Introduction to Probabilistic Method in PDE

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Lecture 24

Zaremba's Cone Condition for Regularity

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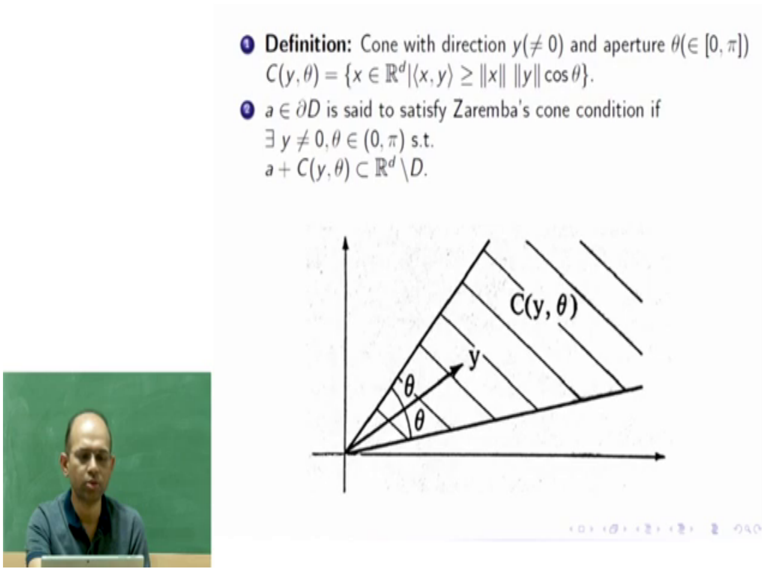

So we define. What is cone? a cone with direction y is a subset of the d dimensional Euclidean space and aperture theta, theta these side these side theta is been 0 to pie. So C of y, theta that is a notation we are going to use is the set of points in d dimensional space such that the inner product of x with y, y is like direction. x and y the inner product is greater or equal to the norm of x into norm of y into cos theta.

Let us understand this cone better manner, imagine that theta is 0 if theta is 0 and cos0 is 1. So, then Cauchy Schwarz inequality says that this must be exactly equal to this because this is always less than or equals to and that equality happens when x and y are the same. So, basically that would be just you know y itself, there be no other point the cone will be just exactly this line but now if theta is not 0 but theta angle increases then you are going to add more and more points I mean in this C y theta.

So, this is cone a point a in the boundary of the domain is said to satisfy Zaremba's cone condition, Zaremba's is a mathematician. So following the name of Zaremba we call this a Zaremba's cone condition because condition these are the things which we are studied by him. If there exists any particular direction y nonzero of course when victor y nonzero so that you can give direction and theta some theta between 0 to pie such that imagine a is attached here and you are attaching a cone. a plus C y theta a plus C y theta would be then you know the cone where the vertex is a is a subset of Rd minus D that means subset of the complement of D.

So what is the point? The point is that. We are talking about the complement the shape of the complement of the domain. So first example of irregular point was the compliment was disconnected one was outside the ball another was another compliment was just the origin itself. Then second example their complement is approaching very narrowly to the boundary to one particular point of the boundary, now we are saying that, if we can attach a cone theta is strictly positive so more than 0 some angel on strictly positive angle.

If you can attach such type of cone on the complement of the domain, then we would call that I mean it called a satisfy Zaremba's cone condition. Imagine that when surface you know the boundary is very smooth then actually you can take theta as large as say Pi by 2. I mean close to pi 2 like you know with like a full thing full one side is there. Like you know, if your domain is a ball then you can actually take theta is equal to Pi by 2.

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- **Theorem:** If $a \in \partial D$ satisfies Zarembas cone condition, then it is regular.
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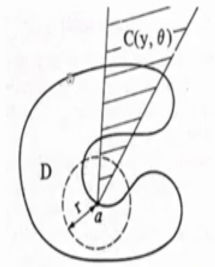
$$P[W_t \in C(y,\theta)|W_0 = 0] = \int_{C(y,\theta)} \frac{1}{(2\pi t)^{d/2}} \exp\left[-\frac{\| x \|^2}{2t}\right] dx$$
$$= \int_{C(y,\theta)} \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{\| z \|^2}{2\theta}\right] dz$$
$$= q(\text{indep of } t) > 0$$

where,

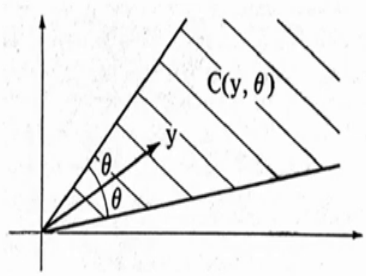
$$\left[z = \frac{x}{\sqrt{t}},\ \frac{\partial z}{\partial x} = \frac{1}{\sqrt{t}} I,\ \left(\frac{\partial z}{\partial x}\right)^{-1} = \sqrt{t} I;\ \det\left(\frac{\partial z}{\partial x}\right)^{-1} = t^{d/2}\right]$$

$$P(\sigma_D \le t|W_0 = 0) \ge P(W_t \in C(y,\theta)|W_0 = 0) = q\ \forall\ t$$
$$\Rightarrow P(\sigma_D = 0|W_0 = 0) > 0$$
$$\Rightarrow P(\sigma_D = 0|W_0 = 0) = 1.$$

