Introduction to Probabilistic Methods in PDE
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Lecture 21
Dirichlet Problem and Bounded Solution

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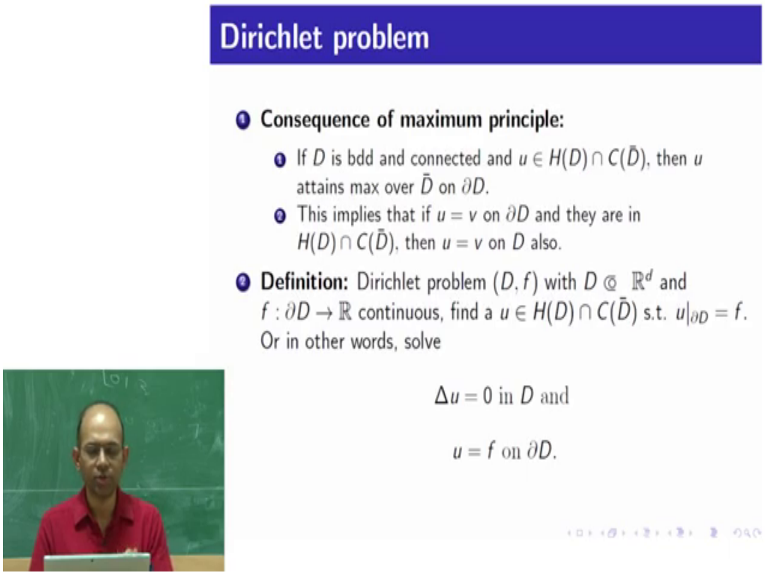

So, next we go to another slide, so next we start the discussion of Dirichlet problem. So, the kind of discussion what we have done just now about maximum principle that is very much relevant to define this Dirichlet problem. What is the consequence of maximum principle that if D is bounded and connected and u is harmonic function on D and continuous on D closure then u attains maximum over D closure on only the periphery only on the boundary, because the maximum cannot be inside if u is non-trivial.

So, this implies that if u and v two different harmonic functions on D, so they are harmonic function on D and also they can be extended continuously on the boundary, so they are continuous on the D closure. And they agree upon they both coincide on the boundary, imagine so they coincide, then this maximum principle dictates that u should coincide with v on the domain itself.

Why is it so? Because u is harmonic v is harmonic then their difference is also harmony, because the differential operator is linear and then u minus v would be 0 on the boundary because both are agreeing on the boundary. And then we are having one example of a harmonic function which is 0 on the boundary, so inside what should be its value? It value cannot be a non-zero, because then maximum would be achieved inside the domain.

So, then u is equal to v on D. Now, we define what is Dirichlet problem, so Dirichlet problem is actually harmonic functions, you know the Laplacian u is equal to 0 on the domain D and on the boundary it is actually satisfying a prescribed function, a function which is already given to us. So, Dirichlet problem needs specification of the domain D the boundary data f, so given D and f this pair where D is an open domain of Rd and f is a function define on the boundary of D real value function which is also taken as continuous here.

So, the problem is to find out a function u which satisfies this or in other words which is harmonic and which is extendable to the boundary, so which is continuous extendable to the boundary, so the continuous on the closure of D such that its restriction on the boundary coincides with f, that is the Dirichlet problem. And we know that from the above result that if solution exist is unique, because if both coincides, so because if u satisfy f on boundary and it has two different solutions both coincide on the boundary, so therefore they should be the same in the whole domain also.

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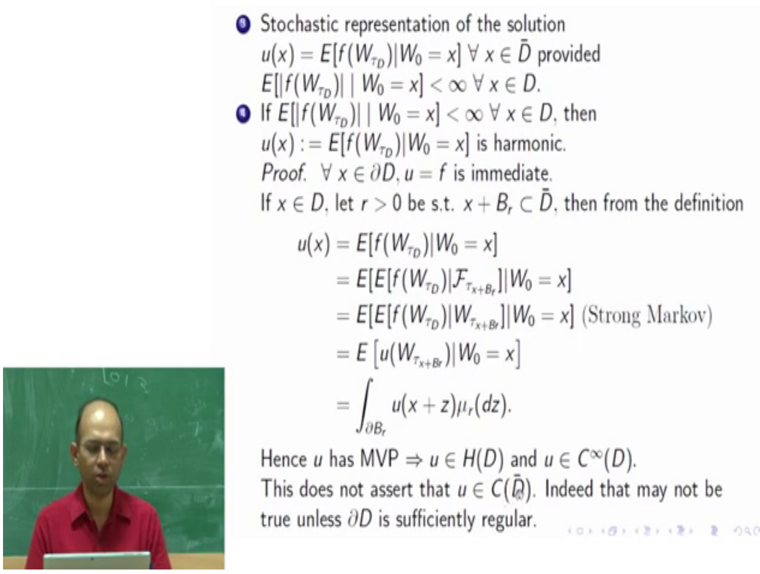

