**Introduction to Probabilistic Methods in PDE**
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**Lecture 26**
**Continuity of candidate solution at regular boundary points - Part 1**

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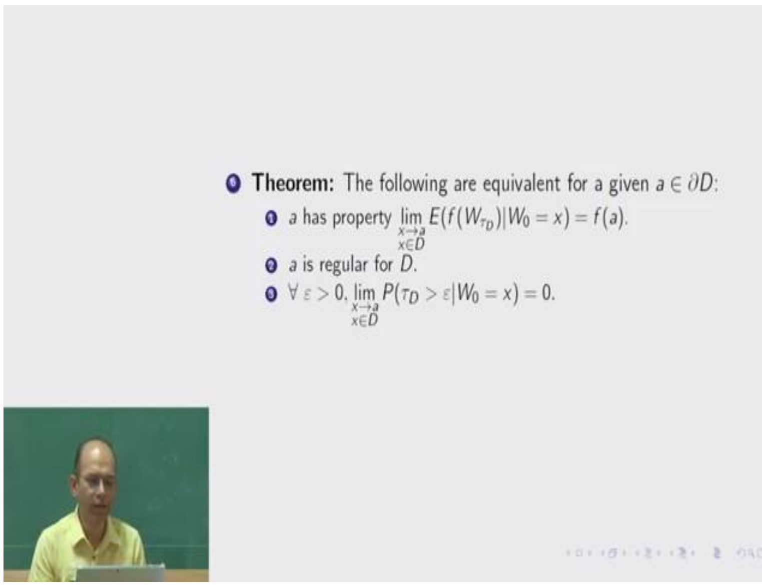


Next, this theorem we have not stated earlier, possibly we have stated but we have not proved it. This theorem is saying that the following statements are equivalent. So, here a point a, which has this property, this continuity property at the boundary is a point in on the boundary.

And if you have this continuity property, so this left-hand side is part of let solution of Dirichlet problem. The solution former form we cannot accept that as a solution because it is valid only inside the boundary, inside the domain. So, this is the continuity property at the boundary.

And the second is a is a regular point on D, for D. And third part is that that if you have a fixed epsilon then stopping the exiting time is more than epsilon, the probability decreases as your initial point approaches to boundary. So, we are going to prove today only part 1 to 2, where that 1 implies 2.

So, what is there? a has property, this continuity property and we have to prove that a is regular for D. We are going to take the counter positive method, proof by contradiction. Which consider that a is irregular for and then we have to get contradiction here. So, we have to get that this is not equal, so that is our goal. And that you need to get for any function f which is bounded and measurable and continuous at point a. So, I have not written here the nature of f, but I think I have written in earlier slides. In this slide or maybe in earlier slide.

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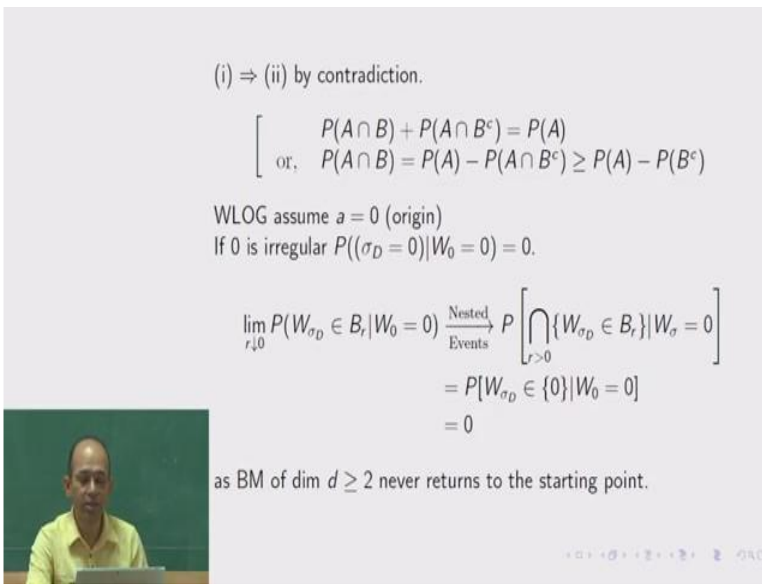


So, the first part is that we use this inequality that probability A intersection B is greater equals to probability of A minus probability of B complement. So, the division is noted here. It is the simple result that A intersection B and A intersection B complement, these two are disjoint events. So, sum of the probability is same as the probability of the union, so the probability of the union is a, probability of A.

And then probability of A intersection B is equal to P of A, so this part is on the right-hand side now, subtraction sign, minus sign, A intersection B complement. So, this part we are replacing by a larger set, but negative set so we are going to get the lesser, greater or equals to sign, probability of A minus probability of B complement. We are going to use that somewhere here.

So, we as I have mentioned, that we are going to take the proof by contradiction approach. So, we consider 0 is irregular, so without loss of generality we assume the point a is 0 here, origin. And we assume that 0 is irregular. What does it mean? The meaning is that if you start Brownian motion from 0, so W0 is equal to 0, then given that the probability that positive existing time is 0 that has probability 0. So, for regular point this is 1.

