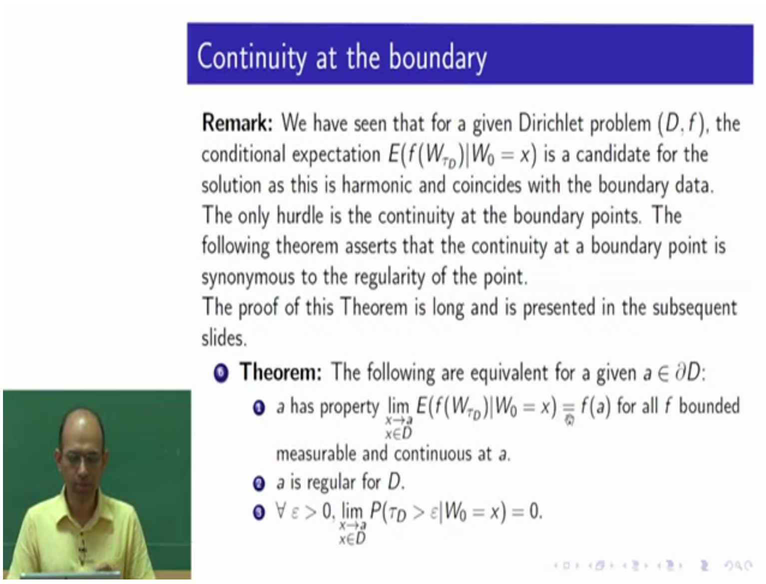
Introduction to Probabilistic Methods in PDE
Doctor Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research Pune
Lecture 27
Continuity of candidate solution at regular points - Part 2

We have already seen one part of the proof that the first statement implies the second statement. We are heading to prove equivalence of all these three statements or in other words we are going to prove that 1 implies 2 and 2 implies 3 and 3 implies 1.

(Refer Slide Time: 00:35)



And this is I mean why is this theorem important. So, let us again recall why are we doing this? Here it is, the first property is saying that the conditional expectation, this term has some continuity property at the boundary. And what is this conditional expectation? Why is it important? Because this is the candidate solution for the Dirichlet problem. We have seen that this conditional expectation is a function of x is harmonic and also it satisfies the boundary data.

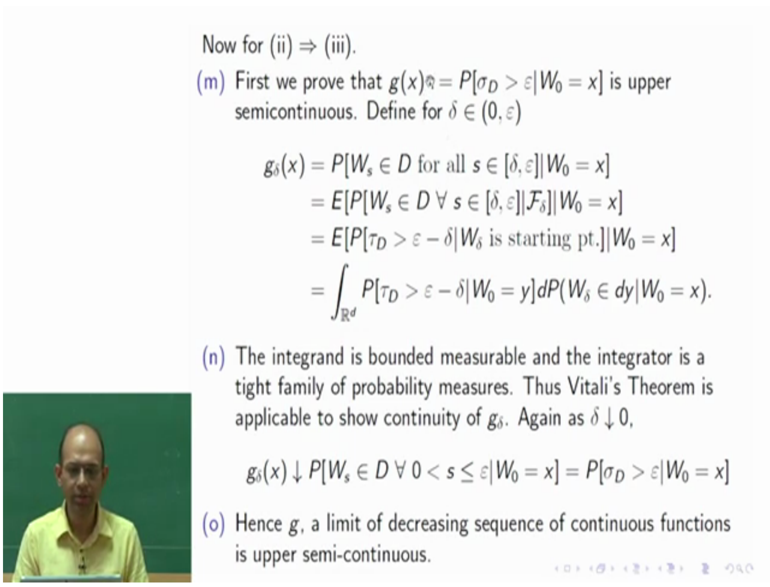
However, it is just a candidate solution because it is not assured that whether this solution can be continuously extended to the boundary. So, the main remaining you know missing puzzle piece is that showing that this is indeed continuous, but that might not be true all the time because we have seen that examples of Lebesgue Thorn, etc when the point is not a regular point we cannot assure this.

So, here this first I mean this you know this first statement of discontinuity implies second statement. So, that means that regularity of a point on the boundary is a necessary condition, but when we would be able to prove this equivalence, then this becomes a necessary and sufficient condition. So, if a point on the boundary is regular, then that function can be extended continuously to the boundary.

