Introduction to Probabilistic Method in PDE
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Lecture 23
Regular points at the Boundary

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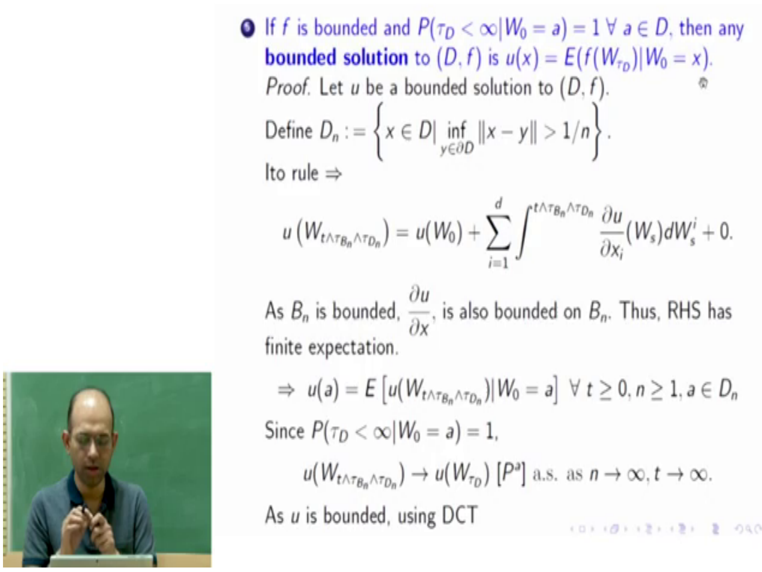

Let us recollect the result this blue colored test is the theorem. If f is bounded and probability that exit which time this exit time correct because there is stopping time I mean that hitting time of the boundary, exiting time from the domain. If that exit time is finite with probability 1 for all a in the domain. Then any bounded solution of this Dirichlet problem D, f is written as u of x is equal to expectation of f of Brownian motion evaluated at the hitting time given W0 is equal to x.

That means, if you have Brownian motion at point x starting from there and then run the stimulation of Brownian motion till the time Brownian motion hits the boundary and when it hits look at the location, at that location find out the value of f evaluate f at location. And you do that repeatedly possibly say million times and take average of these numbers, that is the expectation correct. So, that would give you another numerical approach to you know find out the solution u, so that would give you u of x.

This is a proof, which we have already seen in the last lecture. So here we have crucially used that u is bounded. So here we did not put the condition that D should be bounded. We did not put that condition, we just put one condition above the domain of the of course that is this condition that, it is such that exit time is finite with probability 1. So this is one of course property of the domain D.

So, we just had this assumption, and then we say that okay. Not every solution but every bonded solution can be written this way. So, this is just a recapitulation of the statement of the theorem, what we have seen in the last lecture. Basically, we used dominated converges theorem here. For that we crucially required u is bounded, if u is not bounded we could not have use this theorem and we could not have concluded this result.

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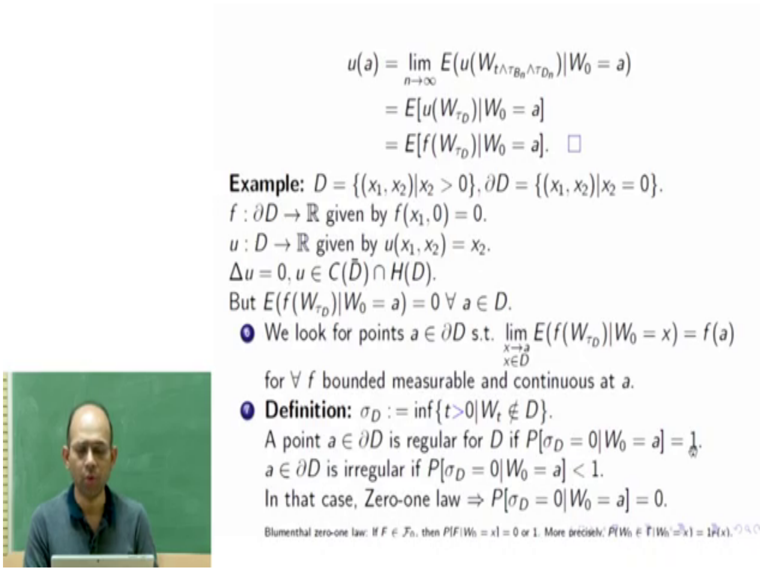