- **Remark:** If for $a \in \partial D$ there is $r > 0$, s.t. a satisfies cone condition for $(a + Br) \cap D$, then a is regular for the domain D.



- **Definition:** Cone with direction $y(\neq 0)$ and aperture $\theta(\in [0, \pi])$ $C(y,\theta) = \{x \in \mathbb{R}^d | \langle x, y\rangle \ge \|x\|\ \|y\| \cos\theta\}$.
- $a \in \partial D$ is said to satisfy Zaremba's cone condition if $\exists\ y \neq 0, \theta \in (0, \pi)$ s.t. $a + C(y,\theta) \subset \mathbb{R}^d \setminus D$.



Now this theorem says that this is a good thing to have Zaremba's condition because if Zaremba's cone condition is true, then it is regular. As I was telling that for you know sphere spherical domain. The Zaremba's cone condition cone you can actually attach a cone of theta pie by 2 etcetera the large and that is smooth also we have that that understanding.

However, the example what we have constructed there we know very well that there is no we cannot attach such kind of cone because it is becoming narrow So narrow there no particular fixed angle can be attached here. So if a is satisfies the Zaremba's cone condition then it is regular this is a theorem. The proof is not very complicated. It is Trivial actually so takes the condition probability the Wt touches the cone C with the direction y and aperture theta. So we would like to find out this probability, we know, that Brownian motion has you know normal is normally distributed for fixed t.

So here we have fixed t, a particular time t and there it is in C y theta. So to find out this probability we need to integrate the probability density function of the normal random variable over these sets C y theta to find out the probability. So C y theta is a domain where we are integrating the Probability density function for this random variable Wt.

So what is the distribution the normal means 0 and variance t. So and it is in the d dimension space correct so the density appears like 1 of 2 pi t to the power d by t d by 2 e to the power of minus Norm x square by 2 t, here we get x minus mu correct mu if the mu is the mean but here Brownian motion has 0 mean because W0 is equal to 0 it started from 0 So it has 0 mean, so norm x square. So now we would take this transformation so z is equal to x divided by square root of t.

And if you do is substitute that manner, so x is a vector z is also Vector, but 1 over square root of t scalar so this scalar multiplication of that and then if you take this Del z Del x this

Jacobian so then 1 over square root t and then identity Matrix would appear here. And then del z Del x inverse if you take this the square root of t times identity Matrix.

And if you take determinant of that the whole thing, then this constant would come out but with the power. Because there are d number of rows. So to the power d so d by 2 would appear and determinant of the identity Matrix is 1 so we are going to t to the power d by 2, but that is t to d by 2 would also cancel with this. So the Jacobian would appear here due to this transformation that would cancel with this and then x by this things would become you know, that's z. So where from this line we are going to get this line, but t disappears totally.

So it does not depend on t we get some number we call that q it is of course positive number because you know, if C y theta has positive measure and that is true because theta is positive so it has positive measure. So on that this value is q which is positive. Next what we do is that we take the probability. So we are going to take this probability. But actually our intention was to find out the probability that sigma D is less than or equal to t. What is the probability of that? Given W0 is equal to 0, so this probability is greater or equal to that probability that Wt belongs to C y theta. Why is it so? Because here C y theta is in the complement of the domain is the outset of the domain, so Wt is in C y theta.

So, a belongs to a satisfies Zaremba's cone condition. So that means that this is in subset of Rd minus D is outside the domain. So this thing that for if particular time that Sigma D is less than or equal to t. That means that Brownian motion leaves the domain at our Before Time t sigma D is sigma D is less than or equal to t

However here we are just saying that Wt is touching this cone. This is so I mean, this is a subset of this you know because this can happen in many different ways this is only one particular way because if this happens surely this happens because Wt is in that cone that means Brownian motion already left the domain and entered into the cone.

And if that happens then of course that you know, it left before time t. So see this is a superset of this. So this probability is larger or equal to this probability, but this probability is already completed as q and that implies that the probability and this is for all t does not depend on t. So does not matter what t you choose Sigma D is equal to 0 given W0 is would be positive q is positive and this is true for all t.