And if all the points on the boundary are regular, then you can do that, then actually this conditional expectation would give you the solution of the Dirichlet problem. That is the background. So, this is really most important part of this topic, because that would allow us to write down the solution of the Dirichlet problem in terms of conditional expectation of a function of Brownian motion.

And then that would give us another computational way to find out the solution. One can actually simulate Brownian motion and find out these conditional expectations. So, this would also give all computational technique also.

(Refer Slide Time: 03:00)



So, we have already seen these parts, that 1 implies 2, we have seen. Now, we would say that 2 implies 3. So, let us read 2 and 3 again. 2 is just the point is regular, a point a on the boundary is regular and third is saying that if you fix epsilon positive and ask that what is I mean ask the probability that exiting time is more than epsilon given that you are starting from point x and x is as close as possible to the boundary.

And of course, that exit time would become shorter and shorter but you have kept epsilon fixed. So this probability would go to 0. The most intuitive things, but that intuition works only if a is regular. So, we are going to see that if a is regular, then that implies this. We first consider function g of x which is conditional probability that sigma D, it is not tau d, sigma D is greater than epsilon given W0 is equal to x.

So, starting from x that sigma D is more than epsilon, what is the probability? That we are going to call as g of x. We would like to prove that g of x is an upper semi continuous function. Why? Because that property will be required to prove the 2 implies 3. So, to prove that g is an upper semi continuous function, what we are going to do is that we are going to write down g as limit of decreasing sequence of continuous function.

We know that decreasing sequence of continuous function, if that converges, that converges to the limit would be at least upper semi continuous. So, we do that here. So, define g delta x for any delta which is between 0 to epsilon. So, a small number, even smaller than epsilon, g delta of x. That we write down as conditional probability that the Brownian motion stays inside the domain for the whole duration between delta to epsilon and given W0 is equal to x.

So, this is basically you know, asking that the Brownian emotion stays inside the domain for the entire period which is like you know here say for example, sigma D is greater than epsilon that is saying that, exit time is you know more than epsilon positive number more than epsilon.

And here, this is also similar kind of, but it is a different event that inside the domain, at least between delta to epsilon. And W0 is equal to x. So here, this thing I can do using the tower property I can take condition with respect to a finer sigma algebra F delta, time delta. So, when I do that, so this is conditional probability and then expectation of these random variable, because this is now F delta measurable random available.

So, Ws is indeed for all s in between delta to epsilon given F delta. And then this conditional expectation of this random variable given W0 is equal to x. So, that we know that these 2 are equal using the tower property of conditional expectation. Now, we try to understand what does it mean. So, if I say that, first we observe that since W is a Markov process, so given F delta, so that conditional probability would be same as if I replace this by W delta.

So, just the knowledge of the Brownian motion will be sufficient, W delta. And then we observe that this event is determined by the path between delta to epsilon and W delta is given. So, W delta is given and delta to epsilon. So, basically we need to look at the path for this epsilon minus delta time length.

And then we can think that the Brownian motion is as if starting from W delta itself, one another Brownian motion you can imagine W delta itself. And then this is basically saying that, it does not exit the domain before this time interval that epsilon minus Delta, it does not exit. So, tau delta is tau D is more than epsilon minus delta given W delta is starting point.

So, these inside conditional probability is exactly same as this conditional probability. Now, I have written g delta in terms of tau delta. So, now we write down this expectation. When we do this expectation, we do in the following manner that W delta is having some particular value. Say y. So, here we write down 0. Why do we do that? Because it is starting point. Correct?

Since it a starting point, so now I mean it is like abuse of notation, you can because this tau D is actually you know exit time of the Brownian motion say W prime, which starts at W delta. And then we are not using W prime notation, we are abusing of notation, wrtiting that as W itself. So, we are writing W0 is equal to that, this point, W delta. So, and then this is, the whole thing is function of W delta.

So, this is just a function of W data. So, the expectation would be the function evaluated at y and then integrate with this law of W delta. So, that is how we find out the expectation. So, this probability tau D is greater than epsilon minus delta given W0 is equal to y, integrate with respect to the law of W delta. So, here this is conditional law because W0 is equal to x should always remain here.

