Introduction to Probabilistic Methods in PDE
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Summary of Bounded Solution to the Dirichlet Problem

So, let us see what we have finished till now, we have addressed the Dirichlet problem on open domain and where boundary data is given and then we would like to solve the Laplacian of u is equal to 0 that equation where boundary data is also given. So, for that we have seen that we can find out stochastic representation, that means that we can write down the solution of the problem as a conditional expectation of random variable.

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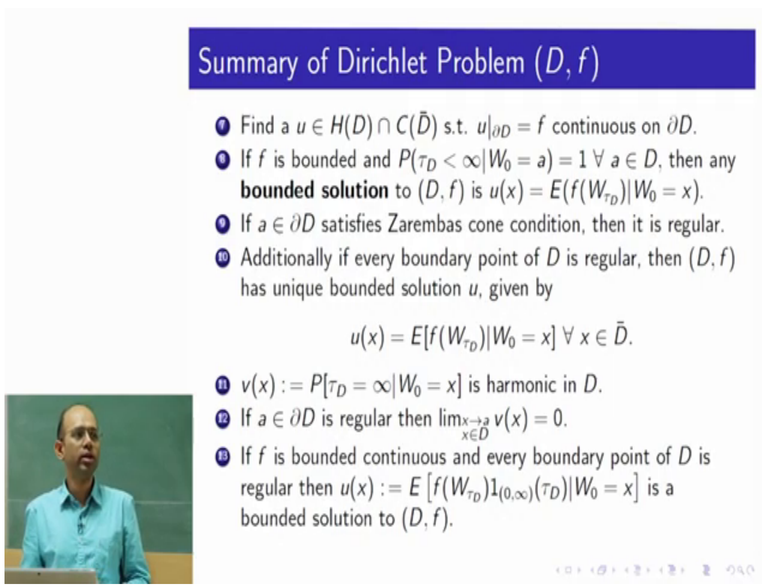

So, let us summarize whatever we have done yet, till now this is the statement of the Dirichlet problem that find a function u which is harmonic on the open domain D as well as continues on the closure, such that on the boundary del D is the boundary. So, on the boundary u satisfies the given condition that function f where f is a continuous function on the boundary. So, that is the Dirichlet problem.

For this problem what we have seen is that if f is bounded in addition to continuous as continuous and bounded and the probability that exit time is finite is one that means, exit time is finite, almost surely, then we can evaluate W tau D because this then this is meaningful if tau D is infinity with some positive probability, then this is W infinity which has which makes no sense.

So, we need of course, this condition to write down W subscript tau D that Brownian motion at time tau D. So, in this case, we can I mean any bounded solution to the Dirichlet problem D f is written as u of x is equal to expectation of f of W tau D given W 0 is equal to x. So, this is the result what we have seen earlier, but this does not imply that given any domain D and any bounded continuous function f if I mean, this function would be a solution of the given Dirichlet problem, it does not say that, it says that other way around it says that is the Dirichlet problem has a bounded solution, then that solution can be written in this way.

However, there could be an occasion that the Dirichlet problem does not have any bounded solution, it does not have any solution. So then also, this function may make sense, why? Because this makes sense because f is always bounded, so expectation is finite, and tau D is finite with property one, so this makes sense. So this function always makes sense, this function always exists. However, this is just a candidate solution therefore. So, this is a candidate for the solution, I mean, so this statement just says that, if the Dirichlet problem has a bounded solution, then the solution can be written this way.

This also says that the bounded solution is unique. Whenever we have this condition true, that this condition is purely depending on the structure of the domain, it does not depend on the boundary data. So, if this condition is true, then the Dirichlet problem has only unique bounded solution, I mean if the solution is there then it is unique, it does not talk about existence it talk about uniqueness.

However, this result although this result is in interesting, but this result does not help us to answer to the question whether given Dirichlet problem has any solution or not. So, we have discussed that that we have figured out that the scenarios when existence fails, that fails only when that the candidate solution is not continuous on the boundary. So, what we have done is that if a is in the boundary which satisfies Zarembas cone condition, this cone condition also depends purely on the nature of the domain.

So, then this a point is called regular and additionally that means if you have all these conditions true like f is bonded continuous and tau D is finite with probability one and also if boundary if every boundary point of D is regular, then this Dirichlet problem D f has a unique bounded solution u. So, this is talking about existence, then it has a unique solution u which is given by this conditional expectation.

