Introduction to Probabilistic Methods in PDE
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Lecture 20
Maximum Principle of Harmonic Function

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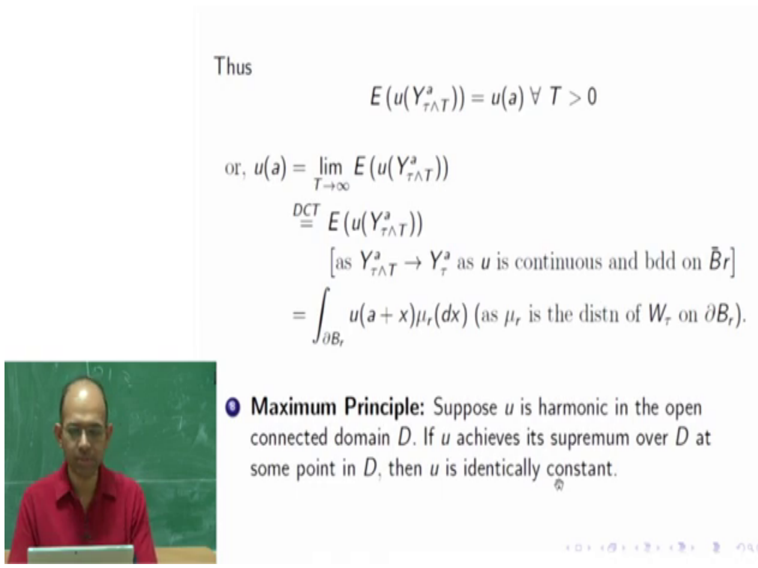

Today we are going to see Maximum Principle for Harmonic Function, suppose u is harmonic in the open connected domain D, so for this part of discussion, we would consistently use this notation capital D would be used to denote an open connected domain of d-dimensional Euclidean space. If u the harmonic function which is defined on open D, if u achieves its supremum on over D at some point in D, then u is identically constant, so this is the statement of the maximum principle, we are going to see the proof of this fact.

Basically before going to the proof let us try to understand what does it imply, It implies that only constant function constant harmonic function has the property that maximum is value is achieved in the domain of D, but other than constant function, why constant function is harmonic? Because harmonic functions are the functions whose second order derivative is 0 that Laplacian of u is 0, in one dimension case it is just second order derivative in higher order case it is trace of the second order derivative.

So, constant function is of course harmonic in that sense because its derivative is 0, however if you have a trivial harmonic function, then it actually goes up and down etcetera and then the maximum of u over this open domain that value would be something strictly more than all the values you have taken inside the domain. So, if you extend this function over the domain over the boundary then possible you know you would see the maximum values achieved in the boundary on the boundary, if you can do that.

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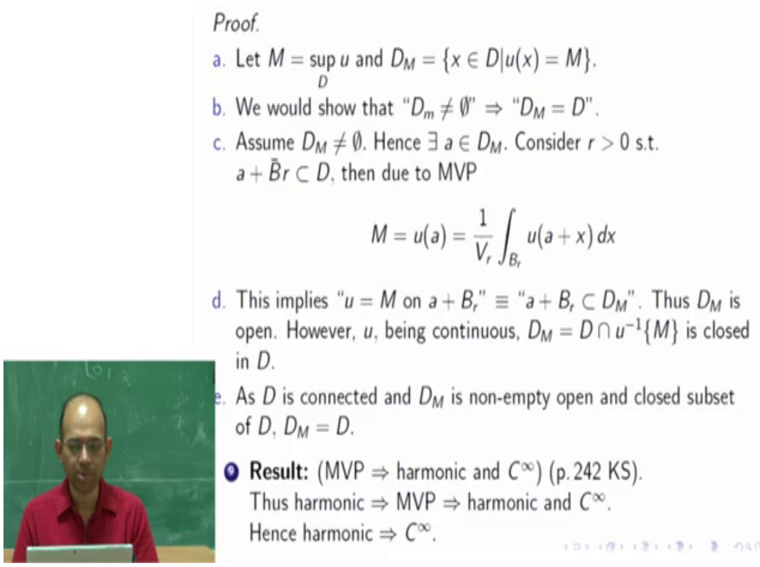

