Introduction to Probabilistic Methods in PDE
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Lecture 18
Application of Ito's rule on Ito Process

In the earlier lecture, we have seen the definition and properties of Brownian motion and today we are going to see some more results of Brownian motion, we are going to see first definition of Ito process, Ito process is a stochastic process which is written in term of Brownian motion and some other appropriate processes.

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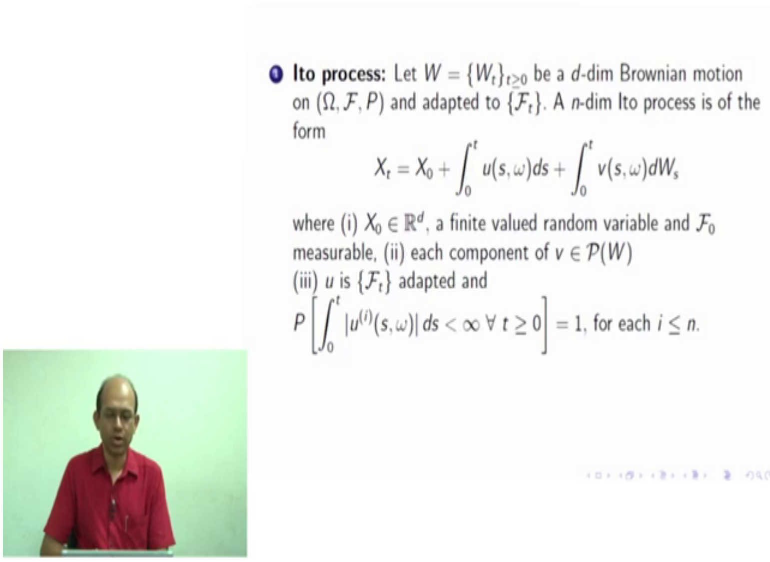

So, let us see that definition, let W be a d dimensional Brownian motion, here Brownian motion is denoted by the letter W, this W stands for the initial of the mathematician Wiener, so this W we take on the probability space omega F P. And we assume that this W is adapted to the filtration Ft, n dimensional Ito process is of the form Xt is equal to X naught plus integration 0 to t, u of s ds plus integration 0 to t, v of s dWs, here u and v are function of time and omega, omega is actually the sample points.

So, that means these are random variables, however it also depends on s, s is time. So, this is time dependent random variables so in other words u and v both are stochastic processes, so we are integrating here the stochastic process u with respect to time and here we are integrating the stochastic process of v with the respect to the Brownian motion W from 0 to t.

And then the integration what we obtain at the end and that if we vary over time we get a stochastic process, so that process is going to call Ito process. So, Ito process is a class of processes, so this is a large class of processes which can be written in this manner that the process Xt can be written as sum of these 3 terms, 1 is X naught, where X naught is F0 measurable finite value at random variable.

And then we have integration of a stochastic process u which is Ft adapted and the probability that integration of mod u ds is finite with probability 1 and then v is also there where this integration of v with respect to Brownian motion is taken then we call this and v should be you know integrable with respec to this Brownian motion, so we choose v to be a member from that script P of W. So, this class of processes we have already discussed when we have developed the theory of integration.

So, these are the conditions on u and v if we can write down a given stochastic process in this manner where u and v are satisfying this conditions, then we would call this stochastic process as an Ito process. So, remember this adaptiveness is very important here u and v both are adapted to the same filtration where Brownian motion is also adapted.

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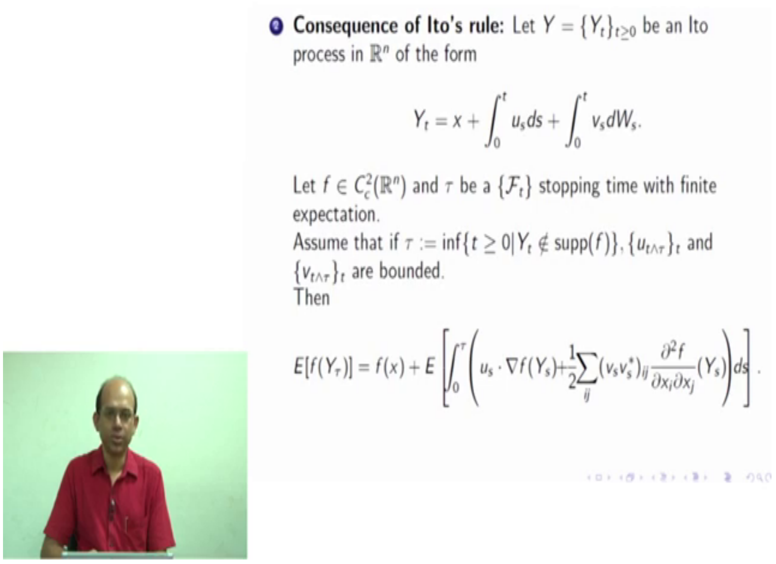