So, now what do we do? We consider this probability conditional probability that probability of W tau sigma D belongs to Br. What is Br? Br is a ball of radius r arround 0, given W0 is equal to 0. Why are we talking about this? Because using this help we can actually come to this point.

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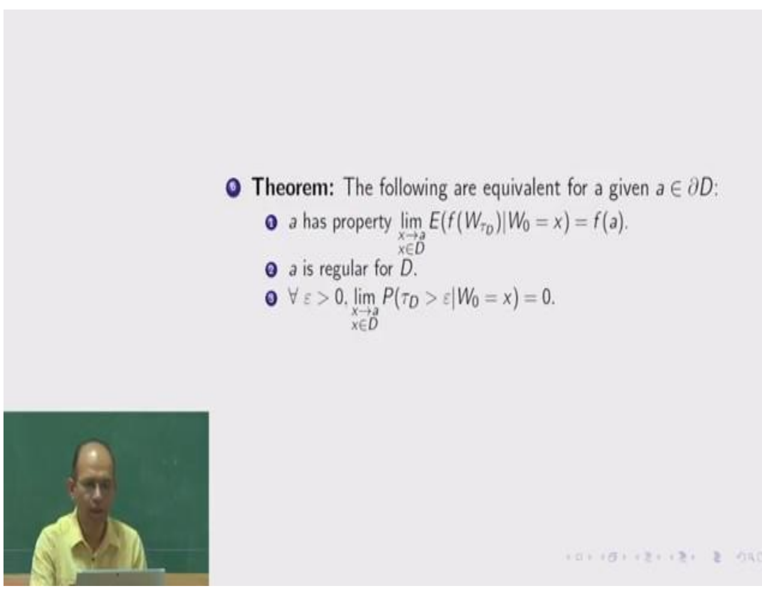


So here, we basically want to prove this correct, prove that this is not true. We need to prove this expectation of this true about W tau D and we need to do this.

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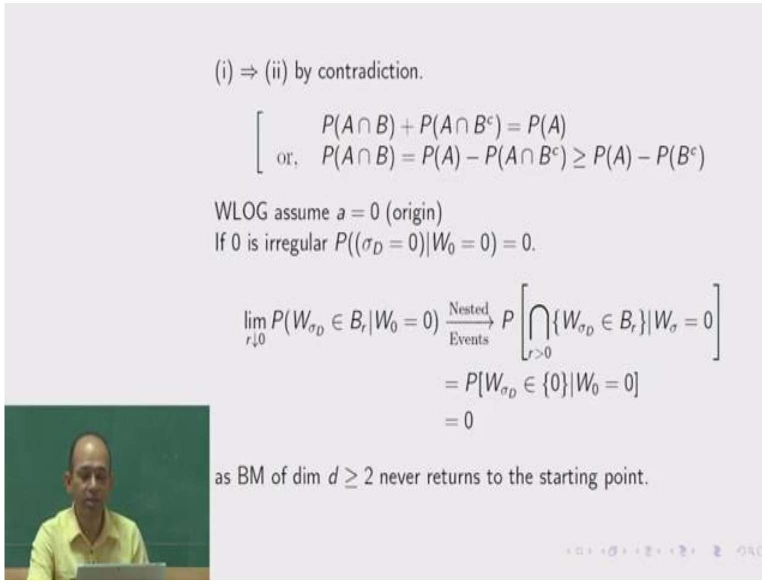


So, we starts from here. Probability that W sigma D belongs to the ball of radius r when W0 is equal to 0. So, here imagine that ball of radius r and then r you have several different values, decreasing values. Then, so if you say that sigma D is a number that when it leaves a domain, if your r is smaller then the ball is smaller, then this event might not remain true.

But however, if it is true for some particular r, for larger r also this is true. So, as r is decreasing, this is actually nested decreasing events. So, when we have such type of limit, so nested decreasing events, the limit, r tends to 0 limit of this type of family of events is exactly equals to probability of intersection of these events.

Because when, we have nested events. That is true actually, not only for probability measure but any finite measure. But for infinite measure, it need not be true, but for any other finite measure also this is true. So, this is probability of intersection, r positive, so here r is going to 0, positive of the same event. The W sigma D is in Br given, so there is a typo, this is W0. W0 is equal to 0. So, this is corrected here.

So, now what is this intersection that W sigma D Brownian motion at the time of this sigma D stopping time is in the ball of radius r and where this is in r for all positive r.

That means that this is exactly 0, because 0 is the only one single point so that these things would be satisfied. So, W sigma D is in the singleton 0, basically this is 0.

But what is the probability of this event? We note that in Brownian motion for dimension d, dimension more than or equals to 2, the Brownian motion starts from a particular point, then there is a 0 probability then you return to that same point. However, this is not true for one dimension point, because one dimension is like going right or left.

So, when it goes right, there is a probability also that it would come to left and it would go to the negative side only by crossing 0. So, if you start from 0, it would always cross 0. But in two-dimension it is not the case. The probability that another time, future time it would be again hit 0 and this sigma D is defined using the boundary of the domain.