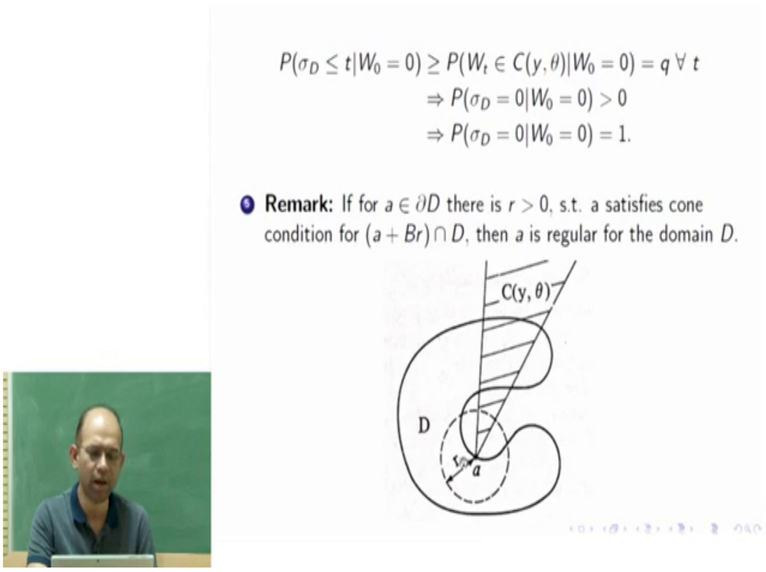

So, next we go to the point 6, in this we look at point a on the boundary of the domain. So, these I mean this del stands for the boundary, so del D. Such that when you approach to the point of the boundary a, from within the domain that means x belong to D and x approaches to a, so within the domain you approach to the boundary of the domain and for each x we evaluate this function and find out the limit of this function this is function of x correct because this is conditional expectation of f of W tau D that means Brownian motion at the hitting time.

So, evaluation of the function at the position where Brownian motion hits the boundary, so then that is a random variable of course we will see f of W tau D would be random variable real valued random variable and you take expectation of that random variable. But this is not standard experience, this is conditional expectation because we start Brownian motion from point x. For another point a, I mean because if for various define x we are taking for say another point a on the boundary we should of course consider another family of point x so that x would be neighborhood of that point a. So, depending on a, we are going to get this you know neighborhood of a in the boundary and there we are going to take this limit.

So, if this limit is to equal f of a. So, that need not be always true if that is the case. For all possible f bounded miserable, because you know we cannot say that this is the case for some specific f but for all possible f were f is continuous at point a. Then that criteria gives some property of a correct because this would be property of a. And not only that its the property of what is a property here which property it is highlighting, it is highlighting the positional property. Then how that a is located in the boundary?

So, basically it is saying that the property of the boundary or the regulate of the boundary at the point a. Let me go through this again, that what does it say. It says that, this equality holds, this equality holds for all f bounded measurable and continuous at a. If that holds for some particular point a in the boundary. Then that is well behaved that is well behaved in the sense that then the kind of results what we are looking at would go through.

So, we are going to categories or we are going to select such points a, and we are going to call that a as regular. So, let us see, that how do we do that, this is a proper definition. We define sigma D this is the time first time when Wt exits domain D. However it is a little different, it is close to that but little different because here see greater than 0 is real. So, I have made it blue color to highlight this, t greater than 0.

Often what happen that if you start Brownian motion from the boundary itself than at time zero it might leave I mean infimum what whatever infimum you take, infimum could be zero. So, time zero it might leave. But we are excluding that case that time zero case. I mean even if we exclude, it does not mean that sigma D cannot be 0, sigma D could be 0 because you know this at some positive time it is leaving but is leaving with certain probabilities but that positive intake could be as small as possible with certain positive probabilities and infimum if you take that could be zero also. But we are not preventing that that is allowed, but what is not allowed is that it is leaving the domain at time 0 that is not allowed.

So, here Wt not belongs to D, when t is positive. So, and then take in infimum of that that is define as sigma D. This sigma D is very close to the exit time except that it is consider so sigma D is strictly I mean tau D is less than or equals to sigma D. Tau D which we have already defined earlier it is less than equals to or equal to sigma D. So, a point a in the boundary is regular for D if probability that sigma D is equal to 0 given Wo is equal to a is equal to 1. What does it mean? It mean that when point a is in the boundary point a is in boundary.