Now, what we observe is that in this integration the integrant is just a probability. So, it is between 0 to 1. It is bounded. Correct? It is bounded measurable. Correct. This is measurable function of y. And then this integrant, so this integrator is a measure. But this measure is not a single measure, it is a family of measures because it depends on delta. It depends on delta, it depends on x also.

So, this family of measure has one nice property which we would require for the convergence that this is tight. So, this is a tight family of probability measures. What does it mean? That for Brownian motion here, if you take a compact set and then beyond this, I mean given epsilon, you can always find out compact set such that beyond this the probability measure of that would be less than epsilon for all the members in the family.

So, that is the thing. And that is true here because here delta is between 0 to epsilon, you are not changing delta much. You are not changing delta, what does it mean? That you are not changing variants much. So, that means the family of you know, normal random variables, that there that family has bounded variance. And x is also like, you know, we are going to take if x tends to 0.

So, it is in the neighbourhood of 0. So, that means the mean is also bounded. So, we are using I mean, I mean, this is tight families coming due to the nature of the family of normal random variable. If you have family of normal random variables, and that is coming from, with bounded mean and bounded variance, then corresponding family of the distributions is tight. So, we are using that property.

So, this is tight. So, now we can use Vitali's theorem. We can use Vitali's theorem because this is tight and this is bounded measurable, to take theorem, Delta tends to 0. So, if delta tends to 0 so then that limit can go inside, using Vitali's theorem. Does Vitali's theorem is applicable to show continuous g Delta? Because, you know, if we put Delta tends to 0, what we are going to get?

I mean this would become P Tau Delta greater than epsilon this thing. So, here I am writing here, the g delta is going to this Ws belongs to D for all s between 0 to I mean open 0 to epsilon closed because here delta, this becomes 0. And so the tau Delta is more than epsilon. So, basically saying that the Brownian motion should stay inside the domain at epsilon and less than epsilon.

And here, this W Delta we have is, so this delta is going to 0, so we cannot assure that equal to 0, but more than 0, we can assure, on the right-hand side of 0. So, the Ws is in D for 0 to, open 0 to close epsilon, because you know that anyway we have actually started from delta to epsilon, and that is the main thing.

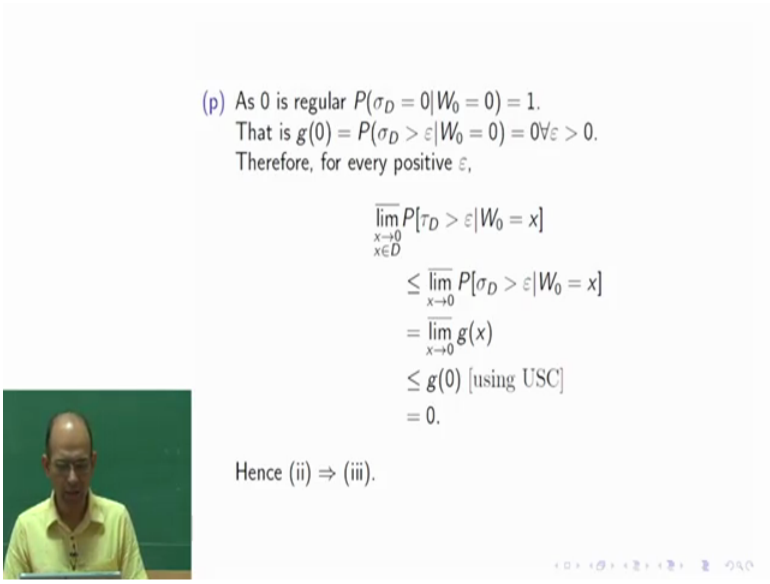
So, as delta tends to 0 we are going to get I mean, actually, this you can see from, directly from here. See here. So, as delta goes to 0, you get open 0 epsilon closed here. So, this part would leave us here, the g delta x converges to this. And what is this? This is basically you know, leaving 0 aside and then talking about this, so this is not tau delta but this is sigma delta.

Sigma Delta is positive. So, this conditional probability is conditional probability this Sigma delta is more than epsilon given W0 is equal to x, but this probability is nothing but gx. So, gx is defined this way. So, g delta decreases. Why does it decrease? It decreases because whenever I am decreasing Delta that means this interval is becoming larger and larger.