So, that probability is 0, so this is 0. So, what is the point? The point is that this limit is 0. So, that means this condition probability could be as small as possible. So, if I choose sufficiently small r, I can make this probability smaller than any predetermined number good.

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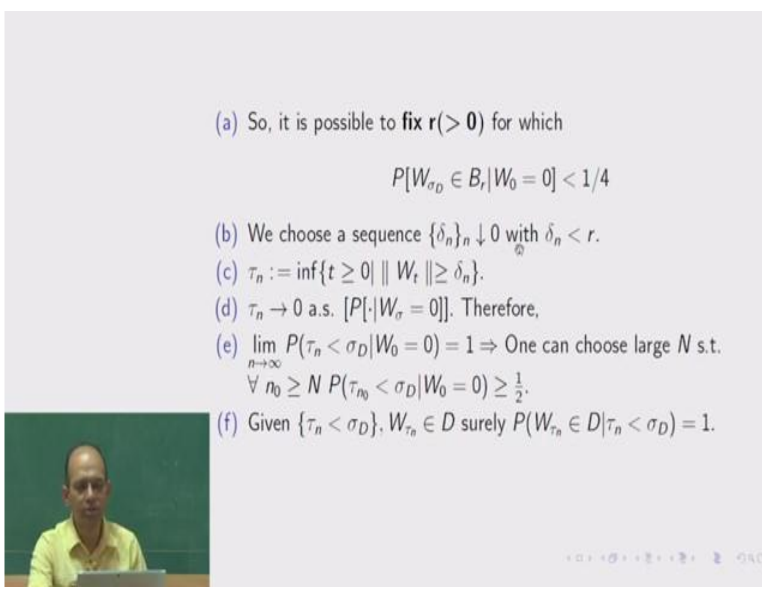


So we started with, say, 1 4 th is a small number, not too small than 1 4 th. And we fix r positive such that this thing is less than 1 4 th. Any of this thing goes to 0 as r tends to 0 but we choose r, a particular r such that this is less than 1 4 th. Why 1 4 th? That would be clear that it would appear in the steps of the proof.

We choose a sequence of delta n. Delta n is decreasing to 0 and we keep delta n smaller than r. Then we define the time when the Brownian motion leaves the ball of delta n. This time would decrease as n increases, because delta n goes to 0. So, delta n will be smaller and smaller boundary and the time for Brownian motion to take, the time the Brownian motion will take to leave the ball would be shorter and shorter. So, sooner the Brownian motion would leave I mean as delta n, it would be the smaller.

So, that is my tau n, so tau n goes to 0 almost surely. So, tau n is the stopping time that would go to 0 almost surely. So, we are constructing several different you know random variables and sequences for our proof just because it is required. So, it is unavoidable, so tau n converges to 0 almost surely, almost surely with respect to the this condition probability that again, there is a typo, this is W0, so given W0, so conditional probability. With Respect to this conditional probability, this tau n goes to 0.

This is clear. Now, we ask this question that what is the probability that tau n is less than sigma D, given W0 is 0? For a particular n, this probability could be some number which need not be 1 it need not be a sure event. However, as tau n goes to 0, so for a large n because sigma D does not depend on n. So, whatever is the realization value of this, for every realization there would be a large n such that tau n would be even smaller than sigma D, whatever the value of sigma D.

So, as n tends to infinity for each and every sample point, it is that we are going to say that will be, go to 1. This probability would converse to 1. It would become sure event that it would of course be smaller than sigma D. So, we get probability that tau n less than sigma D given W0 is 0 is equal to 1.

Now, since it is converging to 1, so we can make this thing as large as possible like less than 1, but any number which is less than 1 it can be you know above that. We can make it, we can find out that such n. So, here what we do? We choose one large capital N it. So, see here, we have fixed one r and that we have not changed here.

And here we are fixing again another thing that is capital N. And now for all n naught, small n naught which is equal or more than capital N, the probability of tau n naught less than sigma D is more than a half. So, we are taking the threshold half. So, since this thing is converting to 1, increasing to a 1, I mean it would of course overcome half threshold.

So, capital N is such kind of such number and this is deterministic number, because this is a deterministic sequence. When you find a probability it is the sequence of deterministic numbers. So, we find that, so we fix that N such capital N such that for all n naught greater than or equals to N this probability is greater or equals to half.

Now, what we do? We consider this sequence of events that tau n is less than sigma D for that probability what we have obtained here. And also the random variable, W tau n, tau n is a random stopping time, W is a Brownian motion. So, the value of W at the time of tau n. So, let us recall what is the tau n. Tau n depends on R, the delta n and delta n converges to 0. So, it is like you know hitting time off the ball of radius delta n.

So, given this and this, the probability that tau n belongs to D, where tau n is less than sigma D. So, let us try to understand why is it 1. So, when tau n is less than sigma D, so that means the time tau n still sigma D is still larger that means the Brownian motion did not leave the domain.

Tau n is smaller than sigma D, since it did not leave the domain that means it is inside the domain. So, when tau n is less than sigma D, then if we ask the what is the probability that W tau N is in D, of course it is a sure event it is 1. So, probability that W tau n belongs to D given sigma D is 1. So, here there is no limit in this.

