Introduction to Probabilistic Methods in PDE

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Part – 1

Stochastic Representation of Bounded Solution to a Heat Equation

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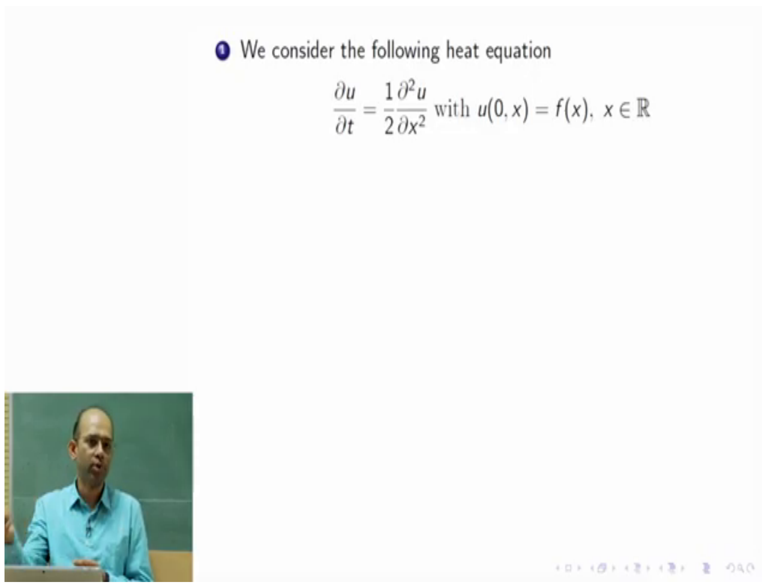

So, next we start heat equation. So, we consider the following heat equation, what is heat equation? Heat equation is a solution of I mean this is the parabolic equation because u is a function of x and t, where for this particular example x is just one dimensional real number. However, one can replace del 2 to del x2 by Laplacian and u would be a function of, you know time t and Euclidean space. So, this is called parabolic equation because with respect to t variable there is only first order operator with respect to this x variable which is we call offen space variable, it we have second order operator and the coefficient is non zero.

So, this is the simplest form of heat equation. This is satisfied on all time positive and all x in R and at time 0, u satisfied one initial condition, we call initial because time is concerned. So, when time t is equal to 0 we call that initial time. So, at time 0, it satisfies the initial data, we call f has initial data, so f.

So, that is the meaning of heat equation. What we are going to do is that, we are going to show that for this heat equation also, we can have stochastic representation. So using Brownian motion, we can write down the solution of the heat equation as a conditional expectation of function this function would be there.

And the Brownian motion would also be involved there. So that we are going to do, so that, I mean that would help one in many different way, one thing is that that would help one to compute. However, for this equation actually the solution has closed form solution one can write down in terms of integrations, it has a close, it is using error function but there are more complicated equations say for example, where these coefficients are not constant but variables etc, that one cannot write down the solution in close form, one can write down as this you know elementary integration.

Then, for those cases what people do, they try to solve numerically, when one solve these equations numerically, then one has to take the whole domain, and discretize the domain and then find out the solution of the on the whole domain there, even if someone does not need to find out the value of the solution on the whole domain, but say I just want to find out the value of the function only at a single point, but still the numerical procedure involved the solving on the whole domain and then only one can get.

On the other hand, if one takes the stochastically representation as we have seen for Dirichlet problem, there we do not need to consider the whole domain because the solution is u of x is equal to a conditional expectation of f of W tau D given W 0 is equal to x, that means you just take the point x where you want to find out the value of the solution and then you simulate the Brownian motion and wait till that Brownian motion hits the boundary. And there the location you do find out, at that location you see that what is the value of f, the boundary data.

And you continue recording that values you know, maybe a large number of time, capital N number of times and take average of these values, that would give you an approximation of the expectation. So that would be a numerical approximation of the solution at that point and for finding out these you do not need to actually compute the, approximate the solution at any other points.