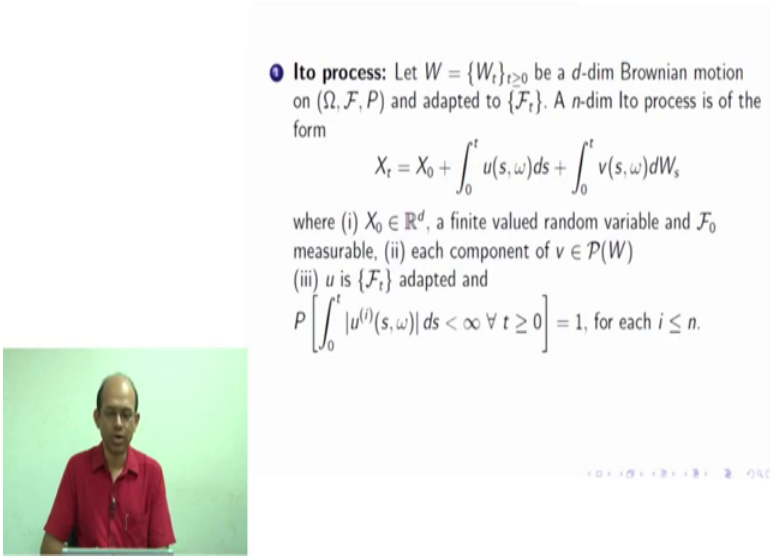

Next, we see one formula, which is consequence of Ito's formula. So, here we start with process Y, so Y be an Ito process of the form Yt is equal to x plus integration 0 to t uds plus integration 0 to t vdWs. So, we are suppressing here the dependence of u and v on omega, because that is the thing we do generally, but in the earlier slide I have written explicitly the dependence of u and v on omega to emphasize that u and v are allowed to be random variables, allowed to be stochastic processes.

Now, we consider a function f which is twice continuously differentiable and has compact support. So, compact support functions and continuous and compact supports functions are bounded and it's all deliveries are also bounded, the mutch of deliveries it can pauses. So, and Tau we consider a an Ft stopping time with finite expectation. Assume that if Tau is the time of exiting the support of f, what is support of f? It is a compact subset of Rn, here we have consider f is a function from Rn to R real value function or Rn with compact support.

So, now if you count the time when Yt this process which is coming from here hits the boundary of the support and goes beyond the support of f for the first time and you call that tau then tau becomes a stopping time Ft measurable to Ft stopping time, why? Because Yt is coming from here and then u, v and W all are Ft adapted processes, so in return we get Yt is also Ft adapted, we are just counting the stop the hitting time of the boundary of the support of f by the process Yt which is Ft adapted, so tau also becomes stopping time Ft stopping time with respect to the stopping time Ft.

Now, if we assume a further condition and u and v namely that given the stopping time the stopped process u and v that is ut minimum tau and vt minimum tau they are also bounded, so this is an additional condition on u and v. So, we assume this additional condition for this following result. So, under this assumption we can obtain the following result that expectation of f evaluated at Yt, so f of Y of Tau sorry not t, Y of tau is equal to f of x plus expectation of integration 0 to tau us dot this is dot product this is you know gradient of f, so this is derivative of f in multi-dimensional so f is for Rn to R, so that the total derivative of f is also function for Rn to R so of Ys and then we are taking this is a vector v is a vector and we are taking dot product of u and the derivative of f plus half times summation over ij vs v star s here star stands for transpose, v star.