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$$\frac{1}{4} > P(W_{\sigma_D} \in B_r | W_0 = 0)$$

$$\geq P(W_{\sigma_D} \in B_r, \tau_{n_0} < \sigma_D | W_0 = 0) \text{ where } n_0 \geq N$$

$$= E\left[E\left[1_{\{\tau_{n_0} < \sigma_D\}} 1_{\{W_{\sigma_D} \in B_r\}} | \mathcal{F}_{\tau_{n_0}}\right] | W_0 = 0\right]$$

$$= E\left[1_{\{\tau_{n_0} < \sigma_D\}} P(W_{\sigma_D} \in B_r | \mathcal{F}_{\tau_{n_0}} | W_0 = 0)\right]$$

$$= \int_{D \cap \partial B_{\delta_{n_0}}(0)} P(W_{\tau_D} \in B_r | W_{\tau_{n_0}} = x)$$
$$P(W_{\tau_{n_0}} \in dx | \tau_{n_0} < \sigma_D) P(\tau_{n_0} < \sigma_D)$$

$$\geq \frac{1}{2} \int_{D \cap d B_{\delta_{n_0}}(0)} P(W_{\tau_D} \in B_r | W_0 = x) P(W_{\tau_{n_r}} \in dx | \tau_{n_0} < \sigma_D)$$

$$\geq \frac{1}{2} \inf_{x \in D \cap \partial B_{\delta_{n_0}}} P(W_{\tau_D} \in B_r | W_0 = x) P(W_{\tau_n} \in D \cap \partial B_{\delta_{n_0}} | \tau_{n_0} < \sigma_D)$$

(a) So, it is possible to **fix** $\mathbf{r}(>\mathbf{0})$ for which

$$P[W_{\sigma_D} \in B_r | W_0 = 0] < 1/4$$

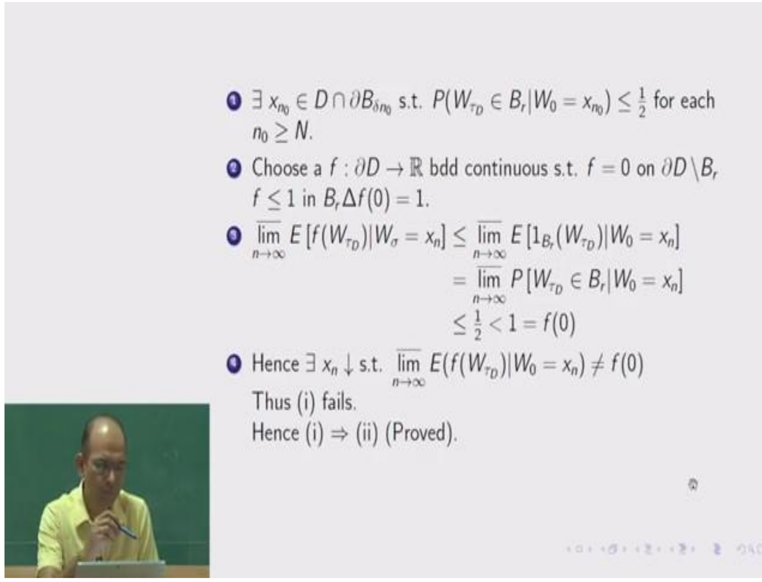
(b) We choose a sequence $\{\delta_n\}_n \downarrow 0$ with $\delta_n < r$.

(c) $\tau_n := \inf\{t \geq 0 | \parallel W_t \parallel \geq \delta_n\}$.

(d) $\tau_n \to 0$ a.s. $[P[\cdot | W_\sigma = 0]]$. Therefore,

(e) $\lim_{n \to \infty} P(\tau_n < \sigma_D | W_0 = 0) = 1 \Rightarrow$ One can choose large $N$ s.t. $\forall\, n_0 \geq N$ $P(\tau_{n_0} < \sigma_D | W_0 = 0) \geq \frac{1}{2}$.

(f) Given $\{\tau_n < \sigma_D\}$, $W_{\tau_n} \in D$ surely $P(W_{\tau_n} \in D | \tau_n < \sigma_D) = 1$.




Now we use this 1 4 th number, you remember that 1 4 th is greater than this probability, W sigma D is in the ball of, open ball of radius r. The conditional probability of this event given W0 is equal to 0. So, now we compute further. What we do is that here we are writing one subset of this, so sub event.

So, we are going to get greater or equals to sign here. What is this sub event? That this is as it is but we are asking even the intersection with this comma stands for intersection. So, this event and also this event is also true. Whenever asked that two events simultaneously true then it is intersection. And then of course this is a subset of that, so we are going to get greater or equals to sign here. So, here the event is tau n naught is less than sigma D. So, where n naught is greater or equals to capital N.

So, as such for defining this tau n, we did not care about whether it is outside the domain or inside the domain, nothing. But, so this is not a sure event or something, this is just an event. So, tau n naught is also we are requiring that is less than sigma D. So, this probability, we can write down as an expectation also.