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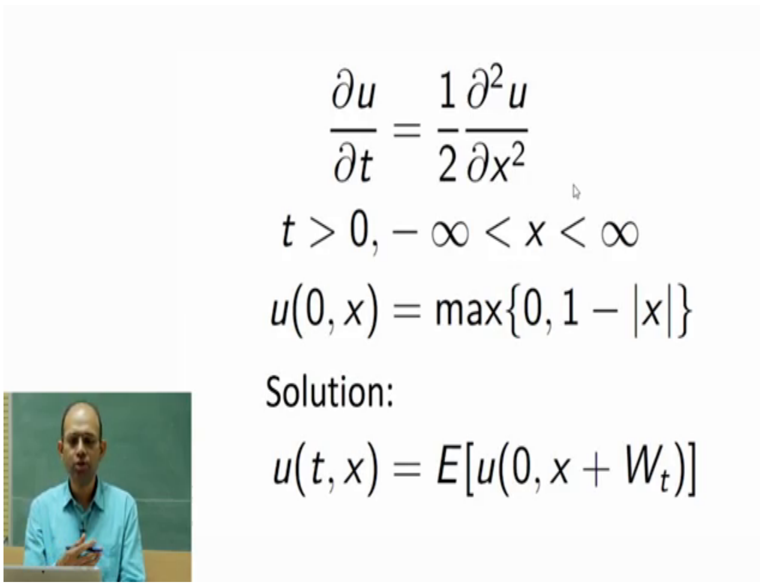

So, that is one advantage of stochastic presentation, so far competition is concerned. For those who are seeing heat equation for the first time, the following example and visualization should help them gaining, understanding of the heat equation. Here, I do not have the scope of actually deriving this equation from the physical phenomena of propagation of heat however, I have taken this equation and then solve these numerically and then the animation might help you to understand that it is really indeed the heat profile propagation.

So, consider this equation del u del t is equal to half del 2 u del x2, so here this time is 0 to infinity. So all positive real time and then x is minus infinity to infinity the whole line, so you can think that the material is a thin rod extended to infinity from both sides to both sides, and then initial data that time 0 it is maximum of 0 comma 1 minus mod x, what does it mean? That x at x is equal to 0 it is 1 minus 0 to 1 and x is equal to minus 1 it is 0. So, heat is like is 0 at the end points of minus 1 and 1 and beyond.

However, inside the interval minus 1 to 1, it is like a triangular shape. So that means the highest temperature is at the origin when x equals to 0, and then the solution we are going to I mean see in our successful lectures that we can write down solution of this equation as expectation of u of 0 comma x plus Wt at time 0 whatever the, you know expression is, so this know is known function x plus Wt.

So, let us see that how does the solution look like. So, here it is not easy to draw such function because it is functional two variables t and x. However, if you fix a particular t there is a function of x we can draw.

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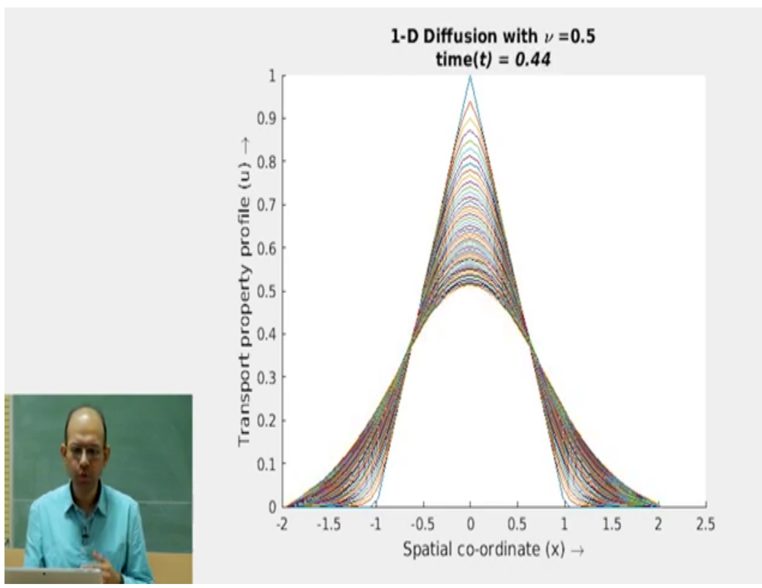

So, this these are the numerical solution of the of the equation at time 0. So, this blue curve is at time 0 and then, if you change time t so your time is increasing 0.1 to 0.2 to 0.3 and it is increasing 0.4 to 0.5 then that function value at 0 is decreasing as it happens because heat dissipates diffusion happens. And then that heat is transferred to both sides and here on other sides and beyond the interval minus 1 to 1 the heat you know temperature amount is increasing, it is increase for some time and then again it will decrease because again you know is the more heat would dissipate and then to decrease and then everything would go to 0 eventually.