So let me see if I have a pictures, see for example a is here, a is here and D is this domain, D is these domain. And now from here, if you start a Brownian motion running immediately it can go outside, or it can stay for some time and go outside. However since this is remember that here locally the half of the part is inside the half of the part is outside the of course there it is a probability that it will go outside so here sigma D if you calculated it will be 0 here, so it is nice thing.

So , here if that probability is 1, so the picture what I have shown that has that because sigma D, I mean so every time with probability 1 Brownian motion would go to the other part correct because you know there is a property of Brownian motion that it would be restrict only one half would be probability zero. Because when ever it touches a point it oscillates their infinite many often. So sigma D is equal to 0 you would be able to get there. Why? Because you know that time when it exits the boundary there, time would be you now as small whatever small you know time you take there is possibility that it would cross before that time also so if you take infimum you would get zero here.

So, probability that sigma D is equal to sigma D equal to 0 given W0 is equal to a, would be 1 at those cases. Most of the cases you know if you draw in a very nice way just like that but of course there are certain cases when this is violated but we are going to see those examples in the one or two slides next.

If a is on the boundaries I mean such that this is strictly less than 1, then it called that it is irregular. That it leaves you know boundary you know this sigma D is 0, that probability is less than 1. However, there is a zero one law this is called Blumenthal zero one law. This says that if this probability is less than 1 this should be exactly equal to 0. And this probability cannot be anything but 0 and 1. If it is less than 1 it should be 0, so what is this law I mean let us read in that footnote, that f is in F0, F0 is the sigma algebra at time 0.

Then probability f occurrence of f, given W0 is equal to x is either 0 or 1. Here what happens the sigma D is that taking time t but taking infimum over all possible these things. So, so the future time zero is present, here is future time and then smaller and smaller time coming coming close to zero there is from you are approaching from right hand side. So this sigma D is actually you know this event is actually F 0 plus measurable, F 0 plus measurable.

However, since the filtration is right continues, so F 0 plus exactly F0 and therefore if we apply now this you know theorem because f is in F0 this theorem, we going to get that this probability of this should be either 0 or 1. Here, this I mean in more precisely we can say that if we ask these question this f the event is of this particular form the W0 belongs to a set gamma belong some set gamma, gamma is some kind of Borel set given W0 is equal to x.

We know how does probability looks like it would be just indicate function of gamma, I mean if x is in gamma then is 1, if x is not gamma then 0. So, we are understand that the rational behind this Blumenthal zero one law and then understand that this probability that sigma D is equal to 0, this conditional probability either 0 or 1. If it is 1, then it call that point a as regular point for D. And if it is 0, we call this irregular point.

Why the name regular irregular comes? That I am going to state later but already the picture what I have shown there it gives some indication that there the point a was on the boundary where locally it looks like this just half plane half side here half there. So, that means the the boundary has no kink has no very sharp edge and for those cases this is very intuitive and this is true. On the other hand that regularity would be validated for very extreme situation.

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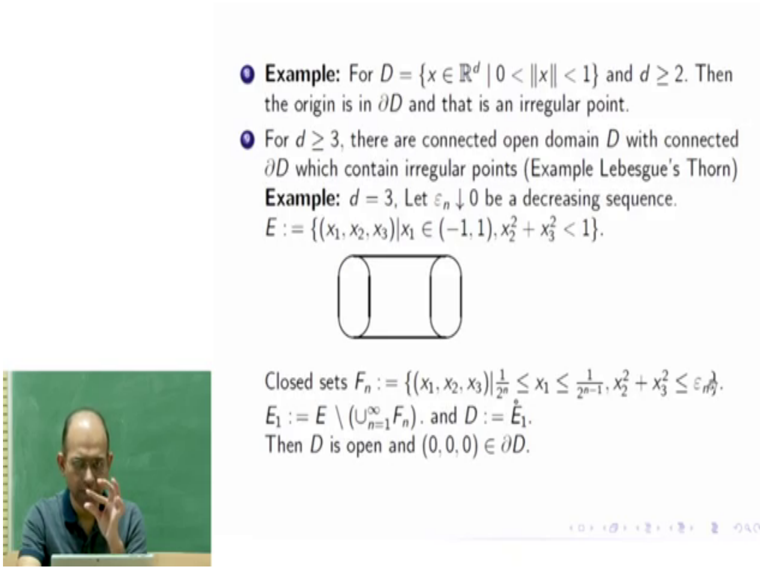

