Introduction to Probabilistic Methods in PDE

Doctor Anindyan Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture 02

Prerequisite Measure Theory (Part 02)

So, now we talk about statements. So, for example when you say that a sequence of real numbers Xn converges to X. This is a statement correct however if my Xn are random variables. So, then the statement need some more clarifications that convergence in which sense because Xn take every n Xn takes various different values for various different sample points. So, these statements would be clarified only if you know we introduced the notion of all the adjective of a statement.

So, that is the thing we are going to do but before that but let us talk about what is random variable? What is the what does it mean? So, we start with the measurable function.

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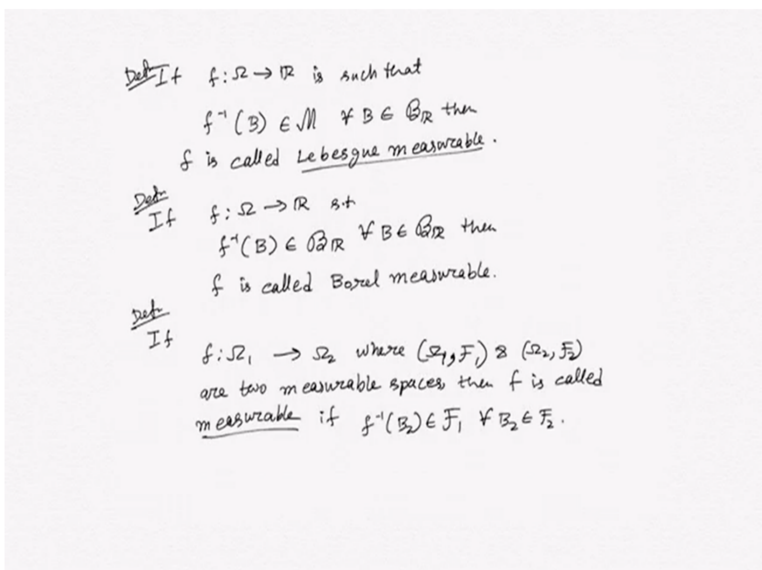

So, if f from Omega to R is such that f inverse of B is in the Lebesgue sigma algebra for all B in the borel sigma algebra of R then you say f is Lebesgue measureable. On the other hand if I have f from Omega to R such that f inverse of B in borel sigma algebra for all B in the borel sigma algebra of R then we say if is borel measurable more generally.

So, this general setting we would not use much but more generally if we have f from Omega 1 to Omega 2. Where Omega 1 F1 and Omega 2 F2 are two different measureable spaces. These are measureable spaces, then f is called measureable if f inverse of B 2 is in F1 for all B 2 in F2. So, that is the definition of measureable functions in the general situation.

So, as we would use for this course, which is which would assume this mathematical model of probability that includes these you know events would be measureable sets and the random variables and measureable functions.

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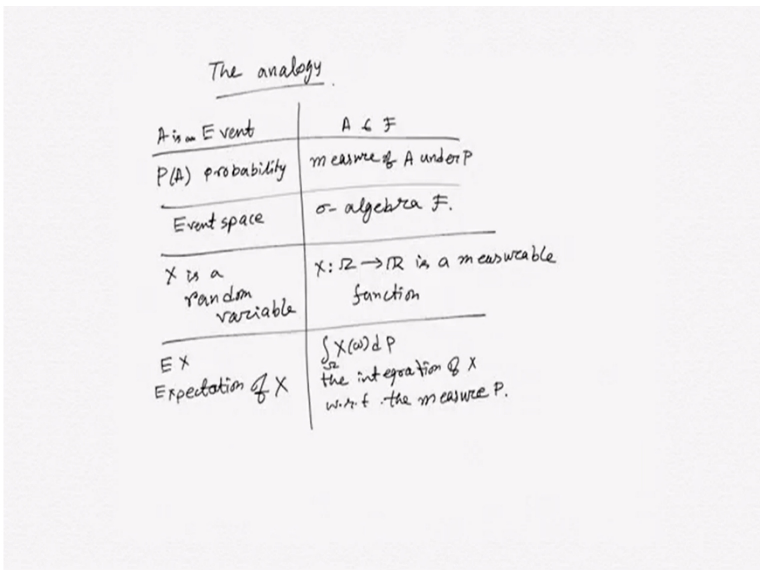