So, this is like you know, the visualization of the solution of the heat equation. And then we know that finally know how this solution look like at finite time, so time t is equal to 1 say, we know how should it look like.

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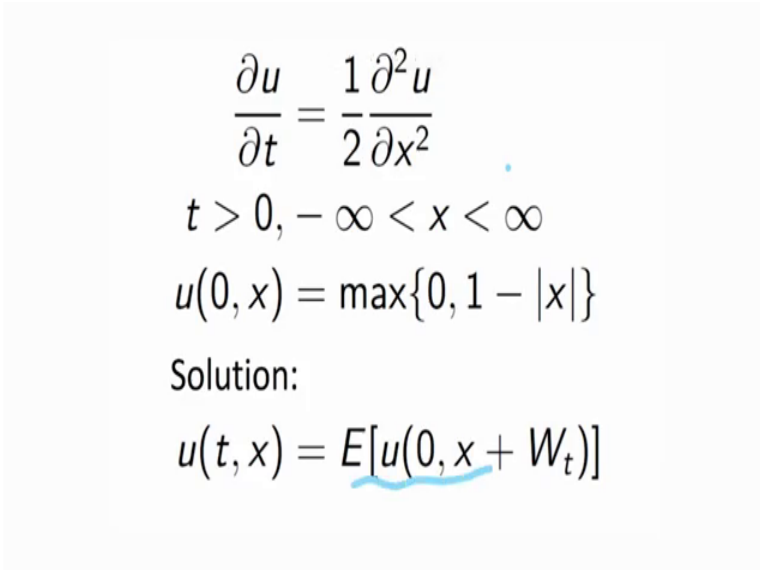

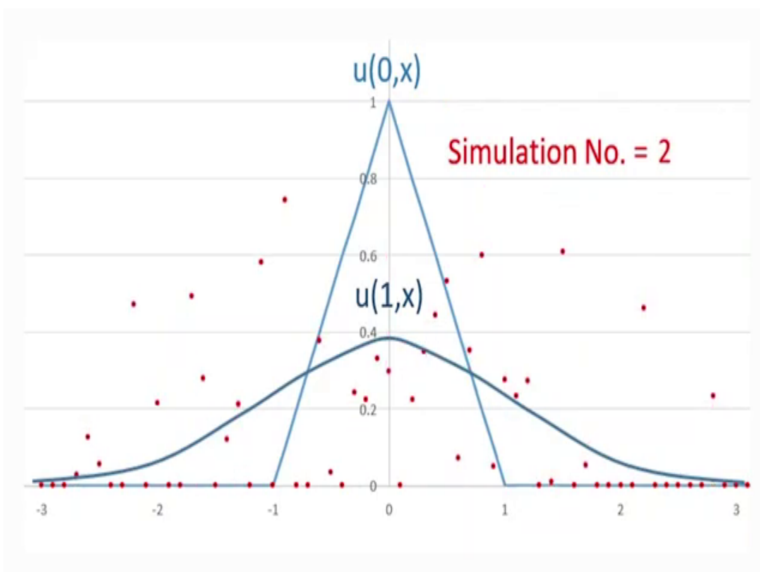

And here, we are now going to see that what is the effect of this formula that expectation of this. So what is the effect of this formula? So, this movie is actually made by finding out expectations by finding out the average. So this Wt is the Brownian motion, so we have fixed t is equal to 1, so u 1x, so this blue line. So, this blue line, so this blue line actually shows the solution at time 1 and this line is the given initial data and then this red points are given this any particular point x, what is the solution value?

Imagine that we do not know we have not computed the solution, we do not know this value, but we want to find out this particular value using this formula, that means we would like, like to find a W 1, W 1 is a normal random variable with mean 0 and variance 1. So, we would like to find out expectation of this..

So, then we take more and more such u of 0 comma x plus W 1 and larger the number of such realizations and then, then we will take the average of those, correct? That is, that is actually sample average, and then the sample average converges to the solution. So let us see how does it converge, so number of simulation. So this is the number of such samples we have considered for considering the average and as the simulation number increases, so more number of samples in increases these you know these points are coming pretty closer to this.

So, this is showing that, one can get a good approximation of the solution one can get using the simulation. If one wants to find out that, exactly on that curve visually exactly impossible one has to run simulation 3000 I mean I have checked that 3000 it is pretty close. So, now we have a better understanding about heat equation. Now we go back to our slide.