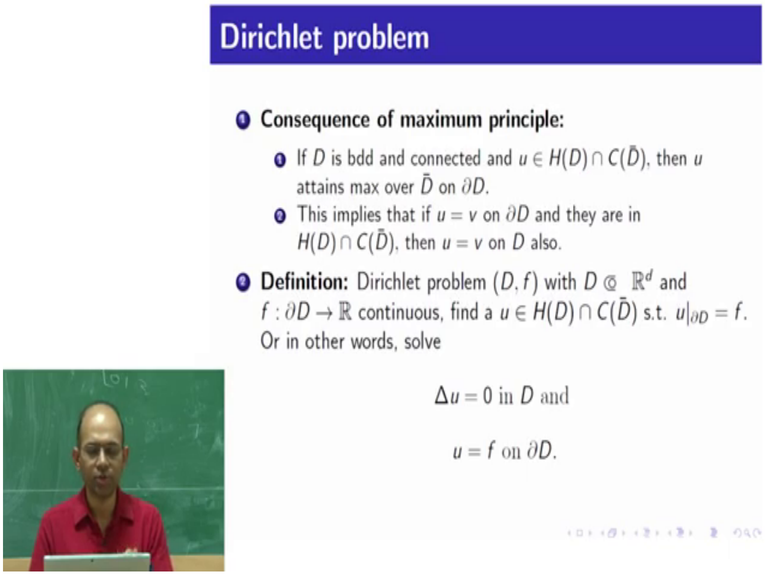

So, here the result what we are going to show in successive slides, so stochastic representation of the solution of this problem is actually given by a conditional expectation u of x is equal to conditional expectation of f of W tau D given W0 is equal to x, what is f here? f is the boundary data capital W is Brownian motion or Wiener process tau D is the stopping time of Brownian motion touching the boundary of the domain or in other words it is the hitting time of the boundary of the domain and this formula depends on a I mean f and given and also D and given if any small x value inside the domain we find we can find out this expectation.

This expectation need not be finite always I mean if we have bounded domain, so then this boundary is a compact set and in continuous function on compact set is bounded function. Then we are talking about taking expectation of a boundary function that would be always finite there is no problem. But when we say that we are considering any connected open domain that open domain need not be bounded, need not be bounded.

So, in that case we cannot assure that this expectation will be finite, we cannot say that. So, we can only write down this way provided that expectation of modulus of f of W tau D is finite, so then we are writing down the justification would be given later so now we just quote the result that we this is our aim our aim is to show that this is the thing.

Now, we start this way consider this conditional expectation and assume that this is finite, so assume that our domain is such that I mean special case already is there as we have mentioned just now that considered boundary domain, however just consider some you know general open domain open connected domain such that the conditional expectation of f of W tau D given W0 is equal to x is finite.

Then we can define u of x in this mean fashion, we are going to show that this is harmonic. So, instead of showing the solution of the PDE has its representation we are doing other way, we are actually considering a special case where this make sense and then we are going to show that this function solves the PDE. So, the in that we have two things one is that Laplacian of u is equal to 0 another is the boundary data.

So, for bounded data is concerned this u define in this fashion clearly satisfy the boundary data, why is it so? Imagine that our x is on the boundary the time 0 Brownian motion is on the boundary, so what should be the hitting time to the boundary? 0, so it is already there, so tau D is exactly 0. So, this conditional expectation is that we are going to read this in the following way, expectation of f of W0 given W0 is equal to x, W0 is no more random, because W0 is equal to x is given.