So, you would see those examples actually you construct some particular example where we are going to compute these probabilities and you show that a particular point is irregular. So consider D is equal to this annulus, norm of x is between 0 and 1 so it is punctured open ball. And d is greater 2. Then the origin is in the boundary del D and that is an irregular point.

Because if you run Brownian motion there, you would always move out with this thing then the origin is in del D and that is, no you always be inside the domain I mean you always be inside domain because the origin is del D but origin is actually D complement I mean D complement but it is the boundary of D but if you start there immediately it would go to the inside the domain immediately it would go inside the domain and from 0 if you will see how long it take to come to 0 that probability becomes 1 if d is equal to 1 because if it touches 0 it touches 0 you know touches 0 infinitely many often.

But when the dimension d is 2 or more then if I mean goes away from 0 so it again returns to 0 with probability 0. That does not return to zero certainly. So, it can only go beyond the boundary only through the outer region and outer region is far away one unit away. So, if you find out this probability sigma D would become zero very unlikely so this probability would become zero. So it would be irregular.

However one can object here for this particular example, the boundary is disconnected. So, it is manufacturing in a particular manner that when it leaves that then there is no other part of the boundary and other part of the boundary is far away. So one can think that ok can one construct a particular example where boundary is connected and a is on the boundary still one can get irregular points.

Yes that is also true, is not very difficult so if we consider so this is the thing for d greater or equal to 3 there are connected open domain D with connected boundary with which contain irregular points so this is called Lebesgue’s thorn Lebesgue’s thorn. So this Lebesgue’s thorn so for construction we first consider cylinder, so bounded cylinder the height is 2 imagine that origin is here and x1 is between minus 1 to 1 so this is 1 these minus 1 and 0 is here x1 till this.

And x2 and x3 the other variables were x2 square plus x3 square is less than one. So is like open disk here, Open disk here and here x1 varies here. So you are going to get one open bounded cylinder solid solid open bounded cylinder. So, now we called that as capital E and what we do is that we take various different copies but different size of such cylinders, and we called that Fn and it is a open cylinder we are going to takes a for example closed cylinder were x1 is 1 over 2n 1 over 2 to power n minus 1 so this distances so 2 it is like difference between this two.

And this is actually basically half of this, so this difference is actually 1 over 2 to power of n, 1 over 2 to the n plus 1 over 2 the n you would be one over 2 to the n minus 1, so this length it be 1 over 2 to the n and then x2 square plus x3 square is less or equal to epsilon n, so this radius is epsilon n, epsilon n is now kept as free we are going to choose later.

And then we take these union of Fn, n equal to 1 to infinity. So imagine that this Fn are like you know consecutive cylinders because F n plus 1 would I mean that right hand side of these would be 1 over 2 to the power n and lift hand side onward 2 to the power n plus 1. So I mean there would be I mean decreasing width and decreasing length cylinders from right to left and would approach to 0 that means toward 0 this cylinders have 0 would be a limit point of these cylinders, zero would be in the closer of this or you take union of all this Fn. All the Fn are close sets, union of Fn is not close because 0 is limit point of union of Fn but 0 is excluded there.

And we take here E subtract of this. So, E subtract of this so we subtract this close I mean whole thing from E because everything Fn everything is inside the E correct and you subtract everything and for sufficient small epsilon everything is inside. And we take this set E1 but E1 is not assured to be an open set, we take interior of E1, E1 interior this you know circle stand for interior that we called as D, D is my domain.

So does (0, 0, 0) origin the domain D? No because this is on the boundary, what happens that of course (0, 0, 0) is in E and not in these so it is in E1 but when we take interior of E1 then (0, 0, 0) is excluded (0, 0, 0) excluded because is on boundary does not matter so this (0, 0, 0) we just take is on boundary. And then we show that this is an irregular point for certain choice of epsilon, if epsilon n are chosen very wisely then we can actually get (0, 0, 0) as an irregular point.