This probability of a is nothing but expectation of indicator function of a. So, we write down that way. So, this event W sigma D belongs to Br. We write down W sigma D belongs to Br indicator function. So, actually better notation is 1 Br and then of W sigma D, because if sigma D is in Br, then it is 1 other wise 0.

And here it is that this event tau n naught is not less than sigma D. So, I have these two, intersection is like product of these two indicator functions expectation. But instead of one expectation, we are taking two expectations. Why? Because this is the probability I need to find given W0 is equal to 0.

However, we are using tower properties of expectation. We are conditioned even further inside with a finer sigma algebra in between that here the sigma algebra is at time 0, but here at tau n naught. So, n tau n naught would be 0 or positive. So, it is finer than this sigma algebra. So, we are getting expectation of expectation of this thing given F tau n naught, given this.

Now, what we do? We observe that this quantity, this quantity is F tau n naught measurable. Why? Because to decide this, you do not need to wait till sigma D, you just need to wait till tau n naught. If you run the Brownian motion simulation or Brownian motion clock, you run the clock and observe the Brownian motion and stop at the time of tau n naught and then wherever you have observe that will be sufficient for answering this question whether this is true or not.

Because by that time you know that whether Brownian motion has layered the boundary or not, whether that touched the you know ball or not, everything. So, that would be clear, so you would be able to say that. So, this is miserable with respect to F tau n naught and therefore conditional expectation of this event given F tau n naught is same as this random variable which is F tau n naught measurable comes out of the expectation.

Then you have only expectation of this random variable given F tau n naught, but this random variable is again an indicator function. So, expectation or indicator function is probability. So, we write the probability directly. So, this part will be probability of this event, W sigma D belongs to Br given F tau n naught.

So, here this part we are taking outside our expectation and then inside expectation of indicator function because of probability. The expectation we need a function of this set, of an event is probability of that event. So here you get, W sigma D belongs to Br given F tau n naught. Then this whole thing this condition probability is a random variable, is F tau n naught measurable random variable and this is also F tau n naught miserable random variable and we have a conditional expectation of product of these two random variables given W0 is equal to 0. This bracket should appear here.

So, now our goal is to complete this expectation. We know how to complete expectation. Expectation of a function of a random variable if you want to compute, say expectation of small f of capital X. X is a random variable. What do we do? We have a integration F of small x times the probability density function of the random variable we consider, if it has a PDF. Otherwise, we just exceed here if it is multi-dimensional etc, then we just take the law. The distribution measure of the random variable. We integrated this with that.

We do exactly the same thing here. For this, since we need this, so what we do is that probability of W so instead of sigma D I am writing tau D. I am explaining why doing, why can we do that, in a minute. So, probability of tau D in Br given. Now, here F tau n naught, so here we are using the strong markov property of Brownian motion, the tau n naught is the stopping time.

So, the future distribution given this filtration at tau n naught is same as the future distribution given the present with that at tau n naught what is the random variable? So, W tau n naught. So, this thing appears here. So, actually here this expectation we are doing in this way and for this part we are doing in this manner that W tau n naught is equal to X.

And then we are going to use the distribution of W tau n naught given W0 is equal to 0, because we need to find out this expectation. And this random variable is now simple because we are using strong markov of property of Brownian motion. So, now it is just W tau n naught is equal to X, X is just a vector in d-dimensional space and then the law is W tau n naught is in dx, so this is just a notation to show that basically I am picking up the distribution measure of W tau n naught.

So and given W0 is equal to 0, but we do not write W0 is equal to 0, we again use Bayes rule, we condition and then take and multiply with the probability. So, this W tau n naught belongs to dx, so given tau n naught is less than sigma D into probability tau n naught less than sigma D. So what is this? This whole product is basically intersection of these two.

The W tau n naught dx and this is true and why are you doing that? Because here we have that tau n naught less than sigma D. We need to do expectation on this. So, we need this, so we are having that intersection of these two things that we are doing in the Bayes rule that conditional probability and then product.

Since we have incorporated this here, so what are left with? We are left with that this X we have to run among each scope. So, what is the scope of X here? So, W tau n naught anyway by definition is on the boundary of the ball of radius delta n. That is its scope. So, the boundary of ball of radius delta n naught around 0, so this.

However, why are we taking intersection with D, thing is that so that we can sigma D to, we can write down tau D here, why because here when W tau n naught is falling you know outside D that means that no ball of radius delta n naught is the part which is outside D, if that is the present space X, that means the sigma D occurred before. Sigma D already have occurred. So, but that is excluded, because that is excluded.

So, only when those cases, it would be of course in D, so that should be excluded. And in this scenario then sigma D is in Br, so that means that is same as tau D, why because sigma D differs from tau D only when that tau D is 0, only when, not tau D is say when, it just immediately it leaves the boundary.

Otherwise, if it immediately does not leave the boundary, then it is different. So, here of course it does not leave immediately in the boundary because here it is tau n naught is in x and everything is inside till some positive time. So, I can replace sigma D by tau D. So, from this step to this step had lots of argument to justify, but yeah so possibly if I write maybe in between steps in lecture notes, so possibly I would write down more steps.