So, nothing random here, so expectation of this deterministic thing would be just f of W0, where W0 is equal to x, f of x, so u of x is equal to f of x on the boundary. So, here u satisfies the boundary data is immediate. So, for all x in boundary u is equal to f is immediate, now we are considering only the case where x is inside the domain it is an interior point of the domain and we would like to show that this u which is define this following manner is actually harmonic function.

But instead of showing that is harmonic we are going to show that it satisfies mean value property, because we have already establish that the mean value property implies harmonicity. So, u of x is equal to conditional expectation of… No actually we have not shown that harmonicity in the earlier slides, we have shown that harmonicity implies mean value properties.

But mean value property implies harmonicity we have not proved, we have just quoted the result and give the page reference number because that is little longer proof so we have not demonstrated here. So, here for all x in the domain D let r positive be such that x this x is an interior point that a ball around x of radius r is inside the closure of the domain D. Then from the definition we just write down u of x is equal to expectation of f of W tau D given W0 is equal to x.

And then we are using tau property of expectation, what we do? We actually put another condition, so this with respect to the sigma algebra which is finer then the sigma algebra at 0, so it is at the time in later time, so this time is stopping time. The first time the Brownian motion touches the boundary of this ball what we have written here that the starts time 0 it starts at x but the first time when it leaves the ball of radius r around x that is tau x plus Br, so this is stopping time.

So, here filtration at stopping time, so this is sigma algebra, so that we condition in between, conditional expectation of conditional expectation of f of W tau D given F stopping time given W0 is equal to x. Now, we us strong Markovity of Brownian motion, what is strong Markovity that conditional distribution of future given past and present where present is stopping time is same as conditional distribution of future given just present.

So, here this filtration is actually sigma generated by 0 to this time, but that we are replacing by just the present one, the present one is W the Brownian motion at tau x I mean at the time of the stopping time tau. So, now here, so this conditional expectation we look into it, so expectation of f of tau D given W tau x plus Br.

So, here we would imagine that that this point wherever it is at this stopping time and then it proceeds further because still it is inside the domain it would require some a more time to leave the domain D and if you start counting time from there and consider that as 0 time then this conditional expectation is exactly u of this correct, because expectation of f of tau D given W0 is equal to x.

Here we have this point, this is a random point, but given that is the random point you consider that as you present that is 0 time and then you consider and this is again still tau D because that would be time to touch the boundary. So, that is same as u of W tau x plus Br, as you know you get ux here that at W0 whatever the location of the Brownian motion was that you get u of x as the result of this conditional expectation.

So, we also get here that W tau x Br is you know u is evaluated at W tau x Br. Now, this conditional expectation of u of tau x plus Br given W0 is equal to x can be evaluated using integration I mean using the distribution of Brownian motion. So, here we consider that mu r is that already in the last lecture we have seen that mu r is the distribution on the Brownian motion on the boundary of the ball of radius r. So, here x is the center and then u of x plus z where z is on the boundary and on the boundary we are having mu r is the density of the Brownian motion.

And we are integrating with respect to on the boundary set, u of W tau x Br, because you know this point is always on the boundary of Br. So, this conditional expectation is exactly this, so this condition x is coming here. So, what is the conclusion? From here we can see that it satisfy the mean value property. Does not matter what my x is if it is inside the domain D I can always write down that ux as average of you know on the surface of a ball x plus Br.

So, using mu r. So, hence u has mean value property therefore u is harmonic on that open domain D and hence u is also C infinity D. So, this is additional remark this is beyond the promise what we have made in the statement. So, this is one remark which is important say we have obtain that u is C infinity in D, but this not necessarily say that u is continuously extendable to D closure.

We have although shown that u is equal to f on the boundary and f is continuous and u is C infinity in D but that does not necessarily say that if you take a point xn inside the bound domain and u approach xn to the boundary u of xn would converse to f of x, so that is not inside the claim that we are not claiming, it is very important to understand that we are not claiming that, so this no way proves that u is continuous on D closure, it just says on boundary it agrees with f and the domain it is satisfying this.

So, let us look at the problem again, so here, so find u is in harmonic function and this thing continuous on D closure and then we would say that that is the solution, that is the notion of this Dirichletl problem. But here what we have done is that when this is finite then you have shown that u satisfies boundary data, boundary condition and also harmonic, but that does not say that in this.