Unless you choose epsilon n in a decreasing manner and very wisely you would not get it. Because if you take epsilon n is equal to one all the time, you know what I mean nothing, this is like a straight cylinder basically. You are not going to get anything but epsilon should be decreasing in a particular pattern.

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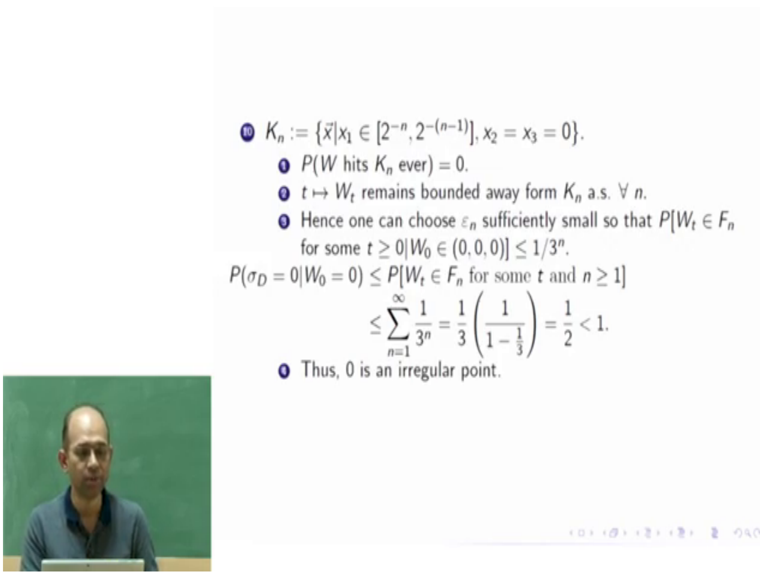

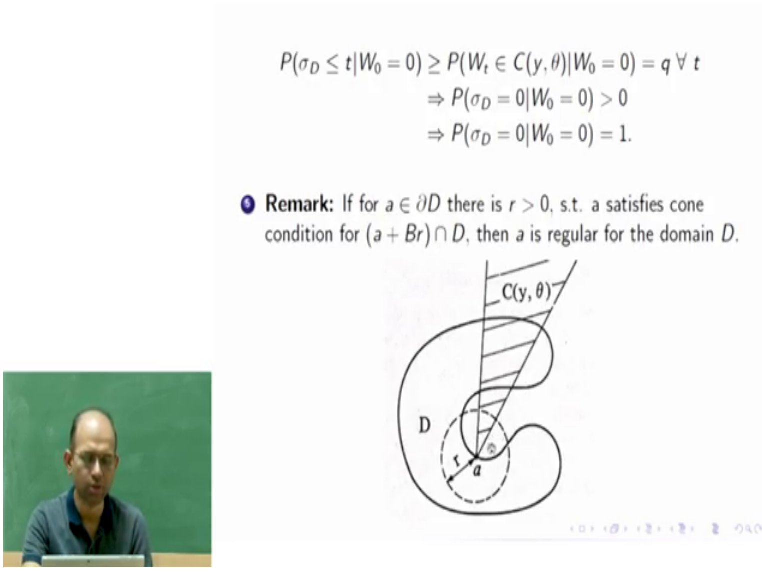

So, let us see so construct Kn where this vector x is such that x1 is between these x2, x3 is equal to 0. This is nothing but the axis of the cylinders. So these are exactly the end points of the cylinders. Left and right and x2, x3 is equal to 0 axis of cylinder, now we are asking that what is the probability that the Brownian motion hits this axis ever for any time.

What is the probability? So we started in the three dimension case, correct? So in two dimension at least you know if it goes here it crosses from, positive to negative and then there is chance that it would cross but in say it is three dimension so one so this k I mean this k et cetera would be just one single line segment in the three dimensional space, in three dimensional space so Brownian motion has 0 probability to hit any line segment in three dimensional space.

Instead of line segment if I had a plane then I could got positive probability but it is just a line segment, in three dimensional space it has zero probability. Now the map t to Wt remains bounded away from Wn almost surely therefore for all n. Because this it has zero probability to hit Kn and Kn is of course close set here and then since Brownian motion path we are looking at so this is you know continuous function and if I take a fixed you know time horizon in a time horizon it would be you know a compact set, because that t would be coming from bounded domain so continuous function image would be compact set.