Now, we are going to estimate this thing, we are going to use the values what we have obtained earlier, probability tau n naught less than sigma D given W0 is 0 is greater or equals to half. So, here tau n naught less than sigma D, tau n naught less than sigma D this appears. So here I have not written but it is there. It I mean basically I should have written but I have just you know presumed that it is there all the time the W0 is 0 because we cannot escape from it, W0 is 0.

So, this thing is greater equals to half, so I am writing half here. And then the remaining terms are here, the D intersection you know boundary of ball here and probability W tau D belongs Br exactly you know tau n naught. So here, so if I start with 0, it is not a typo, this is okay, why is it okay?

Because now we given this, the tau at tau n naught W tau n naught is equal to x, the probability that W tau D is in Br. And instead of that what we are doing is that we are declaring this as our starting time. So, if we consider as a the Brownian motion at tau n naught and there it is regenerating it is starting from there itself. And then since it is Markov it has memory less property. So, from there onward, then anyway this is inside the boundary because D is intersecting this. So, it is anywhere inside the boundary.

So, then we look at that, of the probability that W tau D is in Br. Given W0 is equal to x and then probability, this is exactly the same as before. W tau n naught, this is a typo, belongs to dx given tau n naught is less than sigma D. So, next what we do is that we are trying to again get a lower bound of this value, so how are we doing that?

We are taking infimum over all possible x, because x is anyway running in this domain, D intersection boundary of ball of delta n naught x, n naught 0. So, here if we take infimum over all possible these things, and then the remaining thing would be this, and integration of this measure. So, basically here instead of dx I should write down this whole thing here, D intersection del B, delta n naught.

So, this is integration where this is the measure, this is the function. This function I am replacing by infimum and then we are going to get it. So, we have 1 4 is strictly greater than half into this value. So, we are actually looking at some contradiction. So, now what we do is construct some x n naught because this part is small, so we can always find out x n naught such that this whole thing is less than or equals to half.

So, here we have, what we have obtained is that 1 4 th is strictly greater than this expression and half is already there. So, this product of these two quantities is strictly less than half. But right-hand side expression, the probability is exactly equal to 1. So, this infimum amount is strictly less than half.

The reason that right-hand side probability is equal to 1 can be seen in the following manner that given the fact that tau n naught is strictly less than sigma D that means the Brownian motion at time tau n naught did not leave the domain. So, W tau n naught is in domain D. Here there is a typo, it is Wtn naught, 0 is missing here. And by definition of tau n naught, W tau n naught is on the boundary of ball of radius delta n naught.

So, probability that W tau n, so it is a sure event. The W tau n naught should be on D intersection, the del B delta n naught. So, we have some value here at, and the infimum is strictly less than half, since this infimum is strictly less than half, so I can find out some point x which is also less than half.

If I say infimum is less than or equals to half, then I cannot assure but here I have strict inequality. So, I have infimum is strictly less than half. Since infimum is strictly less than half, so I would be able to find out some x, particular x. So, that is the thing I am writing, some x here, such that W tau D belongs to Br. So W tau D belongs to Br. This probability is less than or equals to half. So, this we can do.

I recall this x is x n naught, the reason we are denoting by n naught is because that we are talking about ball of radius delta n naught. So, keeping that in mind, we are writing it x n naught, this value. So, that we can do, for each and every n naught greater or equals to capital N, because there is nothing specialty about this n naught because this is true for any n naught greater than or equals to N. We can always get this exactly the same thing.

So, for any n naught greater than or equals to n, we would always be able to get, you know I mean of course then as n naught is larger and larger, your ball of radius delta n naught is also becomes smaller and smaller. x n naught is also, is always on the boundary. So, x n naught is also coming closer and closer to 0. However, we are always assured to find out such x n naught such that this condition probability is strictly less than, is less than or equals to half.

So, you got a sequence x n naught along with if we run, this probability would be less than or equals to half all the time. So, choose a function f now from that boundary of the domain to the real value function which is bounded and continuous such that f is equals to 0 on the you know boundary minus Br. So, if you have the ball of radius r, so beyond that.

So, there is a typo here, this and, so f is less than or equal to 1 inside the ball Br. Here comma, here this should be comma, f of 0 is equal to 1. So, we have now just, I mean what is the assurity that such function would be there. Things that I can, this is easy to you know function you are taking that at 0, it is 1.

And then it is decreasing to 0, where I mean so it is always less than or equals to 1 and so 1 is the maximum point at 0 it is the maximum point and 0 is on the boundary because we are looking at 0 as an irregular point, x0 is on the boundary. And there when that boundary part outside the ball of Br, so there it is becoming 0. So, I think it is good if I draw.

