Introduction to Probabilistic Methods in PDE
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Lecture 19
Harmonic Function and its Properties

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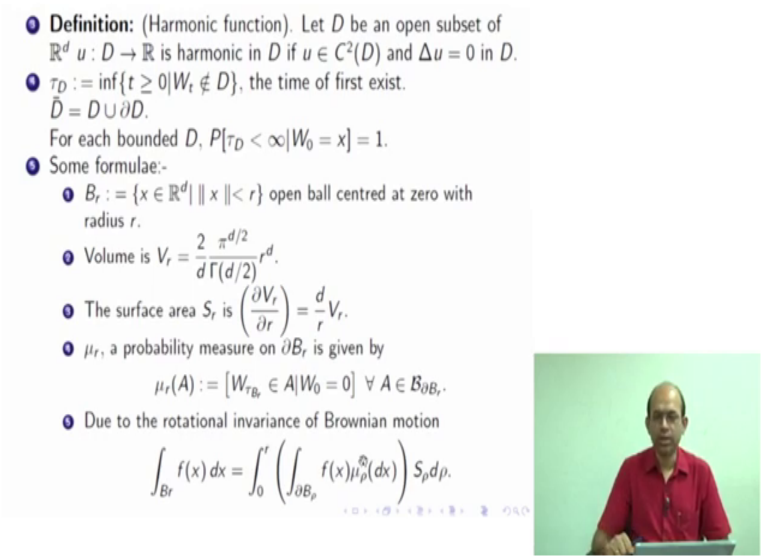

Next, we go to a different topic what is Harmonic Function. So, this is a topic in analysis this course is aiming to discuss probabilistic method in solving problems in analysis including differential equations, so here we are going to discuss few such properties of harmonic functions which are you know sub topics of analysis and we are going to explore those things using Brownian motion here.

So, we first define what do you mean by a harmonic function, let D be an open subset of Euclidean space of d dimension euclidean space Rd, let u be a function from D to R, so D is open domain and u is real valued function define on D, we call this function u harmonic in D if u is twice continuously differentiable on the domain D furthermore its Laplacian value is 0 everywhere in D.

So, what is Laplacian? Laplacian is the second order derivative, if it is coming from one dimension space to R then we just say D2u Dx2 but it is multi say 2 dimensional space then we say del 2 del x del x square plus del 2 del y square of u is equal to 0. So, similarly for high dimension one gets the form of the Laplacian. Now, we consider stopping time tau D, tau D is define to be the exit time by the Brownian motion from the domain D, so this is a stopping time, why? Because Brownian motion is Ft adapted and exit time of Ft adapted process is stopping to with respect to the same filtration Ft.

Now, we consider the closer of D, we write down as D closer and that is equal to D union the boundary of D, so that is there is the boundary we have, D closer is equal to D union boundary of D. For each bounded domain D probability that the stopping time is finite is equal to 1 does not matter where the Brownian motion starts, so if consider the Brownian motion starts from a point which is outside the domain then of course tau D would be 0, because it is already outside.

Student: Sir, if it is D is bounded domain or?

Professor: We have started with any open subset, need not be even connected, but here bounded D.

So, when we assume that it is bounded D then if x is outside the D then of course tau D is 0, but if x is inside D Brownian motion hits the boundary of D and exist D in a finite time with probability 1, so it remains in the domain D forever with probability 0, so it would hit and exit in finite time. So, or in other words we write down the tau D is finite with probability 1. Now, we recall certain formula, so here notation B subscript r is denoting the open ball centered at zero with radius r, this is the definition mod x is less then r when x is in Rd.

The volume of Br we write down as Vr, V subscript r and that is 2 by d pi to the power of d by 2 gamma of d by 2 r to the power d, a very simple exercise that you cross verify for say d is equal to one case, d is equal to one case it would be pi to the power half below is gamma half, gamma half is also you know pi to the power half, so this gamma and this numerator and denominator would cancel each other and r to the power of 1 would appear and d1, so 2 time r.