So Kn is one side is a compact set is the other compact set, now we have the T2 property of the Euclidean space that two compact sets are separated, so there I am going to get that they are separated however here when you have that for finite horizon you can actually extend I mean larger and larger every time you will be separated.

So we can remain bounded away from Kn almost surely with probability 1 for each and every n. Kn is a compact set because it is this thing put a line segment is an bounded and close set in Euclidean space. And Wt if you look at the graph of Wt, I mean the location since it is continuous function and then it is for a finite time interval that is a compact set and continuous function takes you know compact set to compact set so the image would be compact set.

Now Euclidean space has T2 property I mean T2 topology because two different compact sets are can be separated it T2 separation axiom, so with open neighborhoods so you would be able to say that Wt would be bonded away from Kn almost surely with probability 1 with probability 1 we going to see that it is not going touch Kn.

So what we do is that we do following thing we choose epsilon sufficiently small so that the Brownian motion is touching Fn, Fn is that cylinder this type of looking cylinder so I cannot assure that it would be 0 probability would be 0, I cannot assure because it could as close as possible you know it could be you know but I cannot say that because it has epsilon n so it has positive with but for some t and t is also infinite so for finite horizon again say that that for some epsilon n so probability would be small but infinite this thing is here.

So I can assure that with this given n one third of n one upon three two the power n is a very small number and this epsilon n can be chosen sufficiently small such that this Fn shrinks to Kn so close that the probability the Wt hits Fn the probability becomes lower then one over three to the power n. So this is the way we are going to do now as n is increasing so Fn is closer and closer to 0.

And then the chance of hitting that is larger however we are going to take the probability one over three to power n, n is becoming larger and larger probability you are going to set even smaller so the epsilon n should even even smaller even smaller and that is always possible why because you know Kn is the limit of Fn. For Kn the probability is 0 so as you know with epsilon goes to 0 that probability that hitting Fn goes to 0 so from that I would always be able to find out some epsilon n for which the probability is non 0 but less than one over three to the n.

So that is the construction choice of epsilon n and then if we compute this probability that sigma D is equal to 0, W0 is equal to 0 so conditional probability. So this conditional probability is smaller then this probability why because this is a larger event these set is subset of this what is this, this is saying that Ft hits Fn for some t, so here and n for some n for some T and for some n. I mean these is surely outside the domain of D if Wt ever crosses this then of course sigma D you know I mean then of course it goes beyond the domain D.

And for this to happen for arbitrarily small time t the Brownian motion should exit the domain and when exit the domain so that is a subset or sub event then this event the Wt touches Fn for some t and some n. So here we are not putting the constant that arbitrarily smaller t it hits et cetera, we are not putting any constant so it is a much larger scope, just saying that Wt ever you know touches Fn for some n, so these is but here we are putting lots of constraint, because it must leave and E and then when it leaves, it touches then we are trying to find out the infimum and then infimum should be 0 that mean it touches very fast et cetera.

So we have this thing this very crude upper bond, however this crude upper bond is sufficient for us. Why? Because we know its probability, its probability is one over three to the power n, and this is a geometric series, the geometric series sum is one over three to the n, n is running from one to infinity not zero, so one to infinity so one three plus one over nine plus over twenty seven et cetera.

So one third should be taken common so there we get one plus one third plus et cetera and then that series sum is one over one minus one third and the whole multiplication is half, half is less than 1. So what we have obtained is that this is strictly less 1 now Blumenthal zero one law is saying that this must be 0. So thus 0 is an irregular point.

So what is happening here that since all though 0 is in a boundary but the complement the D complement is very narrow attached to 0 very narrow so it would very fast the Brownian motion would leave the domain very fast the probability is also low 0, so that is the point that if your point a is on the boundary such that your domain is like almost like everywhere almost like everywhere.

So first example was that, that boundary was inside the domain actually you know, boundary was just a puncture disk, puncture center then everywhere domain was there so then that was the irregular point, now here it is not everywhere I mean one particular direction of course the compliment the domain is you know reachable however is very very narrow is coming very narrow way so here. So, here say for example compliment is this part so there is a but if you think that a very narrow way very very narrow way arbitrarily narrow way then apparently it is like everywhere the domain is everywhere. So, 0 is an irregular point.