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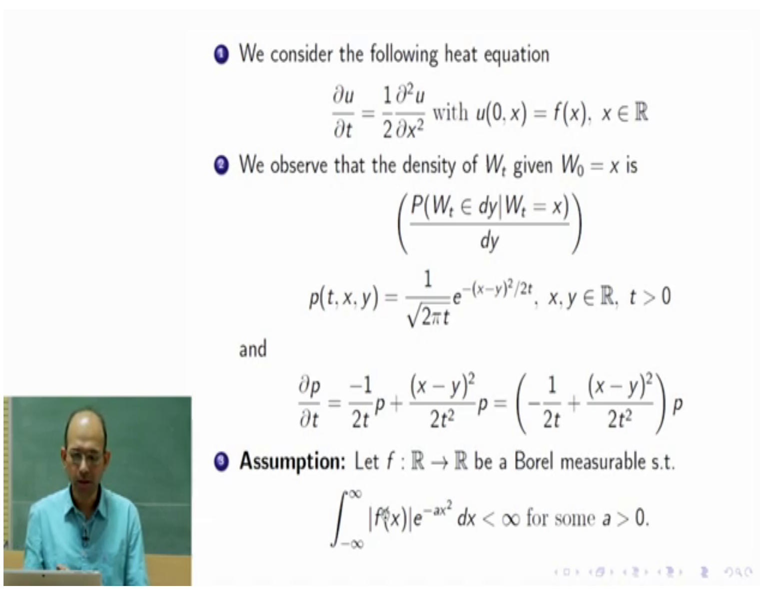

So here, we observe that the density of Wt, the Brownian motion is given I mean given by probability density function Wt is in some nodes. So, that is how we find out the probability density function, probability density is nothing but the Radon-Nikodym derivative of the law with respect to the Lebesgue measure and that is Pt of x y, why? Because you know, we fixed t, we fixed x and the Radon-Nikodym derivative is a function of y. So, p t x y that is given to 1 over square root of 2 pi t into et o the minus x minus y whole square by two t. So, this formula we all know because this is nothing but probability density function of normal random variable with mean 0 and variance t, and that we have already seen for in the definition of Brownian motion, that this should be the density function. And if we now compute the partial derivative of p with this t, we would see that, so this is the product of two terms.

So, from here we are going to go to get minus 1 over two t p. And then from this part, we are going to get x minus y whole square with this part by two t square times p. So we are going to get this, so we record this expression here, we also have one assumptions, which would be obeyed throughout this part of the study that initial data has some kind of growth condition, what is this growth condition? That mod of f of x into e is to the power of minus x square, dx is finite.

So of course, I mean we are not here aiming to answer to this heat equation problem question for all possible given function f, I mean, what does it say? It basically says that if f is any polynomial, not polynomial and I am saying that is growing to the infinity in polynomial growth, then also we know that this integration is finite.

So, this would be violated, that mean it will be info infinite if f you know, grows to the infinity grows means in the sense of mod f grows to infinite it fluctuate minus negative or positive faster than any polynomials, then we might not be able to assure this finiteness. So this theory whatever we are going to derive is under the assumption, that the initial data that is not growing more than polynomial growth.

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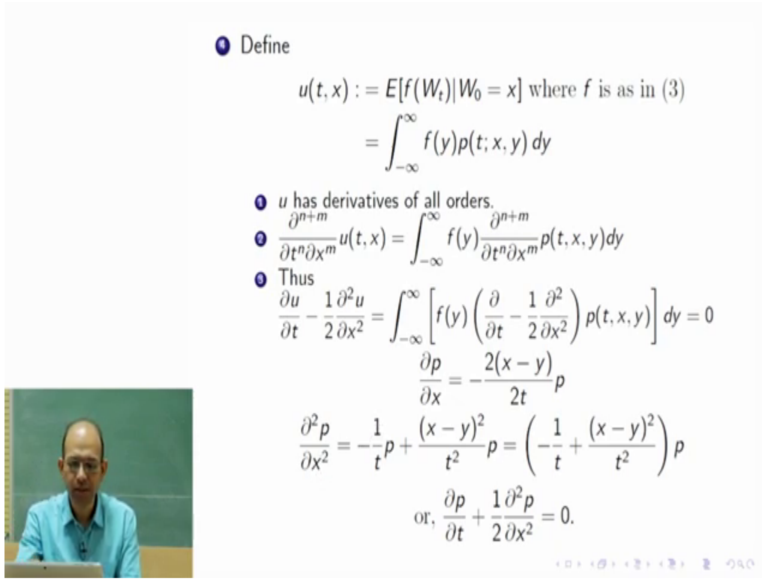

So here we define u of t comma x as you know expectation of f of Wt given W 0 is equal to x, that means you are starting at the point x, this can also be rewritten as expectation of f of x plus Wt given W 0 is equal to 0, what was the way I have written in those you know animation, quite a few is satisfying that condition would give that condition.