So, that is the interval length, so when you have r distance left r distance right then the total length is 2 times r. Next, we go to the surface area Sr, the surface area Sr is obtain by taking partial derivative of V with respect to r, because that is how you get, you integrate surface area with respect to r and you get to the total volume or in other word to get Sr you take partial derivative of V with respect to r. And that is if you do that it is a trivial to check you are going to get d by r Vr.

Next, we introduce a measure mu subscript r we call mu r, a probability measure on the boundary of the ball Br that is given by mu r of A where A is any Borel subset of the boundary of Br, so what does it mean? That you have a Borel subset of Rt and you intersect with the boundary of Br, then what is left that you call the Borel subset of Br? Del Br, so the boundary of Br.

So, you consider any such Borel subset of del Br and then you define the mu r measure of the set A in the following manner this is the probability, probability the P is missing, probability that W tau Br, what is that? That is the Brownian motion value at time when that Brownian motion touches the boundary, so tau Br is actually the exit time of Br ball, so at the time of exit time, so the boundary condition should be found at the boundary itself.

So, we are going to compute the probability, if you compute the probability that this point of Brownian motion is found in the set A, what is the probability that this point of Brownian motion which can exit Br in various different parts of the boundary of the ball but it is found in the set A, so that probability is going to give you a number and that number we are going to call mu r of A, so measure of the set A, so that is the definition of mu r.

Now, ball is symmetric around origin Brownian motion is also symmetric, so this measure is also symmetric, symmetric in the sense that if you have A here and if you rotated A around the origin and the measure would remain the same, measure is not going to change. Due to the rotational invariance of Brownian motion as I have explained that one would obtain that the function fx if we evaluate the function the integration of f on the whole set Br, whole interior and boundary within Br, so whatever we are going to get is exactly same as if we take levels of spheres starting from 0 to radius r of various different radius rho, rho is starting from 0 to r, so that then we are going to get the spheres del Br are the spheres.

And then if we are integrating on the sphere using the measure on the sphere that is mu rho and then that thing whatever number we are going to get that we are again integrating from 0 to r multiplying as S rho d rho, what is S rho? S rho is the measure of the area measure of the sphere. Why do we need to do S rho because mu is normalized it is probability measure, it as you know the surface is growing the measure of the surface would be S rho times mu rho, because mu rho is just the fraction, just the probability.

So, mu rho dx S rho times d rho integration 0 to r, so that is the identity. And now if you replace f by 1, there is special case f by 1 then left hand side you are going to get the volume Vr and here you are going to get f is 1 if this is a probability measure this the whole integration is 1, so you are going to get 0 to r S rho d rho, so we are cross verifying, you can actually retrieve the this you know this thing from here. So, this is a generalization of the above. We are going to use these notations and these relations in the next slide.

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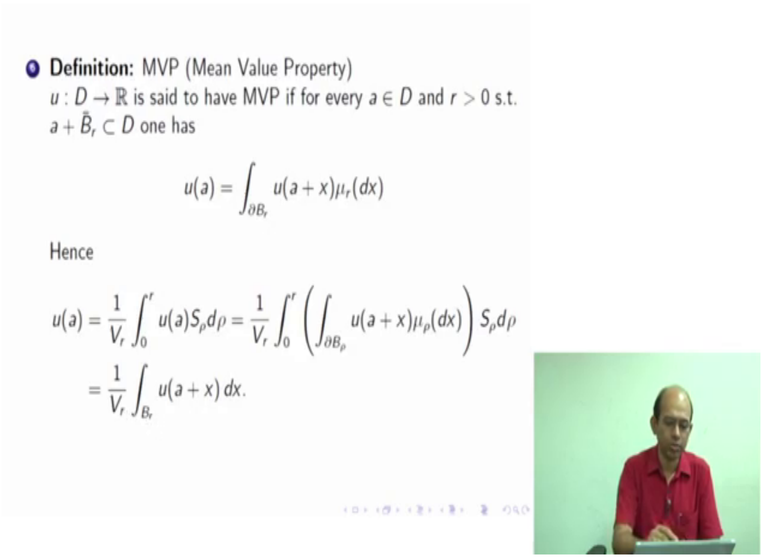