And here this event is putting condition on Brownian motion path on the interval this. And then if you increase the length of interval that means you are putting more and more conditions. So, the event is becoming smaller and smaller, so probability is becoming smaller and smaller and g delta decreases as delta goes to 0. So, this decreases and g delta is so from here as I told that, it is Vitali's theorem can be applied to show that this is continuous function and then we know that this is converging to gx.

I think I did not just say it quite completely that I mean when we use Vitali's theorem we change x also, I mean to show that is continuous in x, then that limit goes inside because I did not talk much about x, but that is important because that would give me that g delta is also continuous. Since g is the limit of a decreasing sequence of continuous functions, therefore g is upper semi continuous. So, now when we have obtained that this function is upper semi continuous, we can use the property of upper semi continuous function.

(Refer Slide Time: 14:52)



So, that we are doing it in here. That, namely that when we take lim sup x tends to 0 gx, that would be less than or equal to g0.So, that is a property of upper semi continuous function, so we are going to use here. So, let us see how do you use here. So, as 0 is regular the regular point, so we know that from the definition of regular point, the sigma D is equal to 0, that probability would be 1, that is a regular point.

And that implies that g of 0 which is sigma D greater than 0 given this thing, where epsilon is fixed. I mean I am not writing g epsilon because although it depends on epsilon, I am fixing epsilon something. So, I am not to repeatedly using it but of course, g depends on epsilon but epsilon we are not changing.

So, now this thing would be 0 because sigma D is equal to 0 with probability 1, so it is strictly more than epsilon. That would be having probability 0. So, that is true for all epsilon positive. So, does not matter whatever epsilon we choose, g of 0 is 0. So, any positive epsilon we choose, g of 0 is 0. Now, for every positive epsilon again we are fixing epsilon.

And then lim sup of conditional probability that the hitting time is more than epsilon, that we are counting here. And then that is less than or I mean basically you know the W0 is equal to x, and you are saying that tau D is more than epsilon. And if I replaced tau D by sigma D, I know the sigma D is always greater or equals to tau D.

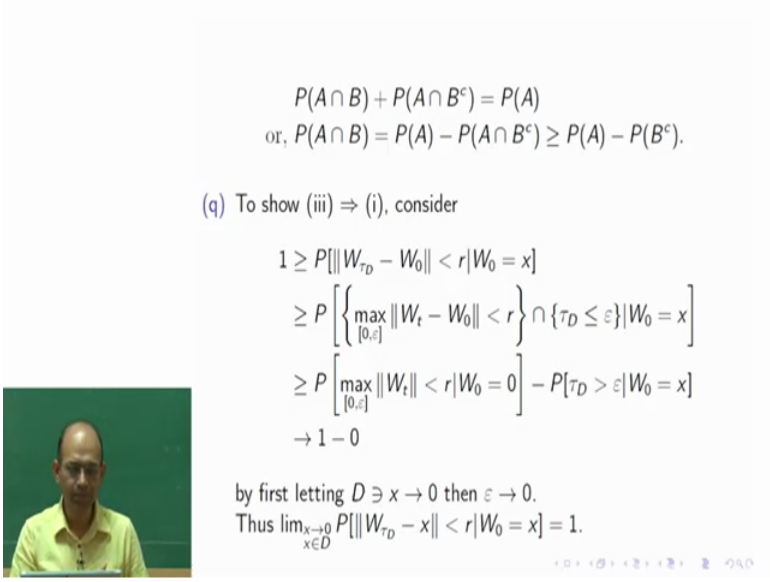
So, that would be more than I mean this implies this So, since this implies this, so this is a superset. So, I would get that less than or equals to sign here, so here x belongs to D is all over the places, but I am not writing here. So, we get this probability, but this is exactly gx, this is a definition of gx, so lim sup gx and then we are using the upper semi continuity property of G to get this is results to g0.