So, v is a matrix here, because W is Rn value stochastic process and v is a matrix here, matrix valued stochastic process, so v v star is also a matrix also a matrix and we are considering ijth component of v v star and then we multiply that with the second derivative of f with respect to i and j, so del 2f del xi del xj evaluated at Ys and ds. And then s is we are integrating with respect to s from 0 to tau. So, we would prove this a result.

So, before proving the result let us again revise it the terms more carefully, here we have only two terms, one term is the first involve the first derivative of f and second term involve second derivative of f with half term. So, we can figure out that this is the term which appears in the Ito's formula for the I mean which is attach to the quadratic variation process of Brownian motion and we have seen the quadratic variation of the Brownian motion is time itself identity, so that ds is coming.

And this first order term first order derivative term comes with the dYt term, but dYt term is here usds plus this thing plus the local martingale part and this us is appearing here ds is appearing here. So, you can understand that basically if you write down Ito's formula for function f and the process Y and then you are going to get without expectation the terms this terms and one more additional term, what is that? That is gradient of f and then the matrix v and dW, so this term should appear but that is absent here, so only reason that to be absent is that possibly the expectation of that is 0.

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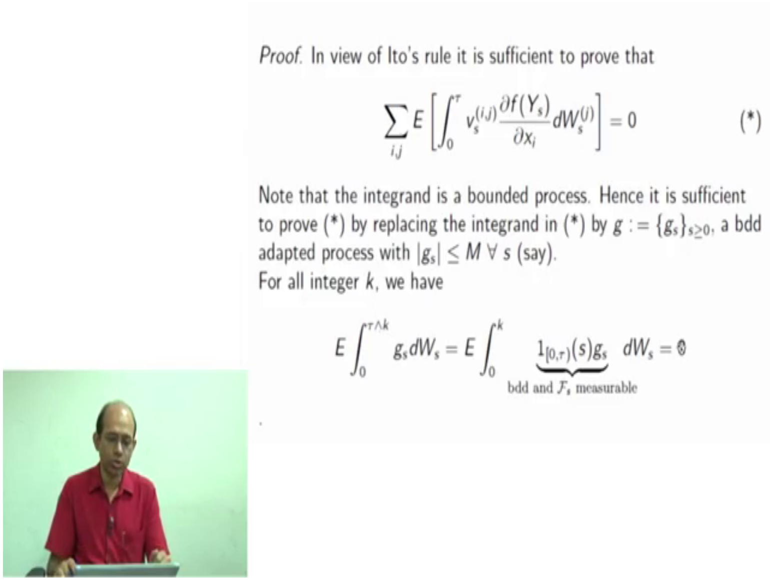

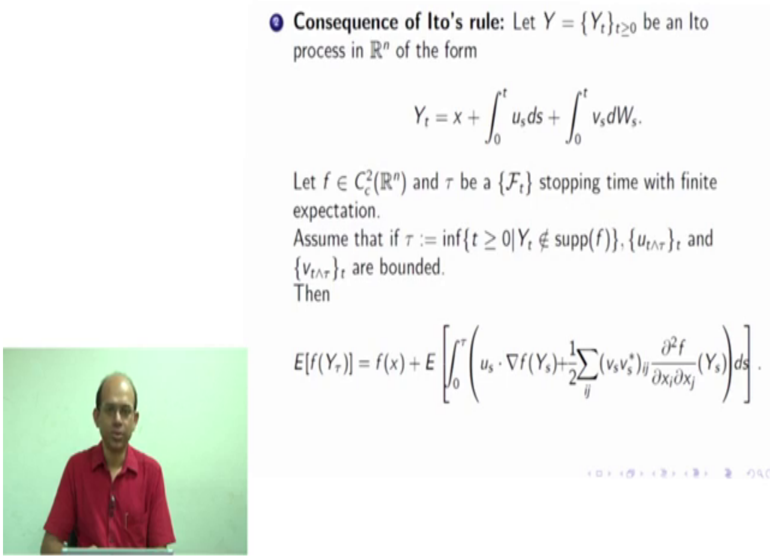

In view of Ito's rule it is sufficient to prove that expectation of integration this v del f del x dW is 0, so this is the thing we need to prove. Earlier I had this summation that I took out of the expectation. So, let us see the indices more carefully, so v is a matrix, vij ijth entry of v matrix and then del f del xi, so ith component of the derivative of f and then jth component of the Brownian motion. So, if you write down as a in terms of matrix notation then del f del x come as a vector you know gradient of f be on the left hand side in between you would have the matrix and then the Brownian motion possibly left hand side del f del x you would written down as a transpose of that so that it is like a row vector.