So, we have not proved that u actually solves this Dirichlet problem, we have shown that this two things separately I mean this line we have proved, this line we have proved and this fact we have proved but this is not done, this needs additional condition on the geometry of the boundary, so that is required. So, this needs some sufficiently regular.

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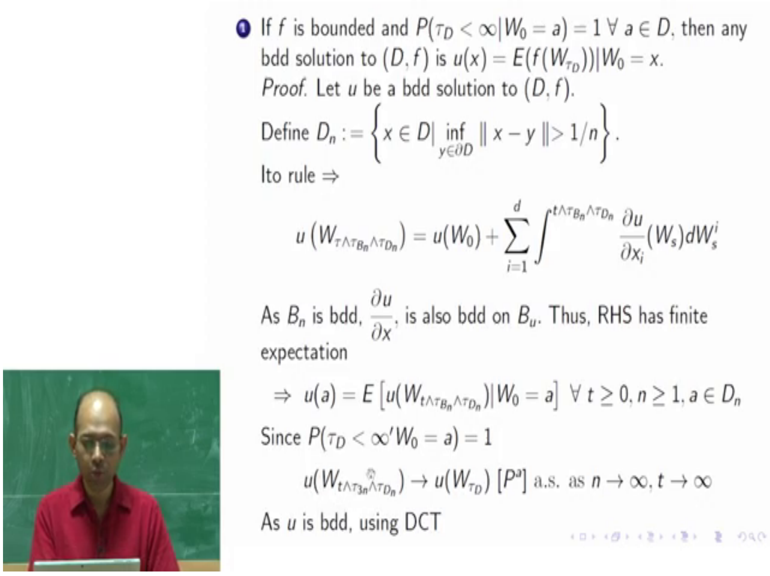

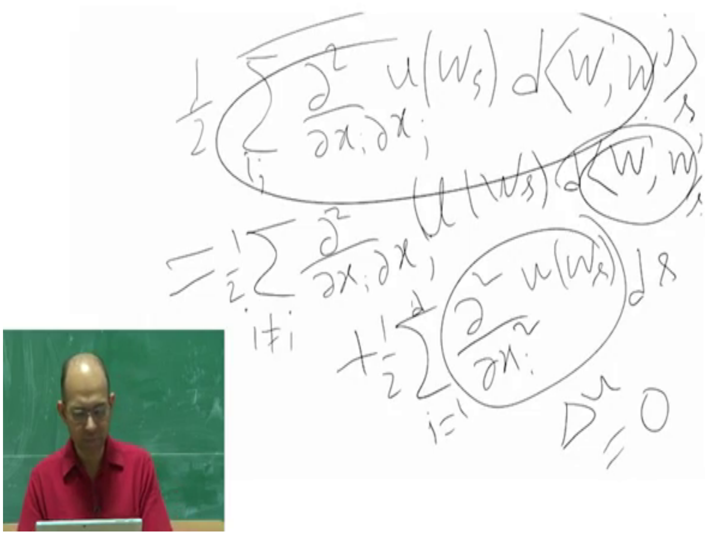

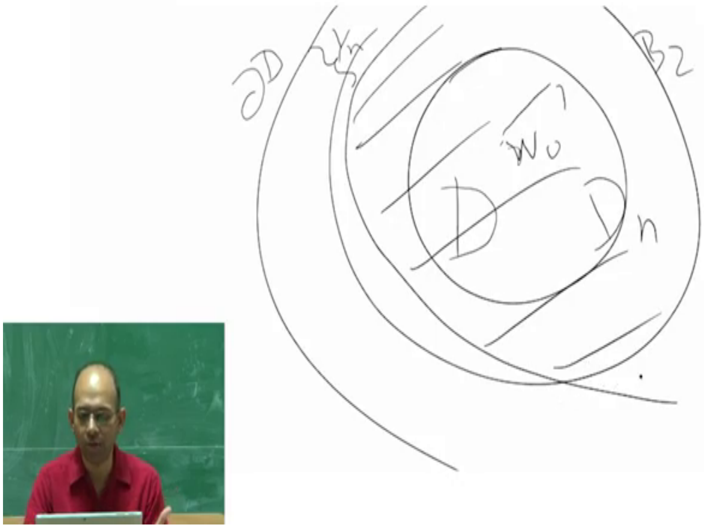

We are going to see more detail, I mean it would take some time to go into the deeper discussion about that. If f is bounded so now let us go to even smaller classes some special cases, if f is bounded and the stopping time or hitting time to the boundary is finite with probability 1 for all a in D, so this is clearly the case when domain is bounded, if domain is bounded the Brownian motion starts here and of course it touches after some time I mean it never touches and stays there forever the probability is 0.