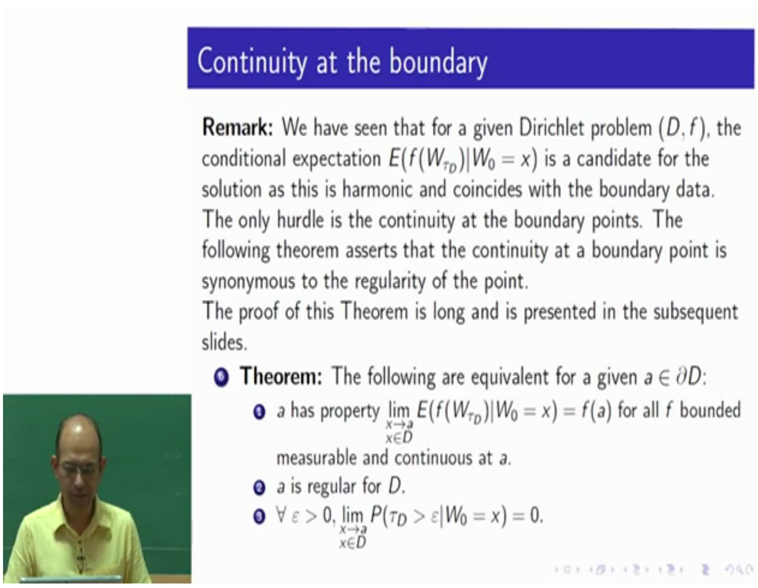
But g0 is argued here that it is 0 for any epsilon whatever you choose. So, we got that this lim sup tau D greater than epsilon, this probability goes to 0. That was precisely the statement in 3. So, that is proved here. So, I mean this proof part is you know, a little roundabout way because actually just to you know, passing to the limit and arguing that this goes to 0 we had to cook up this you know, function and then using this property of the function and to achieve that we had to construct g delta.

Otherwise, basically this is the main part. So, 1 implies 2, 2 implies 3 is done. So, the whole proof would be complete if we can establish that 3 implies 1. So, or in other words that this limit you know this tau delta greater than epsilon goes to 0. Actually in that I do not have lim sup. But here we are using that because you know if lim sup is you know 0 then everything would be, even if it exists that would also be 0.

So, assuming this we have to prove that the continuity of the conditional expectation of f of this thing where f is any bounded measurable functions continuous at 0. So, that is you know, for any such functions, if there is a large class of objects.

(Refer Slide Time: 18:44)





So, here we actually require that conditional probability is you know, this inequality which we have already shown in the last lecture. So, since I need it here in this space, so I just pasted it here. To show 3 employs 1. So, it has only 2 slides. 3 implies 1. What we do? We consider this conditional probability that W tau D minus W0 is less than r, given W0 is equal to x.

So, here what are we doing? We are doing that, wherever W0 is inside the domain, if x is inside the domain. I am not writing every time but x is always inside the domain. So, because if x is upside the domain then this will be exactly 0, correct? W0 and this would be exactly the tau D would be immediately same as this.

So, inside a domain and then consider the exit time of W Tao D, wherever it hits the boundary and you take the norm, the difference? And that is less than r? Of course, I mean, this is just a straightforward because this is a probability after all. So, it is less than or equals to 1, but this probability would be significant or I mean non trivial only if you are very close to the boundary because you know, otherwise it would not be in the, you know ball of radius r, inside that.

So, what do we do is that we now consider intersection of another event. So, tau D minus W0 is less than r. So, basically that is saying that that wherever it hits the boundary, the distance of that point from the initial point is less than r. So, instead of that what we are doing here? Another smaller event there we have there that Wt and W naught, so the maximum distance, that is less than r during epsilon interval.

So, then here it is a random time, but we have taken a fixed time epsilon. During this small interval 0 to epsilon, W to t never exits the r radius ball of W0. So, that is this event. So, of course this is a smaller event and we also say that tau D is less than or equals to epsilon. So, tau D is less than, otherwise this is not smaller, the intersection of these 2.

The tau D is less than or equal to epsilon. And during 0 to epsilon Wt never exits r ball. So, this implies this of course. So, this is a smaller event. So, we get greater or equals to sign here, even W0 is equal to x. Now, A intersection B. So, P of A intersection B is greater or equals to P of A minus P of B complement. So, we are using that.

So, P of this event we are writing here. So, here while writing, W0 we are taking as 0 here because you know, it does not matter because for so far this event is concerned. So, I mean, Wt minus W0 less than r given W0 is equal to x. If I replace x by 0, and then this W0 also be 0, and then this norm of Wt less than r we would ask, it has the same probability. Why?