So, let us write down. So, what are the differences? So, when you say that A is an event A is an event we mean that A belongs to the sigma algebra. So, we have say (Omega,F,P) so Omega F P is the probability space as we have introduced earlier. So, A is an event that means A belongs to F there is the terminology and we say that P of A the probability, and that is nothing but measure of A correct measure of a measure of A under P.

And what we say that event space so event space, space of all events that is sigma algebra script F and what we say random variable. So, we say X is a random variable and that means that X from Omega to R is a measurable function. So, these are the terminologies if we say expectation of X, expectation of X and here we mean that integration of X Omega with respect to the measure P. So, this is integration, so these are the terms and terminologies we were going to use now if we have a statement.

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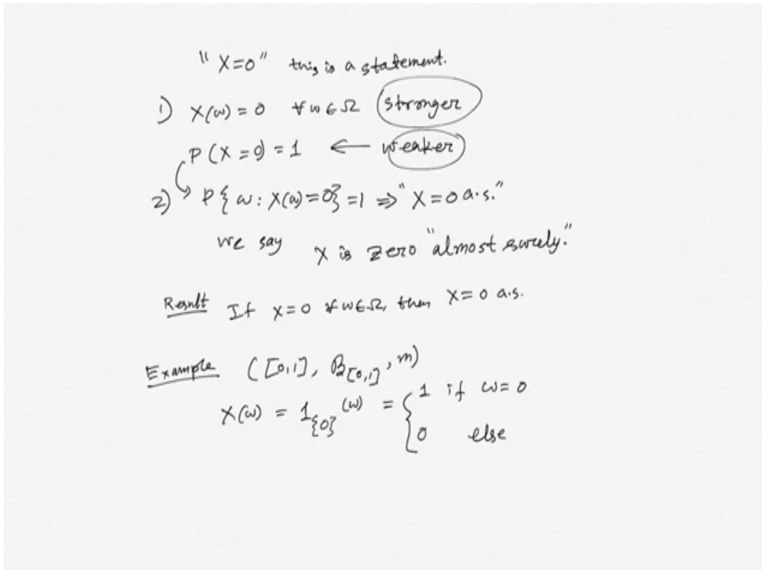

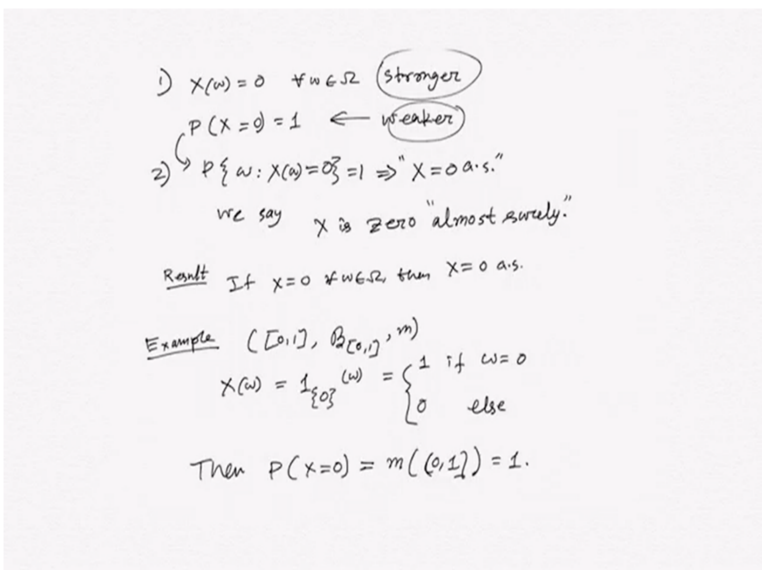

Say for example X is equal to 0 the simplest statement. But now X is a random variable is not a real number just constant. So, that is X of omega is equal to 0 but for what the question is that for all omega or you have another notion that probability that X is equal to 0 is 1 that means probability of set of all omegas such that X of omega is equal to 0 is 1.