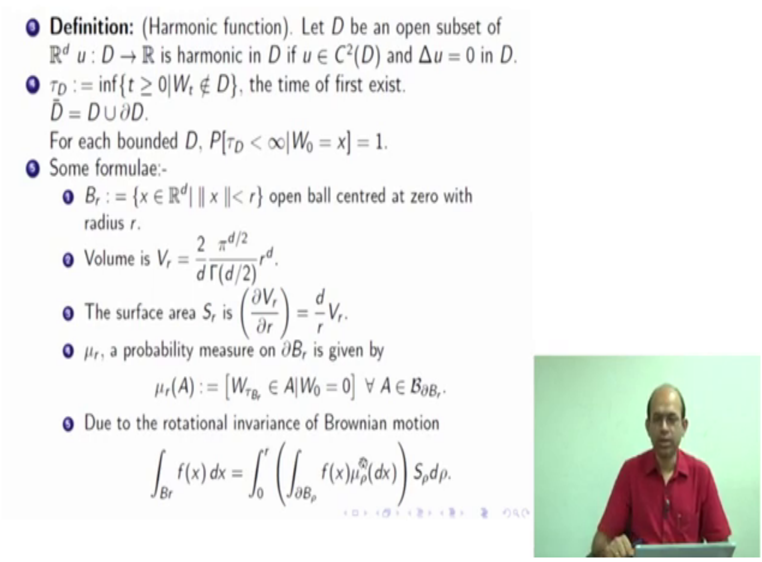

Now, we introduce a notation or a terminology which is call Mean value Property, so let u be a function from the open domain D to R, we call u to have mean value property if for every point a in D and r positive such that you can actually you know include a ball of radius r, r is small enough such that a plus Br bar, so this is nothing but ball of radius r centered around a is inside the domain D, so you can always do that.

So, a plus Br open ball with the subset of D, if it is closer is not in subset of D we can take the half of the radius and then can assure that that would be in the open domain of D. So, we can always obtain such r positive. One has that, u of a is equal to integration of u of a plus x mu r dx where integration is over the boundary of the ball, so this is like average surface area measure u of a plus x mu r dx over del Br, then we call that u is having mean value property.

So, not every function satisfies this property this is like you know saying that if my point a is considered an then we consider a ball around a and then whatever the average value because this is average value where a plus x on the sphere around a, so point a is there but you are considering sphere on the sphere you are evaluating the function u and finding out its average and that is exactly same as u of a there is a very special rigid property, not every function has this property.

So, if a function has this property then we call that function to have mean value property, mean value in the sense the average value, so does not matter whatever sphere you take its average value is assume by the function at the centered of the ball. So, under that assumption for that class of functions we have following relations, so let us see what is this relation. u of a can be written as 1 over Vr, so what is this? Just I have written u of a, but then S rho d rho integration 0 to r is Vr, so Vr numerator and Vr denominator would cancel each other so it would be u of a.

So, u of a is written as u of a multiplied with S rho d rho integration 0 to r and denominator is Vr, then that is equal to 1 over Vr integration 0 to r here we are going to use this formula, this property of ua. So, ua is replaced by u of a plus x integration with respect to mu rho on the sphere of radius rho and S rho d rho. So, here we have obtain exactly what we have seen this relation here so that type of things, so u of a plus x mu rho dx S rho d rho this thing, so this part would be same as u of a plus x integration on the whole ball of radius r, so that is this thing. So, u of a can be written as 1 over Vr integration of u of a plus x dx on the ball Br.