So, probability that tau D is finite that mean that a finite time it touches that is probability 1, the probability is 1. So, this is true for a very special case where D is bounded but we are not assuming D is bounded here we are just saying that let us assume our D such that this is true. So, perhaps some other unbounded domain also may satisfy this condition. So, we are allowing that scope and we also we are assuming f is bounded and there is reason we need assume this I mean if D is already bounded domain we did not need to assume f is bounded, but since we are not putting that restriction, so we are allowing us to be little more general we are just assuming this condition and f is bounded.

Then any bounded solution of D f is indeed this conditional expectation there is typo this bracket would appear here conditional expectation of f of W tau D given W0 is equal to x. So, this is a positive result, this is complete result, so earlier was the intuition behind that why should one expect such kind of representation but under this special condition one can actually get it.

So, we are going to see the proof of that, so let u be a bounded solution of the domain D comma f, why do we talk about bounded solution here? The main reason is that f is taken to a bounded and we are looking for solution of Dirichlet problem, so u is harmonic. So, harmonic function has maximum principle, so the maximum value whatever it can take would be f and f is bounded so u is bounded.

So, let u be a bounded solution to D f that does not mean that D f would have I mean so here we are just taking that if so D f has a bounded solution. Now, we consider Dn, Dn is this is all like you know annulus kind of thing. What are these? We are fixing 1 over n a small number and considering x the norm because this is D dimensional space, so x minus y norm so the distance the D matrix distance of x and y is greater than 1 over n.

So, we are going to get a strip, the compliment of a strip where so all possible x y, but then if I choose y to be on the boundary of the domain then given x I am looking at all possible y on the boundary domain and looking at only those x such that the minimum distance is more than 1 over n.

So, that means that if you have a boundary and if you have annulus of annulus means here like it would not be a circular unless but it annulus sphere or some hollow sphere kind of thing of thickness 1 over n and x is below outside of that, x is even inside that x is inside the domain when also it is not it is little away from boundary it is 1 over n distance away from boundary.

So, that is my Dn, so Dn is actually a subset of D which is away from boundary of D. So, the question is Dn open set? Yes, Dn is open because of this I have greater then strictly greater then sign. But Dn could be actually empty set also because if D is very small its radius itself is a less then 1 over n, empty set, but Dn is an open set.

Now, we use Ito's rule for that we take time as minimum of 3 different stopping times, so here I mean this is t actually this not be tau, I mean t is a time and tau Bn and tau Dn, here let us see the proof, for proof we consider bounded solution even if the boundary data is bounded when domain is unbounded we cannot assure that the solution would be bounded also there could be many unbounded solutions we are going to see a very specific example after completing this proof

So, here we consider the bounded solution only. So, let u be a bounded solution to the Dirichlet problem D f define Dn, Dn is the set of points in x such that which are 1 over n distance away from the boundary. So, infimum of over all possible y in boundary and then x minus y so it like now let me see if I can draw something here if domain if this is D and then this is del D and then if this much distance is 1 over n so this part would be Dn I mean this is this whole thing is D and this is Dn.

So, now what we do is that we consider u value a function u evaluated at the position of the Brownian motion at time t minimum this is typo and this should be t minimum the stopping time of ball of radius n and hitting time of the boundary Dn. So, where did we start from? We start from it W0 at time 0 and then we look at the time t if the Brownian motion position, Brownian motion does leave the ball of radius n neither it leaves the domain Dn and if it leaves then we just take minimum of this whatever either ball of radius n or ball of or the Dn.

So, let me clarify that Dn here is an open set, because we have strict inequality here, it could be an empty set but if it non-empty it would also open set. So, now we use Ito's rule, so Ito's rule would imply that u of W of t minimum tau Bn minimum tau Dn is equal to u of W0 plus integration of del u del x dW i, so it is multidimensional Ito's formula here along all components we need to integrated i is equal to 1 to d, also it has Ito's formula gives another term second term that second del order derivative del 2u del xi et cetera.