Because, you know, like x would be the starting point and which is like, you are starting a new and you are calling that as the origin. And then I mean, you cannot do this kind of, I mean changing when say tau D is concerned. Why? Because when you change actually, you become possibly closer to the boundary or further to boundary, all the probability would change.

But here we do not have tau D. I mean, we have separated tau D basically. So, this is fixed time interval. So, this is fixed time interval. So, it is just saying that, if one starts from point x, you know the Brownian motion starts from point x and Wt minus W0 is you know and then Wt is within the r radius ball, what is the probability of that?

It is the same that of you know instead of x, you start from 0 and you consider a ball of radius r around 0 instead of around x. So, this event, the probability of this event conditional probability would be same as the conditional probability of this event. Therefore W0 is equal to 0. And then B complement. So, this is B. B complement, this complement, tau D is strictly greater than epsilon that is a complement.

Now, we see that we have fixed epsilon and we take x tends to 0. So, here when you take x tends to 0, then we actually are talking about the third condition that when, epsilon is fixed probability of tau D greater than epsilon given W0 is equal to x and limit x tends to 0 that is 0. That is the third condition, is third statement.

So, we are using this third statement. So, here I mean this is the place where I am using third, 3. 3 implies 1, correct. We have to use 3 somewhere, some point. So, here we are using that that this part goes to 0 as we take x tends to 0, but this does not depend on x. So, this remains unchanged and then we take epsilon tends to 0.

So, first x tends to 0, this part would become 0 and then this part would remain and then we are going to take epsilon tends to 0. Now, this of course depends on epsilon but this 0 does not depend on epsilon. But this how does it behave when epsilon goes to 0? This is saying that you are starting your Brownian motion from the origin and asking that between 0 to epsilon time, the maximum distance what the Brownian motion has travelled and that is less than r.

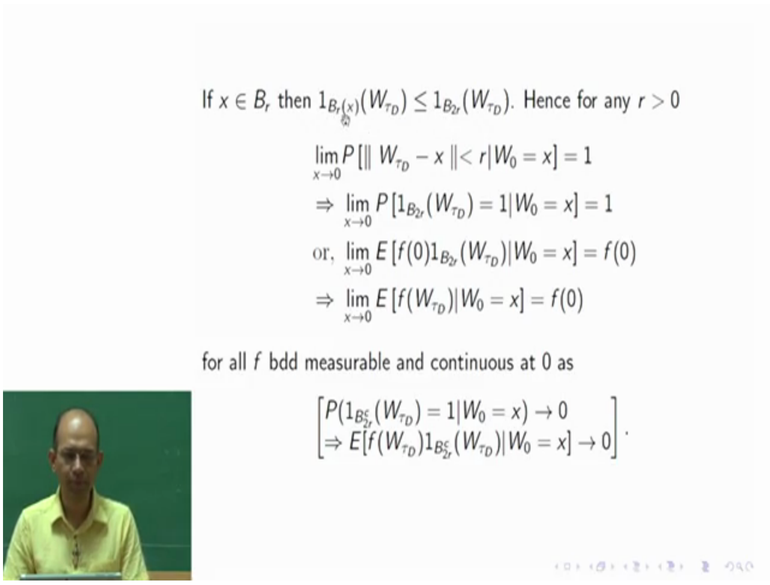
That probability I mean, I mean Wt we know is a non-normal random variable but it is maximum over all possible these things. So, we know that how Brownian motion behaves. Between 0 to epsilon, if you take the maximum of that, so that probability one can actually find out using that reflection principle.

So, that probability would converge to 0, because this would be I mean, beyond the ball of radius r, that probability would go to 0. On the other hand, this should remain in the ball of radius r that probability would become 1, r I am not changing, r is fixed. So, that would converge to 1.

This is true for all positive r. So, by first letting x tends to 0 and then epsilon tends to 0, we are going to get that. So, here in this probability, where r is fixed and x tends to 0, so this is between 1 and 1. So, this limit, so this will, this is 1, so this limit goes to 1. So, limit W tau D minus, instead of writing W0 I am writing x here because it was given that W0 is equal to x.

