0 is 0. So, this is so this implies that we are getting this is 0 which is not equal to this 1 because it is not equal to limit n tends to infinity integration of X n dm. So, here limit and integration does not interchange.

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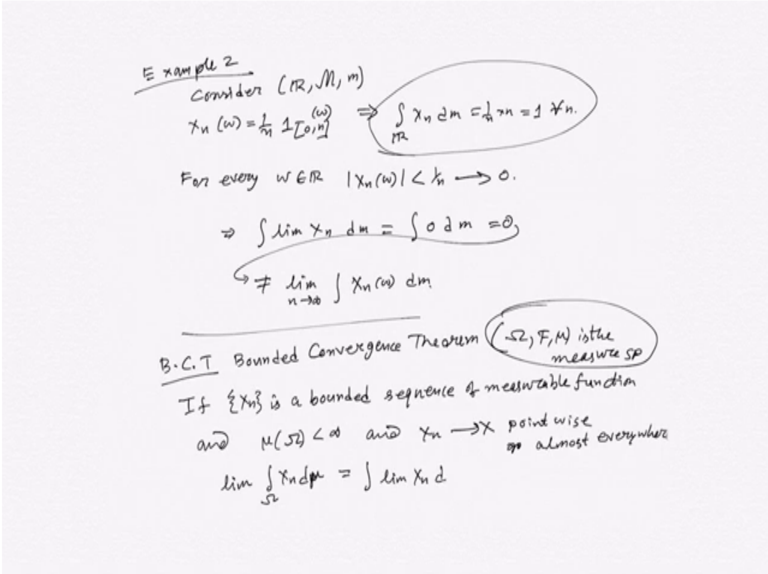

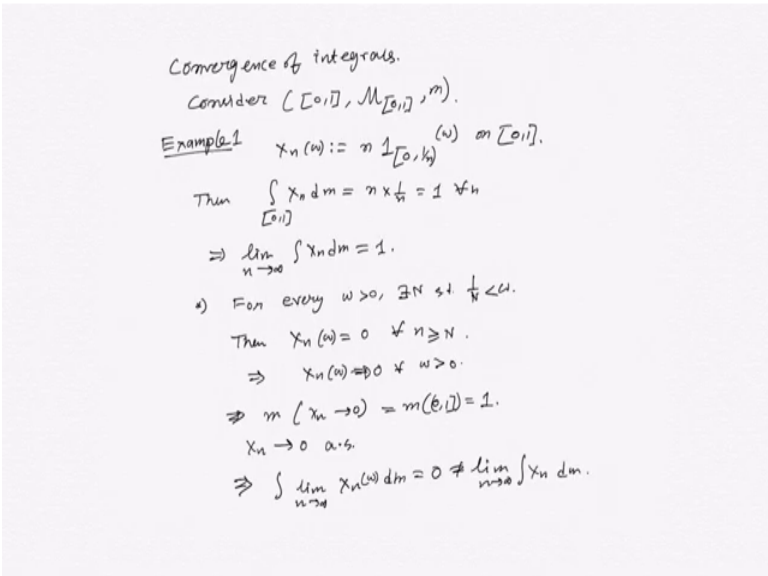

There is another example, so example 2, consider full R and lebegue sigma algebra and lebesgue measure here. And we asked that we consider the simple the indicator function 0 to n 1 over n that is my X n fixing of omega is this, fine. So, here let us calculate what is integration of X n dm? Again using the property of integration, this would be 1 over n times the measure of the base where it is 1 that is n. So, this is 1 for all n, however if we consider where does it converge. So, for every so, every omega if it is less than 0 so it is X n of omega is 0 all the time so it converges 0 anyway.

But if omega more than 0 there is you know some n such that X n of omega would be you know non it will be positive. However as n tends to infinity, so 1 over n so X n of omega is always less than 1 over n, does not matter even it is positive this non negative. And this dominating sequence goes to 0, so X n also goes to 0. So, far all omega positivity is true and omega can be less than 0, it is already 0 all the time. So, for in general I can write down this for all omega in R.

So, this is point ways converges every point is converging here. So if you write down limit X n, so limit X n is 0 if I integrate limit of X n dm dm, so I am going to get 0 only integration 0 is 0, so, which is not matching with limit of integration of X n dm. So, these are a two examples, where we have seen that limit and X n integrations are cannot be interchanged.

So, one should then ask that under what condition it can be interchanged? So, this is called bounded convergence theorem. This is basically saying that these two examples have two anomalies if we can avoid both then actually limit integrations can interchange.