And but there in that term we are going to get dWi dWj so cross covariations, cross covariations of W and then second order derivative of u. Now, there what we are going to observe is that that term would not appear here, why? Because when i and j are different since a d dimensional standard Brownian motion has all components independent so cross covariation of dWi and dWj would be 0, when i is not equals to j.

So, here out of this n square terms n square minus n number of terms would disappear and then the terms where i an j are equal those would remain, however that would also not contribute here for one single reason that because there we are going to get the summation because dW I mean the quadratic where covariations of Wi with is Wi, W quadratic variation we call that is just s for every i it is the same s.

So, the summation would be only on this term which would remain there, so let me tell you write down this the term what we have not written is del 2 del xi del xj u of Ws d <Wi Wj> s this term we have not written, I mean the reason is that in this term as we have already sum over all possible i and j as we have already mention that the sum can be written in 2 parts, i not equals to j part, i not equals to j part.

And in this part this quadratic covariations becomes 0 because these W and Wj are independent, so this part become 0 and another part is i is equal to j part, so i is equal to 1 to d, d dimensional things del 2 del xi square, because i is equal to j now u Ws and then d <Wi Wi>s, so that is like quadratic variations so ds, however this part is Laplacian of u but u is harmonic so this is also 0.

So, this whole term becomes 0 here, so that is the reason that so we do not get the second order term here Ito's rule implies and also that u is harmonic together implies that u of W t minimum, so here it is t minimum tau Bn tau Dn. And what is tau Bn here I mean so if you look into this thing say, you have started from here so that is that you imagine that your W0 and then you take ball of radius say 1 and then ball of radius say 2 so this would be B2 et cetera, consider, consider taking various different ball of radius this thing.

Why are you doing so? Because D itself is not need not be bounded so therefore Dn also need not be bounded however this Bn should be bounded for each and every n. So, here this tau Bn minimum tau Dn which is same as tau of Bn intersection Dn, so this becomes bounded domain. So, as Bn is bounded on that bounded domain, the continuous function del u del x, u is del u del x continuous function, so that is continuous function on a bounded domain is also bounded, because it is continuous function on the whole domain and if you take a bounded domain there and the closure is also bounded, so closure is closed and bounded on a compact set.

So, on a compact set the function is bounded. So, del u del x is also bounded on Bn so this is n, there is this typo. So, here on the right hand side here whenever s is running between 0 to this time, this time W never leaves the domain Dn and Bn and therefore the value of del u del x which are explode are always less than equals to the maximum value where del u del x on Bn.

So, this part remains bounded. Since this integrand is bounded, so it is square integrable class and then this stochastic integration of this bounded integrand with this Brownian motion is a martingale. Therefore its expectation is 0 and finite some of expectation that would also be 0. So, expectation of u I mean this left hand side is nothing but expectation of u W0 but W0 is certain so is u W0.

So, W0 is the starting point of Brownian motion which we take as a the point a and a is in the domain Dn as we have seen in the picture, so this is our a it is in Dn. So, here u of a is expectation of u of Wt minimum tau Bn minimum tau Dn yes now here it is correct, so this is t, so this t is appearing here, given W0 is equal to a. So, this is true for all possible t non negative and all possible n, n is equal to 1 2 3 et cetera.

And also for possible a in Dn this is a vertical line here, so since probability that so hitting time of the boundary finite is 1 because that is a underline assumption we started with this, so since this is the fact, so that means starting from a in a finite time it would surely touch Dn I mean this tau Dn. So, now as we are letting n tends to infinity what would happen? That when n is very vey large so then this time to leave the ball Bn would increase. So, tau Bn goes to infinity, tau Bn because you know tau Bn is a stopping the hitting time of the boundary of ball radius n.

So, n tends to infinity this tau Bn becomes large and larger and also you allow n tends to infinity, so t also become large and larger so this thing. So, if you now consider the minimum of this 3 things then tau Dn becomes the minimum. So, as n tends to infinity this becomes large and then this random variable converges to u W tau D almost surely the P superscript a is saying that the conditional with the conditional probability that we start at point a.