So, probability that W tau D minus x is less than r given W0 is equal to x is equal to 1. So, it is not yet, the proof is not yet done. So, we have just obtained this from this tau D. Just from this you know, this property that this goes to 0 we have obtained this result. Now, from here we need to actually get the statement of 1, the first statement. The first statement is that for all bounded measurable function, f which is continuous at 0, for that it is true. That expectation, conditional expectation is continuous.

(Refer Slide Time: 27:00)



So, how to go for that? So, here from probability to go to the expectation of a function, so we understand that we need to you know, use some major theoretical approach, some extension kind of things. So, what we do is that we take some ball of radius r. So, if x is in the ball of, so radius r around 0, so I mean whenever I do not write around 0 because it is default for my notations that ball is like, if I do not write down the centre that means 0.

Then so first we understand what is this. Br of x is the ball of radius r around x. And B2r is ball of radius 2r around 0. Since x is in Br, so x is less than r distance r from origin, and around x, I have another ball of radius r. So, this ball would be inside the ball of radius 2r around 0.

So, this function is larger than this function. So, it does not matter what I evaluate where I evaluate on. So, we evaluate at W tau D. So, this is immaterial, but this function is more than this function. So, we have this inequality. Hence for any r positive, so what we do is that we are using a, we are writing what is written here, the last line actually and same thing I am writing.

x tends to 0 probability that normal W tau D minus x norm of that is less than r given W0 is equal to x is equal to 1. So, this would imply that, so here r is there, so what do we do is that this I mean this part is same as this part correct? Because W tau d minus x is less than r. Or in other words you are saying that W tau D is in the ball of radius r around x or in other words you are just saying that this value is equal to 1.

So, this value is equal to 1, if this value is equal to 1 of course this value would also be 1 because this is cannot be smaller than this value. So, this event implies that 1B2r of W tau D is equal to 1. So, this is actually larger events, so if this converges to 1, so this would also converge to 1. So, limit x tends to 0, 1B2r W tau D is equal to 1 given W0 is equal to x is equal to 1.

So, this is the way actually you know, to go to the general function first we talk about the simple function, correct? So, here we are doing that. So now, what we do is that probability of a set is same as expectation of indicator function. So, expectation of for the time being, you forgot about f0, expectation of indicator function of this. Indicator function, 1 of B2r of W tau D, because this is the event. I mean this is equal to 1 that means that W tau D happens.

So, probability of W tau D happens is same as you know I mean happens means, is inside B2r. So, that is like a conditional expectation of 1B2r W tau D given W0 is equal to x but this is 1. But then you multiply f of 0 both sides. So, you get f of 0 is equals to this. So, now we are going to use the continuity property of f at 0 and also boundedness of f on the boundary, bounded measurability.

Because when f is a bounded measurable function here, so we can approximate the integration by some another function which is exactly same as f except a small, you know ball around 0 and there it is taking value of f of 0. And since f is continuous at 0, so given epsilon there is a neighbourhood on 0 such that f of x minus f of 0 would be as small as epsilon for every point in that neighbourhood.

And that neighbourhood, the 2r would serve the neighbourhood for me. So, I can and the probability measure is finite there. So, I can actually you know for integration of the whole function, I can actually write down, so expectation of f W tau D, I can write down it in 2 parts. One is an approximation around in that you know, neighbourhood of 0. There I am replacing f of actual function by f of 0.

And by doing so, I have one error bound, and outside, I am keeping the same function and if I do that, so let us see this thing that from here that this is 1, this limit. So, that means that in the complement this would be 0 with probability ability 1. So, in the compliment this is 1 is you know, that probability would be 0, the compliment.

So, if I even multiply a bounded function here f of this you know the W tau D with this indicator function, I would still again get the expectation to be 0 because here the domain here I mean this product, it is multiplied with something and this is bounded and this is multiplied and here this probability is going to 0.

So, this random variable is 1, that probability goes to 0. That means it is 0 almost surely. It is converging to 0 almost surely. So, this would go to 0. So, this is the remaining part basically on the compliment of the ball of the 2r. So, there, this expectation, this part would go to 0. And inside this, this is the scenario and f is continuous at 0. So, with all these things together, we get this result that limit x tends to 0 expectation of f of W tau D, given W0 is equal to x is equal to f of 0. So, this is the continuity property. This is the first step. Thank you.