So, this if this is the situation we understand that with this situation is weaker than the first situation this is the first situation this is the second situation. So, or in other words what I to want to say is that if and we call that this part if this is true. We say X is equal to 0 almost surely with

probability 1. So, if X is equal to 0 for all omega that means X is equal to 0 almost surely but the opposite is not true. So, let us see one previous easy example that we consider closed 0 1 and here borel sigma algebra on closed 0 1.

And the measure we take the living measure and then we consider X of omega this random variable is defined as indicator function of 0. So, what does it mean it means that this value is 1 if omega is equal to 0 0 else. So, this is a very simple function, so I mean this is only you know one on yet one in a point and if we ask that what is it 0 everywhere no it is not 0 everywhere. So, this is not falling into the circumstance which is written in the first one but of course it falls in the second one why it is.

So, because probability of X is equal to 0 is equal to probability or that means here probability is the, is the lebesgue measure. So, we done this is lebesgue measure of the set open 0 closed 1 with all the points which is between open 0 closed and closed 1 for that my random variable X is equal to 0 but what is the measure of this the measure of this interval is the length of the interval that is one. So, if this falls into the second category. So, this clarifies these meaning of almost sure statements.

So, one can therefore use this so this all a.s. is a small abbreviation of the full form almost surely. So, this acts as an adjective for some other statements like Xn converges to a random variable X. This is a set statement one can also consider the statement with almost surely so with probability 1 it happens.

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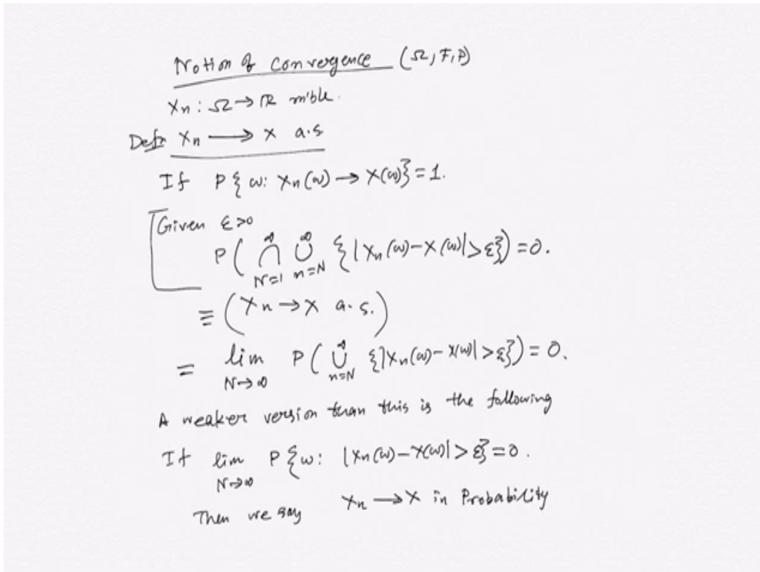