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- If u is harmonic in D, then u has MVP.
  Let $a \in D$ and $r > 0$ such that $a + \bar{B}r \subset D$. Let $Y_t^a = a + W_t \; \forall \; t \geq 0$ and $\tau := \inf\{t > 0 | Y_t^a \notin a + B_r\}$.
  $P(\tau < \infty) = 1$. Then $E\tau \wedge T < \infty$. Again u can be extended to a C_c^2 function beyond $a + \bar{B}r$. Thus by applying (2), we get

$$E\left(u(Y^a_{\tau \wedge T})\right) = u(a) + E\int_0^{\tau \wedge T} \frac{1}{2}\Delta u(Y_s^a)\, ds$$

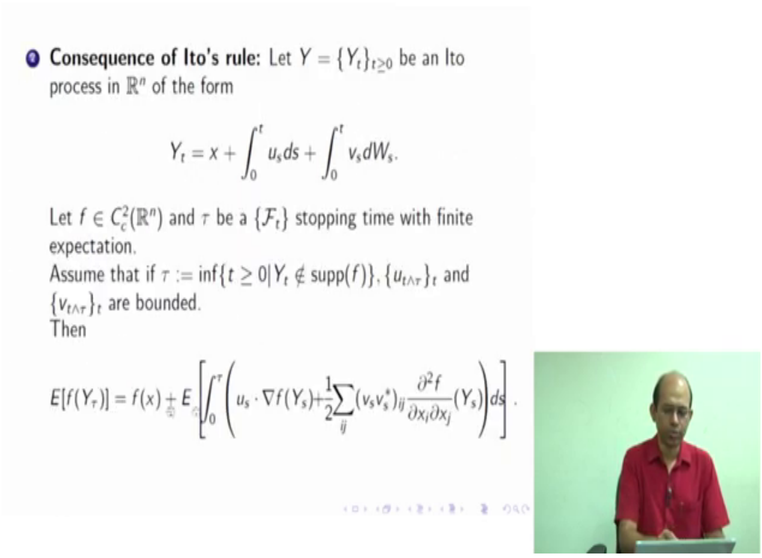
  However, u is harmonic on $\bar{B}r$ and $Y_s^a \in \bar{B}_r \; \forall \; s \leq \tau \wedge T$.

- **Definition:** MVP (Mean Value Property)
  $u : D \to \mathbb{R}$ is said to have MVP if for every $a \in D$ and $r > 0$ s.t. $a + \bar{B}_r \subset D$ one has

$$u(a) = \int_{\partial B_r} u(a + x)\mu_r(dx)$$

  Hence

$$u(a) = \frac{1}{V_r}\int_0^r u(a) S_\rho d\rho = \frac{1}{V_r}\int_0^r \left(\int_{\partial B_\rho} u(a + x)\mu_\rho(dx)\right) S_\rho d\rho$$
$$= \frac{1}{V_r}\int_{B_r} u(a + x)\, dx.$$



Next, if u is harmonic in D that means u is twice continuously differentiable and Laplacian of u is 0, then we are going to show that u has mean value property, for proving let us consider a point a in D and r positive such that a plus Br closer is subset of D, to prove mean value property this is the way to do because for every point a we are going to show there mean value property is true. Remember that this statement itself is a statement in analysis it involves no probability theory because it say u is harmonic in D that means it has some analytical property.

And we want to prove that u has MVP mean value property that is also an analytical property. One can say that in means value property it involves mu r which came from Brownian motion but actually we know that mu r this measure is nothing but normalized surface area measure. So, this is like you know an analytical property, so we are going to prove this statement which is a topic of analysis using the stochastic process theory whatever we have learnt till now.

So, this I have discussed a is a point we choose and r is the radius we choose such that it can be inscribed inside D, so let Y be a process which is starting from a because Brownian motion starts from 0, so this starts from a, so it is translated Brownian motion. And then tau is the stopping time when this process Y leaves the ball, so it starts with the center a and then at some time when it leaves we call that tau time exit time from the ball by the process Y.

Now, as we have already discuss before that probability tau is finite is 1, because it is a bounded ball, then expectation of tau minimum T, so this is finite Y because tau minimum T is anyway bounded random variable and expectation under probability measure, probability measure is a finite measure, so this integration is finite, so expectation of this is finite.