And then, this expectation is of this form minus infinity to infinity integration f of y, p of t x y, and this integration makes sense because of this condition on f and p has e to power of minus x minus y whole square by two t. So, this is finite, so, this makes sense, so we can define this function.

And here, we are going to see these properties that u has derivatives of all orders, p is of course smooth p has derivatives of all orders because it is exponential function it is smooth nice function, but what we are going to say that under these assumption on f, what we have mentioned does not matter what f is, but if f satisfies that condition what is written three then u defined this manner is also a smooth function, it has derivatives of all orders, see infinity function.

Next is that computation of the partial derivatives, how does the partial derivative look like. So, this partial derivative of u can be computed I mean you can actually take this partial differential operators inside the integration, so it is equal to minus infinity to infinity f of y del n plus m, del t n del x m of pt x y, so this is also allowed. So, these are the properties we are going to see the proof of this and also that del u del t minus half del two u del x two u if u define this way that is equal to minus infinity to infinity f of y del del t minus half del two del x two p t x y because this comes from just above, and this part is 0.

So here delta two p del x two is minus, so I have already obtained del p del t expression here del p del x expression is minus two times x minus y by two t times p, this is coming because of this you know e to the power of minus sign is there correct. Del del x equal to you get minus 2, and then this would appear, so this 2, 2 cancels actually, and then you consider del 2 to del x2 of p, then this product. So from this part, you are going to get just 1 over t here and p and from this part again this term, x minus y divided by t, so x minus y whole square by t square p.

So then, if you club this together, then x minus y whole square by t square minus one over t times p appear here del two p del x two. On the other hand del p del t is obtain as minus 1 over 2 t, plus this term. So, we see that exactly same expression that this half is multiplied here half is multiplied here. So, 2 times del p, del t is equal to del p del x 2.

So, here I made a mistake here it should be minus sign correct because, so here is the minus sign. So, this is equal to 0. So here therefore, from here we ignore that this part is 0. So, this u satisfies this equation that del u del t is equal to half del 2 u del x 2. On the other hand, so for initial condition is concerned, you put t equal to 0. So, w 0 is equal to, so expectation of f of W 0 given W 0 equal to x.

So, it is starting from there and so it is no more random it is just deterministic W0 is x so, the expectation of f of x is f of x. So this function trivially satisfies the initial condition. So, we know that from this slide that u defined this way is indeed the solution of the heat equation, but of course, we need to give a details proof of these facts that u has derivative all orders and this expression because this is also used here, correct? We need to prove one and two.

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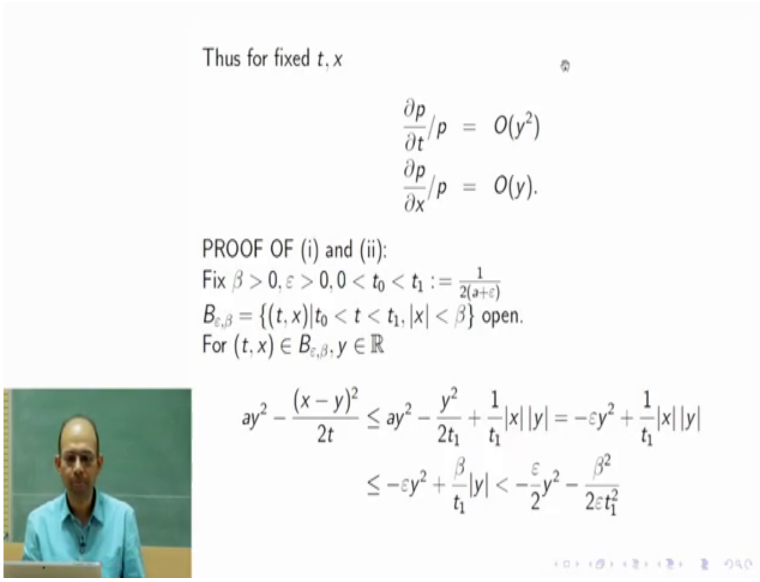

So for fixed t and x, we first observe that del p del t is divided by p is order of y square, so it even if it increases, it does not increase more than of order of y square because del p del t we have written that as you know some terms times p, so let us go back there. So, del p del t is equal to p times this thing, so it does not grow more than the power of y square, it is parabolic at most. So and del p del x by p is order of y that means, does not grow more than that.