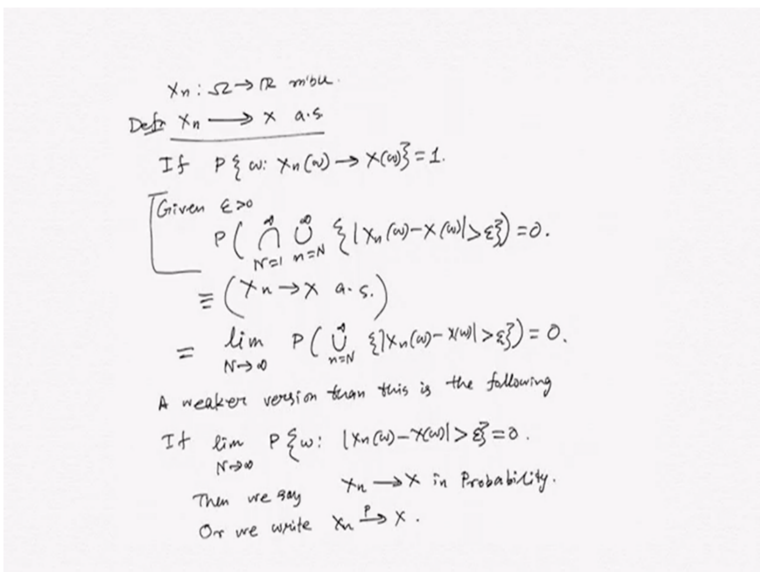

Now, we talk about some notions of convergences so notion of convergence. So, we say that Xn converges to X almost surely. So, if the probability of events so here I have probability space (Omega ,F, P) and my sequence Xn as I am discussing here is a function from omega to R. So, real valued and they are measureable, measureable with respect to sigma algebra F sigma algebra correct you are measureable with this to F sigma algebra I am not clarifying what is my F and some abstract sub-sigma extra sigma you know whatever you start with one can start with the borel sigma algebra or maybe even smaller than that and then probability of omega such that Xn of omega converges to X.

So, this is one so this can also be written in many different ways like you know one can ask that mod of Xn omega the difference one can talk about the difference, of given epsilon given epsilon positive if we consider this difference and ask that when these difference more than epsilon since Xn converges to X. So, of course this difference would after certain time n would be less than given any epsilon.

So, if we consider this thing and consider union of for all n is equal to capital N to infinity. This the set and this set I am taking union N to infinity And whatever I am going to get is dependent on capital N and then we take capital N is equal to 1 to infinity and then this whole thing if that measure is 0 then this is synonymous to the top line.

So, this is synonymous of saying the Xn converges to X almost surely, there is another notion of convergence. Say for example since this set is a decreasing family of sets we call nested events and then intersection is there and probability measure is a finite measure we can actually take this limit outside this limit credit. So, outside so this is synonymous of saying the limit capital N tends to infinity probability of N is equal to capital N to infinity the set of events such that Xn omega minus X of omega.

So, this is greater than an epsilon is equal to 0, however the weaker version a weaker version could be so what is a weaker version say instead of asking the probability of this whole union that goes to 0 we can ask the probability of only this thing goes to 0. There is of course a weaker version because union is a large set you are saying the large sets you know the sequence of the larger that largest sequence is going to 0 and weaker version is that you do not have this you just write down limit.

This is a weaker version probability that omega such that mod of Xn omega minus X of omega this is greater than epsilon. This goes to 0 this is weaker version and it has a name. So, if this is true so if this is true then we say Xn converges to X in probability in probability often we just write down Xn and then P above or below this short very short notation Xn converges to X probability. So, these are the two different notions of convergence we are discussing now.

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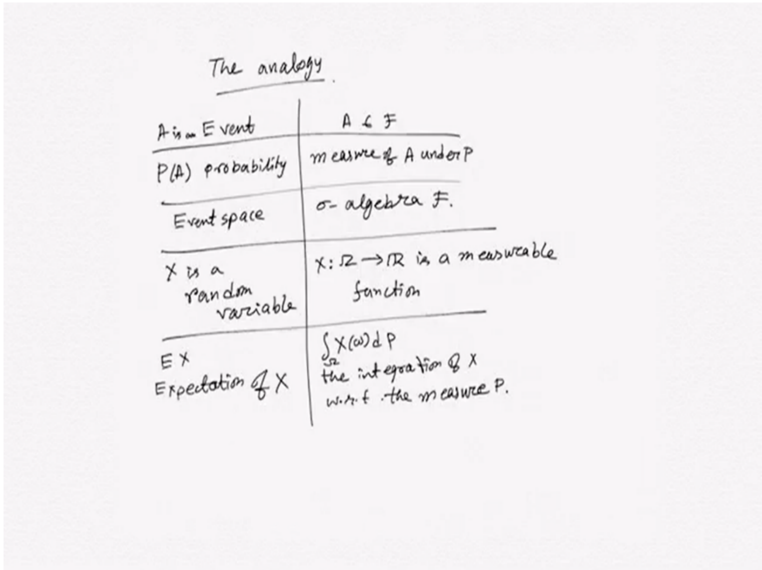