So, that is a final result for these type of this for this discussion given a PDE problem we always try to find out one sufficient condition under which we can assure the existence of a solution and then you also ask that if it exists, whether there is a unique solution. So for existence is concerned, so we have a concrete answer for the Dirichlet problem.

So, we have seen the proof of this result and next the following result we have not proved, but let me state this because that would then you know help us to have a complete picture about Dirichlet problem, we have so, far talked about only the scenario where the exit time is finite with probability one, one can ask that what happens that if I do not have this condition true.

So, one would lose uniqueness. So, this is the precise statement here, if we consider a function v of x, which is defined as probability that tau D is equal to infinity given W 0 is equal to x, so if this is nonzero, if this is nonzero, that means, you know tau D is infinite some positive probability. So, then this would be a non trivial function. Nevertheless, this function would still remain harmonic is still harmonic.

So, like here we had ux defined this way that was harmonic function. And this function is also harmonic function is a very nice result and if a is point on the boundary such that it is a regular point, then limit of vx as extends to a, goes to 0 that is quite natural because you know if you are close to boundary and then you asking that, the exit time is infinity what is the probability, of course the probability should go to 0 because it is very likely that you would hit the boundary, if it is very close to the boundary.

However, a point is an irregular point we know that some examples like you know Lebesgue thorn. So there even if you are very close to the boundary, but still the probability that u would exist in a finite time might not decrease to 0 because it may leave away I mean the kind of scope it can explore in the near here and there it might not touch the boundary and can go away.

So there might be a positive probability that exit time is infinite. So far those cases you know, we cannot assure that these vx would go to 0 as you know you are approaching to the boundary. Nevertheless, if a is regular that means those kind of you know pathological scenarios are excluded, then you would have this result. So this is also a theorem which I am not proving I am just stating it.

Next, if f is just bounded continuous function, so as before, however, like you know the point eight is removed, the tau D defined at this condition is dropped just f is bounded continuous is retained and every boundary pointing of D is regular that is also retained from earlier. Then, this function u which is defined as conditional expectation of f of W tau D and indicator function that tau D is finite and this is actually generalization of the earlier one, even if you put this thing earlier one that will be exactly one almost surely because tau D is finite almost surely.

So, this is just generalization of the earlier one. So, this is a bounded solution to D f. So, this is the extension of this thing and this works. So, this is also a bounded solution to D f, but, are there any other solutions to D f? Of course, from twelve we can see that of course, there and many others, why? Because so if tau D is infinite with some positive probability that means v is non trivial function.

If v is non trivial function and harmonic in D, that means it satisfy is non trivial function satisfies the PDE, that Laplacian v is equal to 0 and here every point on boundary is regular, that means that vx converges to 0 on the boundary. So, this vx itself satisfies the Dirichlet problem D comma 0. So, v satisfies a Dirichlet problem D comma 0 because on the boundary it is taking value 0.

Then, if I add v with u if I add u plus v, u is as before so u satisfies this you know D f and v satisfy D 0.

So, u plus v would satisfy Df, why? Because, you know u plus u would still satisfy the PDE because the operator is linear. And on boundary condition v is adding nothing, so you already had the f value and v has 0 value so addition is f and this also says that does not matter you can actually take any multiple any scalar u plus lambda v, lambda is any real number.

So every bounded solution, so, this is actually a reverse statement. So, from the discussion 12 and 13 we can now conclude u plus lambda v is also solution of D f for any lambda and 14 is saying, furthermore, every bounded solution to D f is of this form if D f is any boundary solution, it is of this form. So, this much is like giving us a complete picture of Dirichlet problem, at least the bounded solution of Dirichlet problem.

It is saying that okay, this special case where exit time is finite for those case the solution exists and is unique and is given by the equation 10 that ux is equal to expectation of a f of f is the boundary data of W is Brownian motion and tau D it is exit time this, this way. And if one does not have this condition 8, that tau D is finite, then also, we can write down the solution, we can assert existence of solution, but you can assert uniqueness however, although we cannot address our needs, but we can classify all the solution, in that case we can write down all possible solutions, these are all of this form.

So this complete answer of the bounded solution of Dirichlet problem.