So Sigma D even if you take you know, take infimum of all possible t for which happens the q is the probability that would always remain there. So this would be greater than 0 positive. So here we can again apply Blumenthal zero one law to assert that this would be exactly equal to 1. So what does it mean that these a would be you know, I mean here we start with zero, correct? 0 is the point we started with so here very particular case here. We have without loss of generality we have considered a to be 0.

So here 0 is the regular point, because this probability is one. So let us see that, this appears to be little, you know, strong condition this Blumenthal's this thing strong condition in the sense that say domain is like this, and then the whole cone is on the complimentary, but however, there are domains which is like, you know which comes like this.

So there may not be any escape to the infinity. And we do not need that much because what we need only local Behaviour. So this much that this is extend to the infinity that thing is redundant. So this is the illustration of that imagine that a point is here the far part of the domain is coming like this, which is resisting to attaching a cone in the compliment should not force a to be regular irregular because if a is regular irregular that should only depend on the local behaviour of a.

So that is a drawback of this theorem. The theorem statement actually does not take care of this type of examples. However, one can extend the theorem very easily. One can take a part of the domain so if this is the domain. So D and then we take a ball of radius R around a and then you take intersection.

So then this you know Moon type, you know, like part what appears would be your new domain. So in that new domain a is a regular point a regular point and a satisfies the Blumenthal's I mean, it is satisfy the Zaremba’s cone condition and then you can apply the theorem to conclude that a is a regular point.

So if a belongs to the boundary and there is a positive r such that a satisfies. Yes, there is a type of this a would be script a satisfies cone condition for a domain which is now the full domain but the part of the domain, a plus Br like this type of thing, then a is regular for the domain D also.

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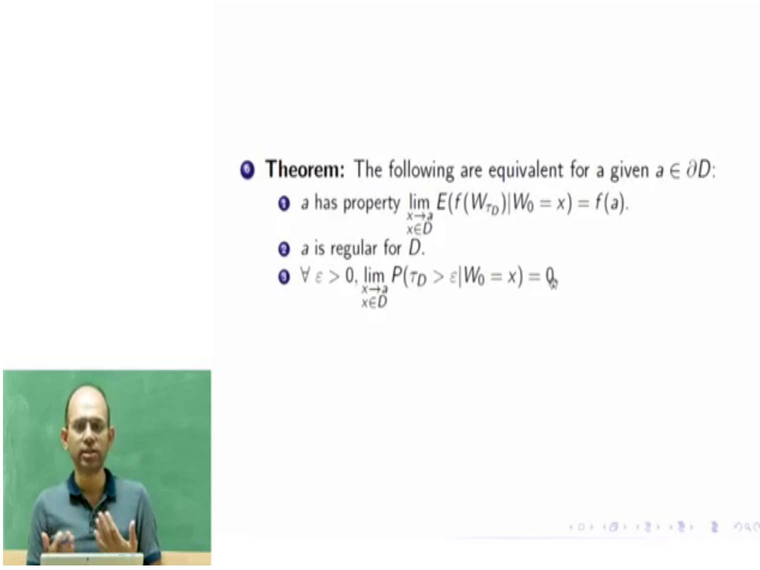

So, this is the remark. So we are going to stop by seeing this theorem, but the proof I am not going to start now because it would take a long time a full more than one hour lecture will be required to finish this the proof of this theorem. So I would do that in the next lecture. The theorem statement is that, the following are equivalent for a given a in the boundary of the domain. If a is in the boundary of the domain the following three conditions are equivalent.

First condition is saying that limit x tends to a, x being in D expectation of f of W Tau D given W0 is equal to x is equal to f of a, as x tends to a. Why are these things so important? Why are because our whole discussion is circling around this thing. So because you know, this part looks like solution of the Dirichlet problem. And this condition is saying that the solution is continuous or the Closer of the domain and unless the solution is continuous on the closed we don't call that as a solution.

And this thing is equivalent of saying the a regular for D, a is a regular point for D and the third point for all positive epsilon limit x tends to a, x being in D probability that Tau D, This is exact exit time is greater than greater than Epsilon given W0 is equal to x is equal to 0. So what is the interpretation of this property?

This property is saying that if you fix epsilon but do not change now fix epsilon and then you take x close to a. Where is a? a is on the boundary. When x is close to a and then for and you start your Brownian motion from x. Then you are checking that when the Brownian motion is leaving the boundary that time and you are asking what is the probability that it would take at least Epsilon time? At least epsilon time, Of course, if you come very close to the boundary the probability that it would take at least epsilon time to leave would decrease. Decrease but thing is that if it is regular, then the that is equivalent to saying that this would decrease to 0.

This is very natural. So regular thing is that what is you know intuitively we understand that should happen. So those are like regular points that is also this is a the another criteria equivalent criteria and this also another equivalent thing and this would enable us to settle down for that you know, what is relevant for the Dirichlet problem. Thank you very much.