So, next we introduce what is the integration of a measureable function that would give me what is expectation correct because in this table we have clarified that if I have a measureable function X and measure P and then if we integrate that would give me expectation of X. So, what do you mean by integration. For integration theory we should first start with the simplest possible function and we also give the name as simple function what is that?

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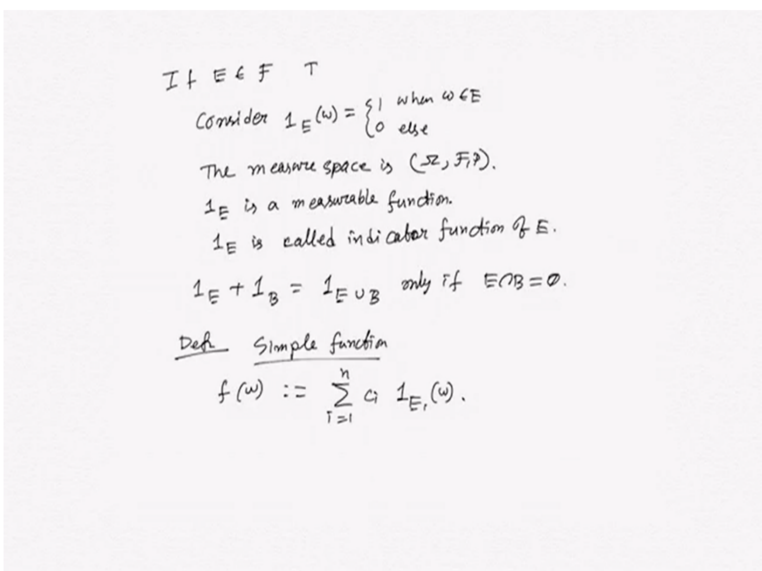

So, if we have E in the sigma algebra F so what is the measureable space we are talking about that Omega F and the measure is P. So, this is the measure space this is the measure space we are starting with measure space and then E is in capital F. So, the E is in the sigma algebra so that means E is the measureable set here that is the definition measureable set correct. So, a measureable set is nothing but a set which is in the sigma algebra so I mean also know I must clarify it depends on the sigma algebra.

So, if we change the sigma algebra same set may not remain measureable. So, always when (())(17:49) is a measurable set there one should also clarify under which sigma algebra. This is measurable sometimes when we know the choice of sigma algebra and this is there is a very you know immediate default choice then we sometimes do not mention. We just say that X is measurable. However when people we discuss about various different sigma algebras and confusion may arise we always clarify that with respect to which sigma algebra this is measurable.

If E is in F that means is measureable consider a function 1E of omega this a function so this takes only two values it takes value 1 when omega is in E and 0 else. So, this is of course a measurable function why because inverse image of 1 is E is measureable and any borel set you take which includes I mean either 1 or 0.

Its inverse image would be either E or E complement if it even includes 0 and 1 both then its inverse image would be Omega itself that is also in the Sigma algebra F if your set B does not include either of 1 and 0. Its inverse under 1 E would be empty set that is also in the same algebra F so this 1E is it a measureable function. So, this measureable function is called indicator function is also called indicator function.