What are the anomalies? First example, the function was not bounded function because it was for any you know, finite number positive number, whatever it is a c, you take, very large number one million. You can always find that 1 n such that with more than one million and then that for that particular n X n would cross that particular number for some omega, for some omega and the measure of that omega is also positive. So, if you know this X n, so contrary to that is a X n is bounded. So, if X n is a bounded sequence of measurable function then I do not have that problem.

However here our X n was a bounded sequence because mod of X n is less then or equals to 1 for all n is bounded. However, that the set of omega is on which X n is non-zero that set is going to infinity because close 0 to n and n is becoming larger and larger. So, it is becoming the you know four positive real (())(17:54). So, if so if we also avoid this, what does it mean? That measurable function and the measure what we are talking about is a finite measure and the mu of omega is finite that means the totality has finite measure.

And then integration so, here X n is from omega, the X n d mu limit this is equal to integration of limit X n, also I should also assume that X n has a limit. If X n is a bounded sequence on measurable function this also and X n converges to X point wise. Actually I do not need point wise I mean converges in converge almost everywhere is also sufficient. So then this is true. This is called bounded converges theorem.

There are some generalizations of bounded convergence theorems, many generalization, important generalizations, so one is called dominate converges theorem. There you do not say that, X n is a bounded sequence less than of a some fixed number but you say X n is mod of X n is less than or equal to another function measurable function non negative measurable function which is integrable.

Then, also you can have the same assertion. There are many other generalizations also we are not going to just have all these but this is the basic fundamental result for convergence of integrals and then the conditions under which we can put the limit of limit inside the integration.

So, next we see one more important result that is important when I have two different measure spaces. So, like you know but those I mean two different measures on the same measurable space and that comparison between these two measures. So, let us go to the next slide.

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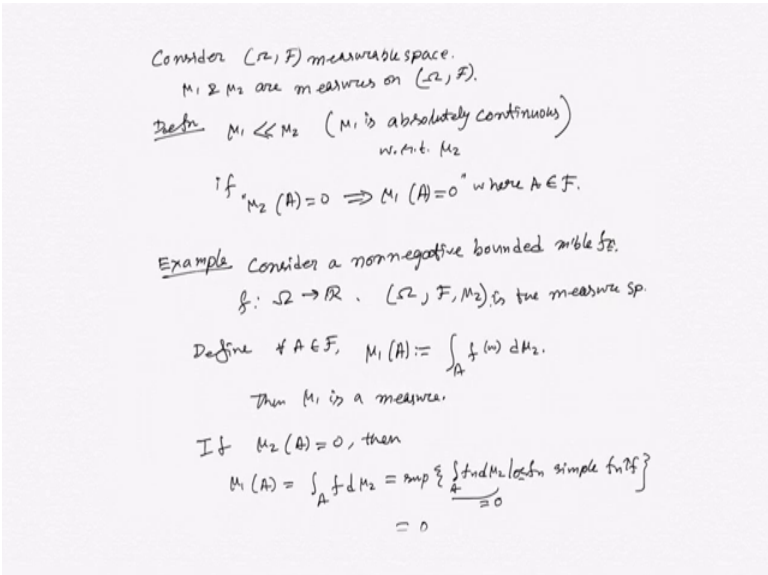

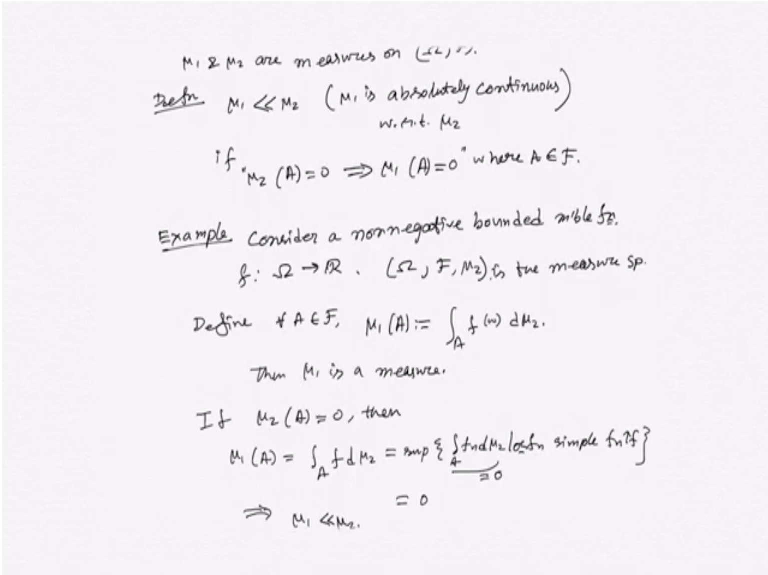

μ_1 & μ_2 are measures on (Ω, F).