So, the proofs start here, this proof has many steps, it is not long proof just one page long. So, let M be the maximum of u, the supremum of u you cannot say maximum because it may not be release, supremum of open domain D of the function u, we also denote D subscript M, D subscript M is a subset of the open domain D, it is basically level set, the level set, it is not lower level set, it is exactly the contour actually, that when ux is equal to M, this is setup points on D where u take the value capital M.

If u does not take the maximum value at any point in the domain then this should be basically empty set, so our goal is to show that this empty set, so that is the statement of the theorem, that it should be empty set. So, for that we prove by contradiction, what we do is that we is not actually by contradiction it is just saying that if we consider that this set is non-empty then we should obtain that DM is equal to D itself, we should yes, it should be capital M, this capital M, this capital M should appear on the subscript.

So, our plan of proof is that, if the set in non-empty we should obtain that this set is exerting the whole set, what does it mean? It means that the function is constant, every point is maximum that means the function is not changing its constant function, or in other words except constant function for any other case this DM would be empty contra positively. So, we proceed in the following manner we assume that DM is not empty, so what it mean?

That mean there exist a point a in the set DM and then consider since a is also in the subset of capital D point in Capital D and Capital D is an open domain, so I can consider a ball of radius r around this point such that the closure of the ball is still contain in the domain D, we do that. So, consider sufficiently small r positive such that of the closure the ball of radius r around the point a is inside the domain D.

Then due to a mean value property, because mean value property we can use now as u is harmonic function, we have already shown that harmonic function has mean value property, so here capital M is equal u of a basically because a is in DM, so we have this. Now, using mean value property we are writing this equality, u of a is equal to the volume of this ball Vr is nothing but the volume of Br and then integration of u over this ball, over this ball a plus Br.

So, how do you do that? we write u of a plus x, x is run over Vr is a synonymous, so we get this. Now, u of a plus x, these values, can neither be more then capital M, because a plus x is always in D, because B this is inside D. So, maximum of u on D is M, so u of a plus x cannot be more then M, however this average is M that can only be achieved if u is equal to M almost everywhere, however u is continuous because u is a harmonic, it is continuous.

So, almost everywhere equal to M is basically saying that it is exact equal to M everywhere inside this ball u of a plus x where x is here that means u is capital M on this ball of radius r around the point a. So, this implies that this ball is subset of DM, because DM is basically the set where u takes value M, so therefore this open ball is inside DM.

So, what does the consequence we have picked up one point a from DM, and now we have obtain that a is an interior point of DM, because a plus some ball that is in DM, so that proofs that DM is open. However DM can also be rewritten in another way that u inverse of capital M, so here this is the definition, u inverse of ux is equal to M for all those x in D that means this is intersection of D and u inverse of M.

So, DM can also be written as D intersection u inverse of M. So, if we look at the subspace topology of D, so there DM is a close subset, because u is a continuous function continuous so inverse image of a close set under continuous function is closed, so we are going to get DM is closed in D. So, we have obtained both, we have obtain DM is open and also DM is closed, open in D and closed in D and D is also connected.

So, all the possibility is that DM must be either D or empty, but we have started with non-empty, so it should be D, so DM is D. So, that is the proof, so this is whatever I just told it is written here as D is connected, D is non-empty and therefore DM is D. So, the result is that let me summarise then that mean value property implies harmonic function and C infinity, so which we have not proved here but I am just giving a reference page number 242 from karatzas and Shreves book which is our reference book.

So, this part and the other part that harmonic function is having mean value property and so this is actually not the consequence of this particular theorem but about our discussion before closing this part and harmonic functions having mean value property and mean value property implies harmonicity and also C infinity infinitely differentiability.