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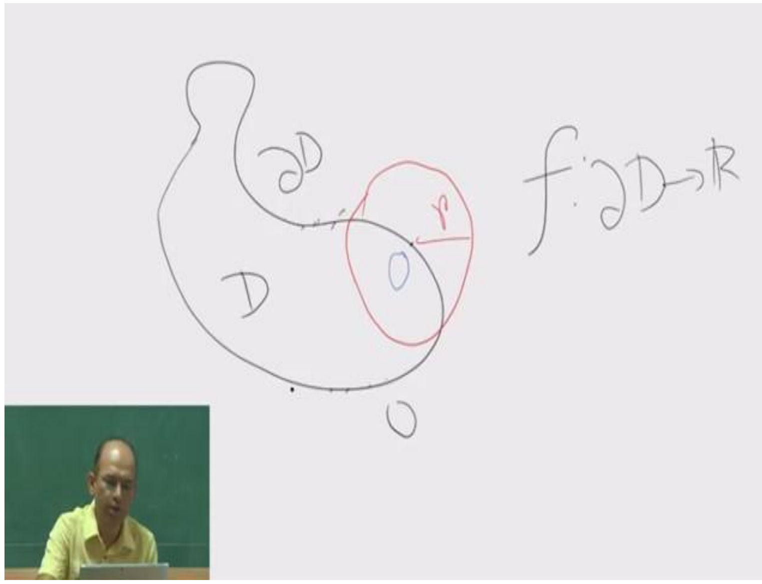


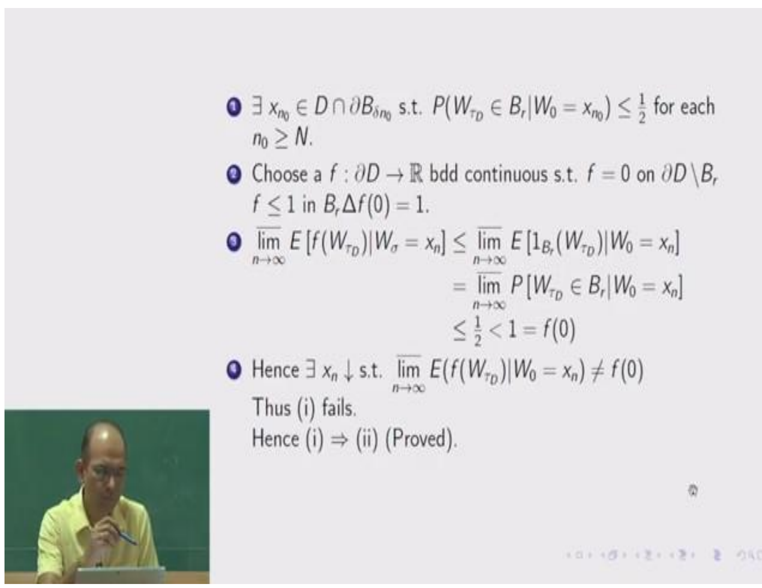


Imagine that this is 0, this is ball of radius r. And this part is the boundary of the domain, this is domain D and this is origin. So now, the function f what we are constructing is 1 at 0 and 0 here. The function f is defined on the boundary of the domain, boundary of the domain of course boundary of the domain is close set here, we are going to get because D is open, connected domain.

And here what we get is that a function, I mean existence of this function is assured correct, we know that this is a close set, this is a close set, they are separated. We can do that, we can assign and we can create a function f which is 1 at this point and 0 here. So, that is choice of the function.

So, we choose function that which is 0 on the complementary part of the ball but on the boundary. And which is 1 at 0 and this is comma and then f is less than, strictly less than or equals to 1. So, you can construct such function. So, now by taking this function f, we imagine, remember that what was our goal? Our goal is to prove, do prove by contradiction.

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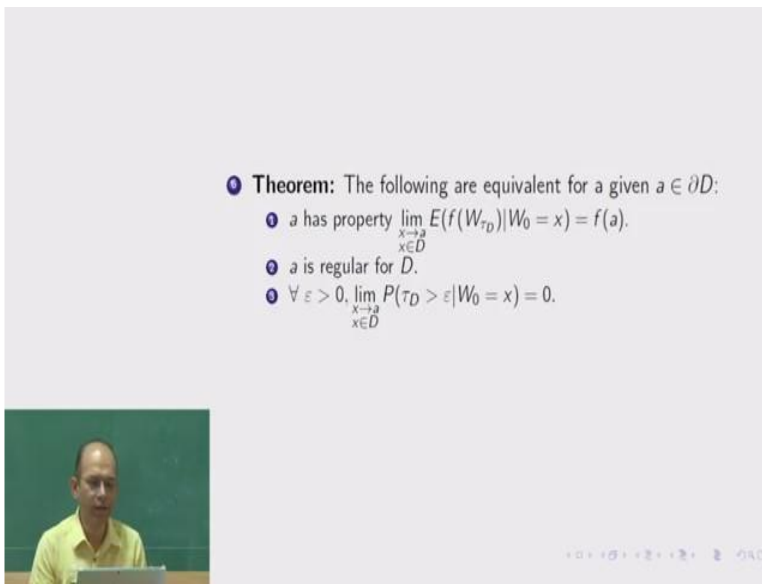


So, for that this part says that this is true for all bounded measurable function f which is continuous at the point a. So, to contradict, either you need to figure out one single example, sufficient for which it does not hold. So, here my function f is continuous at the point a. a here is 0, origin the point on the boundary.