So, these are the growth estimates which we would require for our proof. So, proof of one and two, so we first fix one beta positive and one epsilon positive and also t not positive and t 1 is more than t 1, so t 1 is defined by 1 over 2 times a plus epsilon. So, whatever epsilon you choose we take this way. And we consider a strip, this is a strip because mod of x is less than beta and t is between t naught to t 1, so open interval t naught t 1.

So, this is open strip, so on the time axis it is t between t naught to t 1 and then on the space x it is like minus beta to plus beta. It is basically open rectangle. I should not say strip as this is an open rectangle, what is important is that t naught is positive. So, this is little away from origin and also is bounded, that is also very important. A difficulty arises when it is unbounded. So you are making you know, constructing this set where things are bounded.

So now for every t and x in this rectangle, we call this B epsilon beta because it depends on epsilon and beta. Why epsilon? Because t 1 depends on epsilon, and a is a positive number, a is the number of what appears here is the same number of the assumption, what appears in the assumption. So, now we would use these inequalities, so let us read this inequality a times y square minus x minus y whole square divided by 2 t is less than or equal to a times y square minus y square by 2 t 1 plus 1 over t 1 times mod x times mod y. So this part is equal to minus epsilon square y square plus 1 over t 1 times mod x into mode y. So here, this is y square a minus 1 over 2 t1, so y square times a minus 1 over 2 t 1. However, t 1 is 1 over 2 a epsilon, so 1 over 2 t 1 is a plus epsilon. So a minus 1 over 2 t1 is epsilon.

So from this, we get this. So now we understand I mean initially it was not clear why did I choose t 1 in this fashion? The reason is now clear that we would like to choose t 1 in this fashion so that these difference appears to be epsilon also. So that now I have a negative sign epsilon, so that is inverted parabola just to get that we have chosen t 1 this way.

And now, this is an upper bound of this. So, this is one over t 1 mod x into mod y, however mode x is in less than beta. So, I can have this upper bound that, so this is beta by t 1 on mode y. So, this is even less than, so here see I have still mode y here involved and y square involved here. So, this is basically inverted parabola this you know y square. However, this second term is growing, but eventually this this, you know, for large y this first term would win, it would dominate does not no matter I mean how small epsilon is but for large y this term would start dominating than this.

However, I mean when you have to wait. So for ball of large radius, but we can actually try to you know, find out one estimate which is where we do not have first of a term. So, we obtained that if we consider minus epsilon by 2 y square and here minus beta square by 2 epsilon t 1 square. So then this is an upper bound, and for this, this is below this quantity, so I am not giving the details here, but this is just a simple calculation as you know this t1 is dependent on epsilon one would get it and you just need some algebra to complete this part.

So now as t is more than t naught and less than t 1. So 1 over t is less than over t naught.

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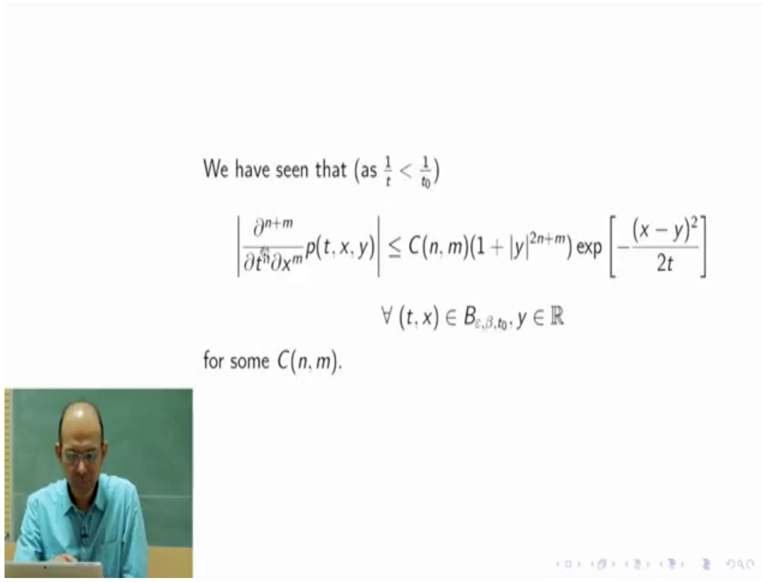

So 1 over t is less than 1 over t naught this is also the expression for p we already have obtained del p del t, del p del x, del 2 p del x 2 and we have seen that how the formula is coming like p time sum terms. And that if you have say nth order then you get sum to the power of n etc that that would appear and mth order with this is space also you are going to get this.