Defn $\mu_1 \ll \mu_2$ (μ_1 is absolutely continuous w.r.t. μ_2)

if "$\mu_2(A) = 0 \Rightarrow \mu_1(A) = 0$" where $A \in F$.

Example Consider a non-negative bounded m'ble fn.

$f : \Omega \to \mathbb{R}$. (Ω, F, μ_2) is the measure sp.

Define $\forall A \in F$, $\mu_1(A) := \int_A f(\omega)\, d\mu_2$.

Then μ_1 is a measure.

If $\mu_2(A) = 0$, then

$\mu_1(A) = \int_A f \, d\mu_2 = \sup \{ \underbrace{\int_A f_n \, d\mu_2}_{=0} \text{ | } f_n \text{ simple fn} \leq f \}$

$= 0$

$\Rightarrow \mu_1 \ll \mu_2$.

Consider I had I have a measurable space omega F and I have two different measures on this, so mu 1 and mu 2 are measures on this omega F. We would call mu 1 is absolutely continuous which this to mu 2 if mu 2 of A is equal to 0 implies mu one of A is equal to 0, where A is from F. So, what does it mean? That if there is any set A measurable set A such that it is null in a sense of mu 2, it is mu 0. So then under mu 1 also it is null.

So, any set which are unimportant, with this mu 2 is also unimportant with respect to mu 1 and then you say mu 1 is absolutely continuous with respect to mu 2. If that is a case, then we have a very nice result. Before that we are stating the result, let us see some other some examples, so some examples where this property is true.

Consider a non-negative bounded measurable function f from omega to R and consider a measure space omega F this omega and measurable with respect to the sigma algebra f. And mu 2 measures. Define mu 1 of A is equal to so different for all A in F, integration of f d mu 2 over the set A. So I have a bounded non-negative measurable function f that I am integrating over the set A with respect to the measure mu 2.

And whatever the value I going to get that we are declaring as the mu 1 measure of set A. So, is mu 1 is a measure? Yes it is measure because for measure you need that measure of empty set is 0 it is true because if you integrate this function over empty set you are going to get 0. And then if you ask that if measure of countable union disjoint countable union of sets is the sum of the measure that is also true because if you have disjoint union here or the integration that you can write down as sum of different integrations and then from there you are going to get it.

So, mu 1 is a measure and then if we have mu 2 of A is 0, then this is the 0 measure set and this f. And now what is the meaning of this integration? This integration this meaning is, we take supremum of integration of simple functions sequence of simple function which converge which increases to f, converge increases to f. However, when you do this, so then integration of A f d mu which is that supremum of all these A f n d mu where f n are non-negative and as simple, simple and increases to f.

If we do this, then we see that for every simple function f n here when integrate over set A which is measure 0, I would always get 0 here. Because simple functions are summation of ci and indicator function finite thing and then whatever indicator functions we have, but on the on the set A and then base you know base that base would be of measure 0. So, this is 0 supremum of 0 is 0. So, this is 0.

So, what does it mean? It means that mu 2 is equal to 0 implies mu 1 of A which is define by this is 0. So, it implies that mu 1 is absolutely continuous with this to mu 2. So, now one can ask that there is an example this is just an example of a an of a PR of measure where 1 is absolutely continuous this another.

One can ask that is it the all class of the measures which are absolutely continuous with respect to mu 2. I mean, does it mean that any a mu 1 which is absolutely continuous with respect to mu 2. Can be written is in this way. So, this is the reverse direction question. So, we go to the next slide.

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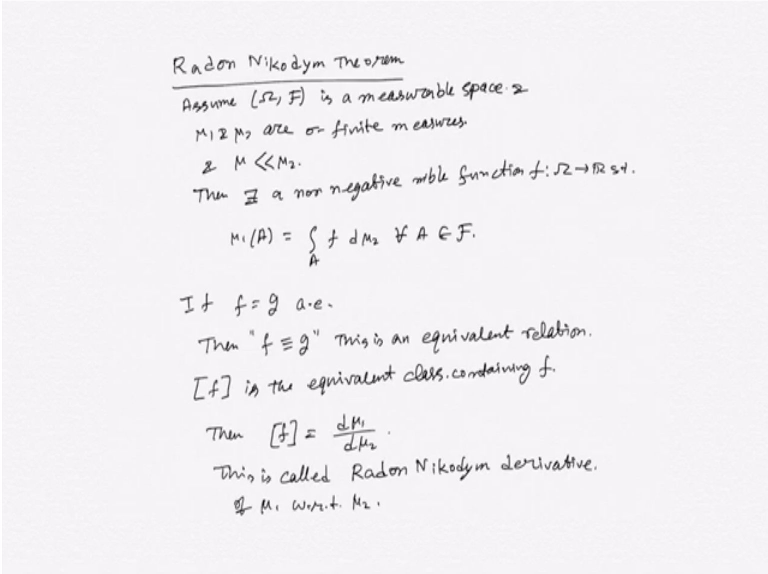