So, this indicator function would be used in the course very often I mean this function and then if I add two indicator functions. Then I will get again another indicator function if that sets are disjoint correct. So, like 1 of E plus 1 of say B would be 1 of E union B only if E and B are disjoint correct disjoint if they are not disjoint then it is not going because you know that E and B are exactly the same then you are going to get 2 on that thing.

So, here we are now we need to extend the class of indicator functions because a addition of two integral functions need that we indicator function if they are in of that is indicator only if there E and B are disjoint. So, we give we consider that collection we call that simple functions simple functions what are simple functions? Simple functions are nothing but linear combinations of indicator functions. So, finitely many so you if you have that summation of Ci 1 of Ei i is equal to 1 to n.

So, this type of function which is finite sum of some constant multiple of indicator functions then we then we call that as a simple function. So, this is a simple function, now for simple function we would introduce what is the integration of the simple functions before that if I integrate a function it is very easy when we try to go to the next slide.

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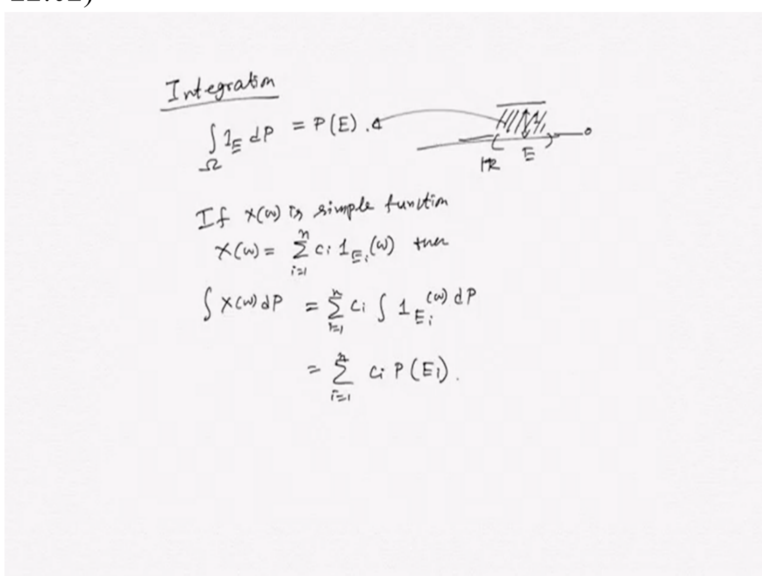

So, here we are going to define integration first we would define integration for indicators function then we will define integration for simple function, for indicator function, integration of 1 E where E measurable set and dP. So, this thing is I need to introduce correct I need to define so whatever way I mean I would define. That should be meaningful in the sense that that should be that should coincide with the purpose of integration.

What is the purpose of integration that integration of a function of a non-negative function gives its area under the curve but got it, if you F is a very nice continuous bounded function non-negative function and then if you integrate that F on interval 0 to 1 say finite interval. So, then that integration value should be area under the graph of that function so that is the basic purpose of integration. So, if we keep that purpose in our mind so integration of indicator function of E so where it say if we draw this.

Say for example this is the real line and E is this set subset of this indicator function would look like 1 here and 0 outside everywhere and then area and the function will be the exactly this and then what will be its measurement. So, measurement will be this side would be the measure of this base and the height multiplication of the height and height is 1. So, the whole area would be 1 into measurement of the base.

So, it would be exactly measurement of the base so P of E. So, that is the that is the definition what we get from the purpose of integration. So, we fix it we keep it and then for any simple function if X of omega is simple then we can write down X of omega as summation finite sum i is equal to 1 to n Ci 1Ei of omega. This way from the very definition and then we also need integration to be a linear function why linear.

Because you know if you have a integration of F and integration G separately calculated the integration of the sum of F and G should be some of their integrations. So, that is true for even area under the curve that notion. So, here integration of X of omega dP would be summation i is equal to 1 to n Ci integration of 1Ei of omega dP from the definition of in integration of indicator function. What we get is that i is equal to 1 to n Ci P of Ei. So, we have now values of integration of in the simple functions.