So, x minus y whole square so this power would increase, So, from that we have we know that this term this the partial derivative m plus nth, derivative the del t I mean, n del x m of this a is less than or equal to some constant times this growth variable 1 plus mod y to the 2 n plus m. Why 2 n appears here because we have seen that with del del t, del p del t, already we have obtained x minus y whole square.

However, for del p del x got just linear term and then second order we got second order term. So, from that by induction you know that we could get 2 n plus m that is the order, and however there is also one exponential term e to the power minus this term which decreases exponentially fast.

So this is the growth property what we are going to use and this is true for all t x in this thing for any y in R. And here this is very important that this constant we can choose to finite when my point t and x are from B epsilon beta, so I am writing t not also to specific the heat also represent t naught. Why is it so? Because you know this expressions is not only depend on y but also depends on x and also 1 over 2 t.

If t is very small then this means are very large. So, there is a reason that I had to take away from 0, so, t is taken from between t naught to t 1. And for this expression also we have this this t in the denominator. So, if I now consider only the B rectangle, rectangle open rectangle B there x is bounded and t is bounded away from 0.

So, this part we can actually you know the remaining the whatever that function is, I can write down as a constant times of this I can bound these by constant times this. This constant of course, depends on many things, it depends on the beta epsilon choice of t naught and therefore, you know t 1 also t 1 depends on epsilon a, of course and also n m, it depends on everything so this is constant.

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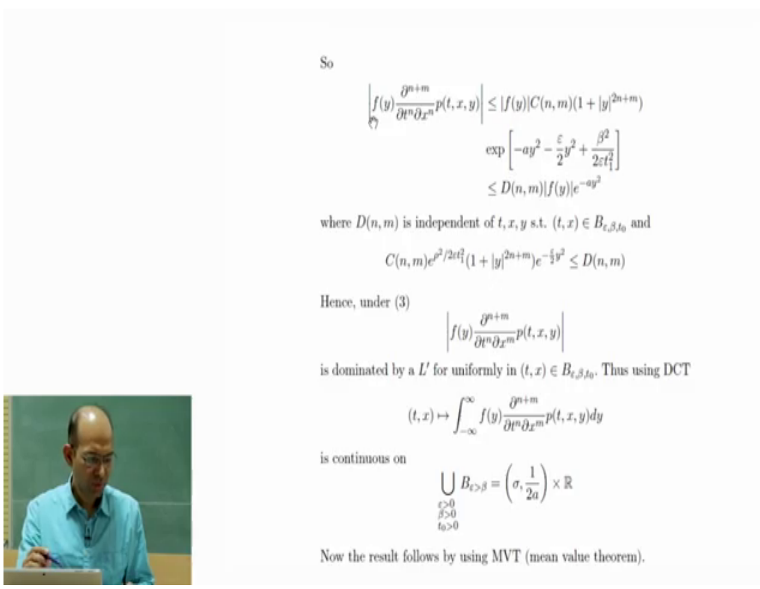

So, next what we see is that if we now have a f of y also multiplied with this, then we have mod of fy on the right hand side also, then this mod of f of y is here and here whatever e to the power of that expression was there in the earlier slide e to the power of minus x minus y whole square by 2 t of this thing. So, a y square minus x minus y whole square by 2 t so this thing is dominated by minus epsilon by 2 y square minus beta square by 2 epsilon t 1 square.

So, I did not had a y square there I just had this expression but of course, I can add and subtract a y square. So if we do this so here I have added and since I have added so, I got I could use this thing and then I have to subtract also so minus a y square. Now, this part is f of y into e to the power of minus a y square, but however I we have e to the power of minus epsilon by 2 y square and this term and we also had 1 plus mod y to the power 2 n plus m.

However, we know that since you know this is like inverted parabola, so this would as y increases this will decrease fast and this would start dominating this. So, this product of this polynomial term and this exponential decay term would be bounded between so to be bounded between 0 and some finite number.