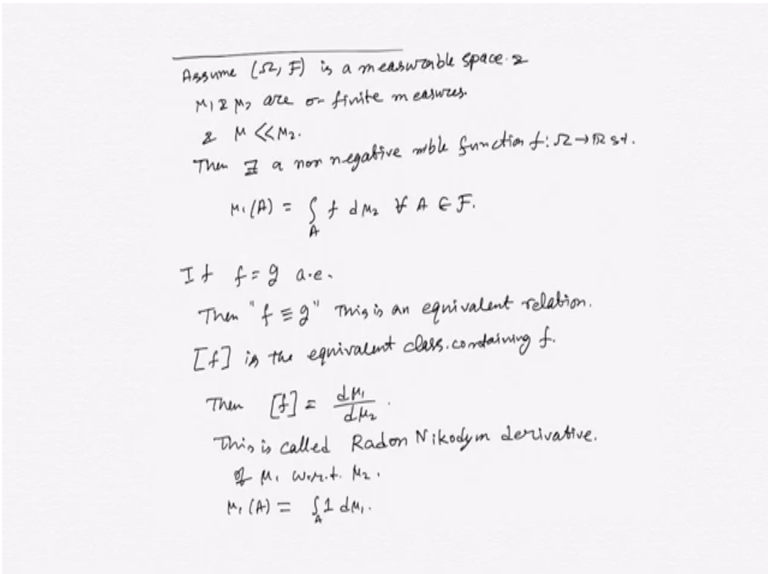

So, this theorem is called Radon Nikodym theorem. So, I am not proving it, I am just stating this theorem Nikodym theorem that says that of course, I mean the statement is not as general as we anticipate it needs some conditions on the measure, we need sigma finite measure. What does it mean?

That this even if the measure of the whole set is finite however, the whole set can be retained as countable union of subsets and those sets has finite measure. So the situation when you have a measure space such that, that the whole set omega can be return as countable union of say finite measure sets. Then we call that measure spaces sigma finite measure.

So, we consider sigma finite measure here. So, assume that omega F is a measurable space and mu 1, mu 2 R sigma finite and mu 1 is absolutely continuous with respect to mu 2 then there exist a non-negative measurable function f such that mu 1 of a set A is integration of f with respect to mu 2 over the set A. So, this is the statement of Radon Nikodym theorem.

And here whatever that f we obtained. So this f if we just change this f at 1 single point or may be (())(29:31) single point we are going to get another function, but both the function would have the same integration values. So, we are going to take equivalence classes of f. So, take say we you say f, if f is equal to g almost everywhere, almost everywhere, then we say that we set f equivalent to g.

So, this is equivalent class and then equivalent relation and then equivalent relation puts one you know partition in the class of all such functions all measurable functions and then if class of that, so we denote that box f is the equivalence so this is equivalence relation, equivalent relation and this is the equivalent class containing f. So, then this equivalent class of f is called d mu 1, d mu 2. So, this class is called the Radon Nikodym derivative.

Sometimes we do not always mention equivalent class we just say this function we understood. So, this is like you know sometimes you know this is not specified explicitly but all the time it is understood that we are talking about the equivalent class and this is unique.

Is unique in the sense that if f and g are in two different equivalent class then we would not be able to have this relation for both f and g or in other words if f and g are two different functions satisfying this equation for all possible A for all possible A in f. Then, f and g should be in the same equivalent class and that is unique and that unique class is called the Radon Nikodym derivative.

So, we call this Radon Nikodym derivative, derivative of mu 1 with respect to mu 2. Radon Nikodym derivative of mu 1 with respect to mu 2, so these Radon Nikodym derivative, so what is the notion of what is the meaning of that? Because, see if you now replace f here by d mu 1 by d mu 2 then you can see some algebra is happening, what you have visiting schools also like you know. So, d mu 2 and d mu 2 cancels like that d mu 1 and d mu 1 of A and mu 1 of A is any way we know that this d mu 1 that we know very well.

So, that is the reason of this notation. Since, mu 1 of A is d mu 1 of A, you have integration of d mu 1 over set A and then that is the reason that we write down f as d mu 1 and upon d mu 2, this is a notation. Thank you very much.