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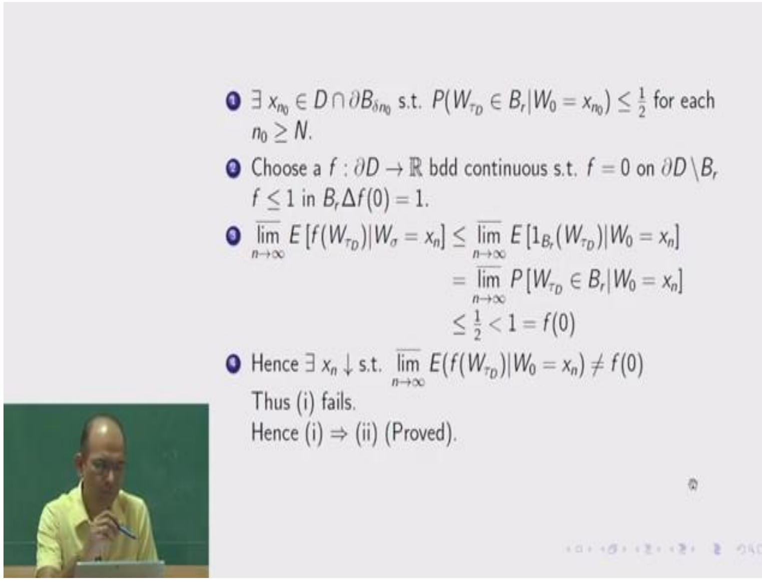


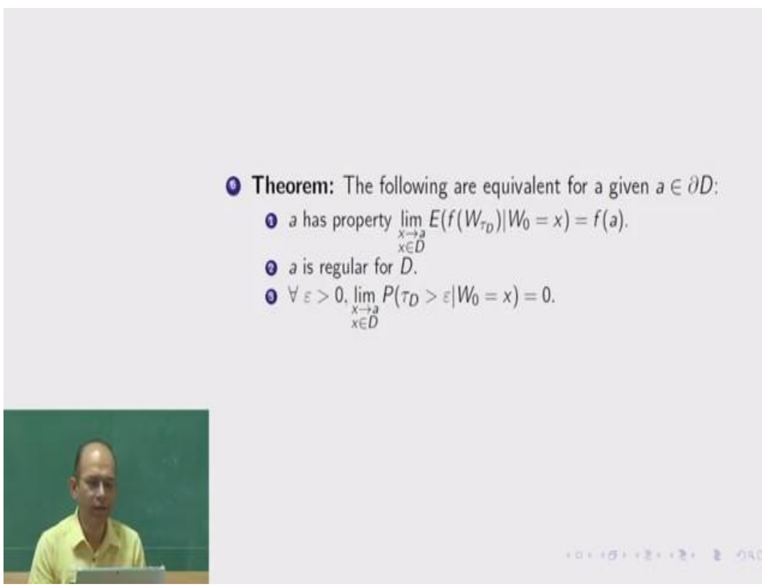


So, now we consider this limp sup because I do not know whether the limit exists. f of W tau D given W sigma is equal to xn. So, let us see what is this factor. This is exactly this quantity, expectation of f or W tau d given W0 is equal to x here.

So, where x tends to a but for us a would be 0 here. So expectation of f for W tau D given W, sorry here it is a typo, it will be 0, W0 is equal to xn. Now, we are replacing this part by this. So, since f is less than or equal to 1, so if we replace f by indicator function of Br. So, then inside Br it is 1 always. But f is always less than or equals to 1 there, replacing that. So, this would be of course, we are going to, of course going to get this less than or equals to sign here. So, lim sup n tends to infinity here expectation of indicator function of W, indicator function of say Br of W tau D given W0 is equal to xn.

Now, we know that expectation of indicator function is probability of this even itself. So, this conditional expectation is conditional probability that W tau d belongs to Br given W0 is equal to xn and n tends to infinity. And here what we have constructed is that after capital N onward, capital N is fixed. For every n naught we can choose some xn naught along which this would be always less than or equal to half.

So, here now we have choosing exactly considering that xn, we are keeping that xn and let n tends to infinity. So, then capital, after capital N onward, this value would be always less than or equals to half. So, this lim sup would always be less than or equals to half. So, that is strictly less than 1, but 1 is a value of f at 0. So, we got that this is not equals to this, it is strictly less than this value. Whereas xn still converges to 0, because that is the point that x converges this to lim.

So, here to get I mean for proof by contradiction if we have assumed the negation of this to conclude the negation of this and here this is true for all possible f et cetera but now we have constructed a particular f and constructed a particular sequence xn converging in a particular manner.

And there it did not hold. So, when a particular, I mean if in the multi variable calculus even you find one particular way in this limit and this do not hold, then of course the total limit you do not have any hope and lim sup is not matching. So, therefore forget about limit.

So, limit would also, even if the limit exists that would not match. So, we get that this equality do not hold, this does not hold. So here, hence that exist xn decreasing to 0 such that this is not equals to f0. Thus, one fails. Hence, 1 implies 2. Thank you very much.