Again, u can be extended to a twice-differentiable function with compact support beyond that ball we can do that, because when you consider u on the domain D from that we have shifted out concentration only at point a and the ball around a of radius r. And then by not changing the function inside the ball, closed ball we can extend we can modify outside that to make that as a continuously to a differential function with compact support we can do that, thus we can apply the relation what we have earlier discussed in the beginning that this is the end of the prove that f of this thing, so here f was Cc2 Rn, so you can use this result, so we are going to use this result now.

So, remember here Y tau is like this but for our application here x is a this is u is 0 and v is just identity matrix, so all these assumptions whatever is written here everything will be satisfied. So, we consider this we apply that theorem, so we get expectation of u of this process is equal to u of a plus expectation of the second derivative term, first derivative term does not appear because that is dot taking dot product with drift mu here u is 0, so that does not appear here.

So, here 0 to tau minimum T appears half Laplacian half times Laplacian of u of Yas ds. So, after obtaining this we now apply use the fact that u harmonic, because we started the u is harmonic, so it is harmonic that mean Laplacian of u is 0, Laplacian of u is 0. So, here this part would be 0, so we are going to get this expectation is equal is equal to ua.

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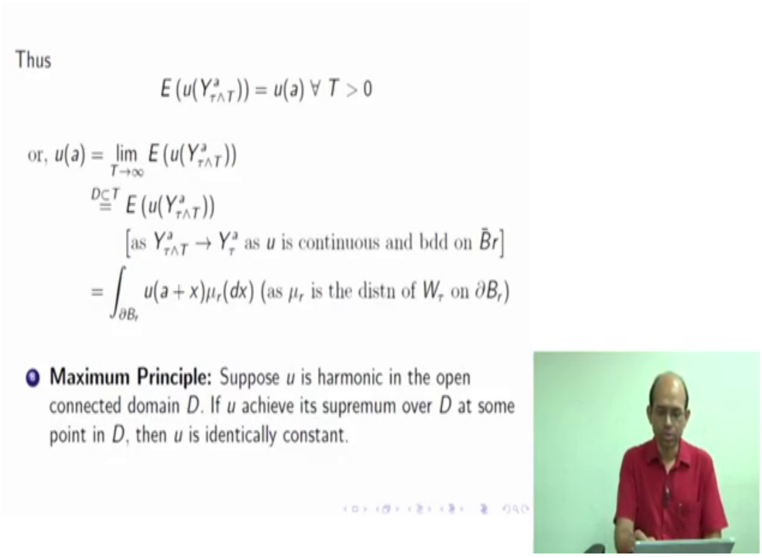

So, expectation of this is equal to this for all T positive. So, now u of a can be written as here capital T is there, we have to get rid of capital T because I should only deal with tau, so we take limit T tends to infinity of this thing, so we use dominated convergence theorem DCT here is C looks like a subset but it is DCT, using dominated convergence theorem is expectation of u of Ya so this is typo it should be just tau there is no minimum T this is going to converge here as this random variable is converging to this as capital T tends to infinity almost surely and u is you know continuous and bounded on the set Br closer so we can use the dominated convergence theorem here.

So, this term this expectation we are now writing here, so here there is a typo it should be just Y superscript a subscript tau there is no capital T is equal to integration on the surface of the ball, so this is basically sphere of u of why is it so? Because the tau at tau I mean the process Y is on the boundary because the tau is the exit time at that exit time Y must be on the boundary of the ball of radius r. So, a plus x and then for this expectation we know that mu r by definition is a probability that it is found on that set on the boundary.

So, this expectation can be computed using that u of a plus x mu r dx, it is coming directly because of the definition of Y, Y is x plus W and the probability the W touches some points that is giving the mu r measure, so it is this u of a plus x mu r dx, that is it. So, this saying that for does not matter what a point we choose all the time whatever you know r positive radius we can find such that the ball is inside the domain I would always get this relation that u of a is equal to u of a plus x mu r dx on the sphere. Therefore, u has mean value property, mu r is the distribution of W tau on del Br, I think I would stop here rest I would discuss in the next.