So that I can multiple that bounded you know that upper bound I can multiple C n m to get a new coefficient D n m. So, this modulus of these things is less than or equal to D n m into mode of f of y into e to the power minus a y square.. So, where D n m is independent of the choice of t x y, such that t x is in B epsilon beta t naught. So here what do we have capital B we have used, so away from origin there, whatever t and x we choose from there we would always independent of the whatever tx, so this does not depend on t x because here C n m is that does not depend on variables t x.

So you would get D n m here. Now, there is a typo here. So this is beta square. So e to the power of beta square and other things are okay 2 epsilon t 1 square is fine. So, we considered this product C n m into e to the power of beta square divided by 2 epsilon t 1 square into 1 plus mod y to the power of 2 n plus m.. So, what appears here, Into e to the power of minus epsilon square by 2 y square, I mean this is actually basically saying that what is D n m. So this is D n m. So, I mean, because this would be dominated by the decay of these, so we can have this.

So now, we know that this term mod of f y, I mean this term, what is d n plus m this thing. Since it is less than or equal to some constant times mod of f y into e to the power a y square and since this is integrable, so I know that this part is dominated by one integrable function, this dominated by integrable function. So we can use dominated convergence theorem for passing to any limits etc here in this case.

So for example, if we want to take a sequence of x, xn converge to x etc so but that is dominated by this thing because here this would be a family of function of y which is dominated by an L 1 function and that family would be family parameter would be xn. So, so this whole family is dominated by this thing because the right inside does not depend on x.

So it is dominated by one L 1 it is not prime dash L 1 for uniformly in t and x. So I just talked about x but t also, you can do t 1, t 2 ,t n and a sequence which converges to t and there also you are going to get this. So, we can use that and therefore dominated convergence theorem and for appropriate choice of sequences to establish continuity of this map that t comma x is going to f of y, I mean del n plus m del t n del x m, p t x y dy.

So, we can establish continuity of this map using the dominated converges theorem. So, this is continuous, but where is it continuous? It is continuous for each and every choice of epsilon positive beta positive and t naught positive. So, whatever beta epsilon t naught we have initially we have chosen those things just to get this finite constant D n m and further we have obtained that this is continuous, but when you have obtain this continuity, then we got this property of this you know this map, this map is continuous on this you know ball B I mean this is also a typo I mean this is B naught greater B epsilon comma beta comma t naught.

So, this rectangle and then it is true that this map is continuous on this rectangle no matter what is your parameters, if you now decrease the parameter or possible parameter epsilon etc, so you are going to get larger and larger you know you know the rectangle and if you now take t naught as close to 0 so, you are going to cover the time t is equal to 0 part, if you beta as large as possible and then you are going to cover the whole area line and if you take epsilon also you know as small as possible, so then t 1 is defined as 1 over a plus epsilon.

So, as small as possible then you are going to be as close as 1 over 2 a because this number is little less than 1 over 2 a, 1 over 2 times a plus epsilon is little less than epsilon positive but if you take epsilon as small as possible then the upper bound of the time limit will be as close as 1 over 2 a, but of course, this time cannot extend to infinity.

I mean it is not going to there and we cannot assure it also. So, what we have observed that B epsilon beta this is t naught is that rectangle and if you take epsilon as close to 0, then 1 over 2 plus epsilon will become 1 over 2 a, 1 over 2 a plus epsilon becomes 1 over 2 a, and then union of these things is equal to 0 it is not sigma this is 0 comma 1 over 2 a and here the strip for x beta is as large as possible so, union of all possible things will be just R real line.

So, what we have obtained is that this map is continuous on this 0 to 1 over 2 a, time interval and the whole space R. So basically this function f of y integration of f of y p t x y d y that thing without the derivative is a solution of the heat equation because that we have seen that, this is the solution and the derivatives are like this.

I mean here one can I ask that where did I prove that the derivatives can go inside, basically this is like you know, like induction kind of thing. So, you put n is equal to 0, m equals to 0 then this is just this then also you get this is continuous, and then you put that n is equal to say it is like the 0 you put one. So, then show that again, this is differentiable and derivative. So, I mean, I mean, this is differentiable, so the derivatives also continuous you get and then one after another you proceed.

So this proves that the solution of the heat equation is smooth, it is infinitely differentiable. So that is end of the proof. Thank you.