So, this is not 3 this is B actually Bn almost surely n tends to infinity t tends to infinity. So, now we would use dominated convergences theorem, why? Because here u is bounded on because here we have started with a bounded solution, since we have started the bounded solution so we have we would take the advantage of that. So, we have now here is sequence of random variables so bounded random variable since u is bounded which converges almost surely.

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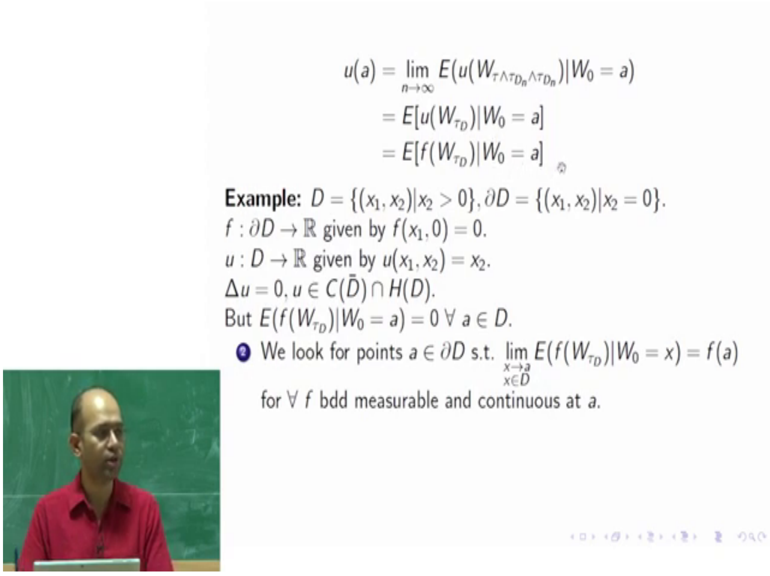

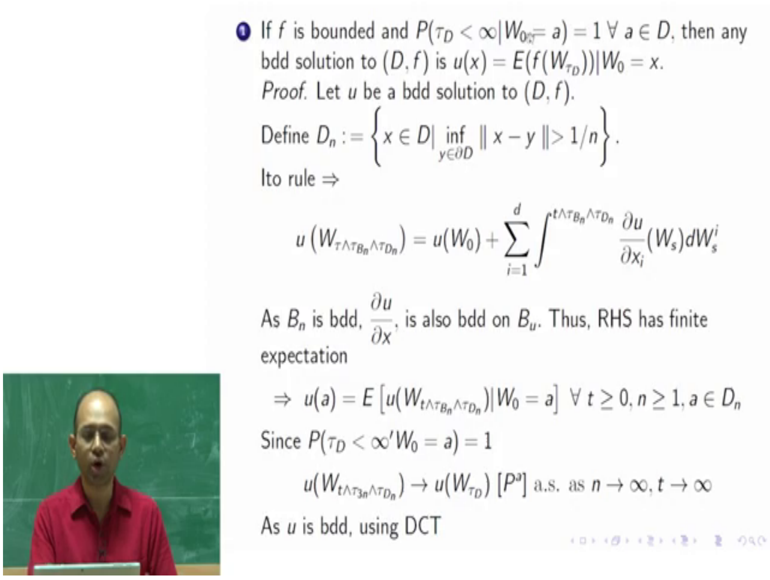

Now, our goal is that expectation of this, sequence of expectation that would also converse to this expectation of right hand side. So, u of a is equal to limit n tends to infinity that I can write, why? Because this is true for all possible n so I can take limit n tends to infinity here, so we are doing precisely limit n tends to infinity u of Wt again t here t minimum tau Dn tau Bn not both the Dn, Bn minimum Dn given W0 is equal to a so that a is expectation of u W tau D using this part here we are using dominated convergences theorem DCT that we can take limit inside, W0 is equal to a, however what is the value of u at the position W tau D?

So, tau D is a time when the Brownian motion touches the boundary of D but when that means this is the locus of at a boundary at that location u is satisfying the boundary condition the boundary data is f, so u of W tau D is f, why? Because u satisfy the Dirichlet problem bounded solution of the Dirichlet problem, this is expectation f W tau D, so what did we prove?

We have proved that if u is a bounded solution of the Dirichlet problem, D comma f where D is such that stopping time or hitting time is finite almost surely and f is bounded then that solution of the Dirichlet problem has this representation can be written this way expectation of f of W tau D given W0 is equal to a, the problem has this solution.