So, that you are going to get a scalar out of it, so this whole thing is a scalar, so this is a scalar 0. So, you need to prove that this is 0, so that is the goal. So, if you do that then nothing more is required then this is immediate. So, in view of Ito's formula we just need to prove this expression is 0. Note that the integrand is bounded process here, why? Because of the assumptions we have made that tau is the time, so within the time tau you and v behaves like a bounded process and the integration will be for 0 to tau.

So, there u and v are all bounded and function f is a twice difference constant differentiable function on the compact support, so the function and its derivatives all are bounded, so this part is bounded, this is bounded so integrand is bounded. So, now we have an integration of a bounded process with respect to Brownian motion and we are now trying to minimize our notations so instead of writing all the details we are going to write down just g. So, it is sufficient to prove this type of identity for some g where g is a bounded adapted process.

So, there is a property of this functions, so just a general g, actually this is true this is 0 for a general g so we are going to prove that. So, let us assume that g is a bounded process with an upper bound capital M, g is less then or equals to M mod of g is less then or equals to M more or less. So, now for all integer k, we have expectation of 0 to tau minimum k, why are you doing tau minimum k? Because tau itself is need not be a bounded random variable, however if we put a minimum all with k that we will put a bound on that, so then this tau minimum k remains another you know stopping time but bound stopping time and if it is bounded then we can actually argue better.

So, we consider a some integer k positivity that k and take expectation of 0 to tau minimum k, gs dWs. Then this integration can be rewritten as integration 0 to k and then indicator function of 0 to tau of s. if s is equal to more than tau then this part would be 0. So, when we integrate with Brownian motion from s to s is 0 to k then for occasions where tau was below k there this part would become 0 after tau onward, so this integrand would not contribute anything.

On the other hand, if tau is more than k this part would be always 1 all the time divisible to k and therefore I would get this integrations form 0 to k. So, this is always you know a finite interval integration this dWs. And here we see that g this is bounded the whole thing is bounded Fs measurable, so when you have bounded Fs measurable process we know that it is in the class of LW, LW is the space of processes which are adapted and it is square integrable, so those conditions are satisfied.

So, this is going to be a square integrable martingale, since it is a martingale so when 0 to t is there, like as a function of t, now it is 0 to k, so that means the martingale evaluated at time k, so expectation of that would be 0 so we get that this is 0. But we get this is 0 for the modified stopping time that is truncated tau minimum k, but this is true for all possible k.

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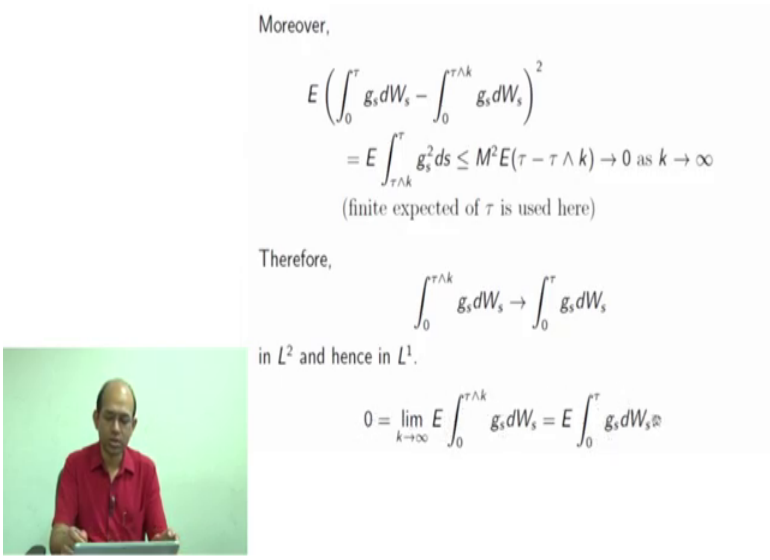

So, that is the thing we are going to use next that we have we actually need to discuss about this integration but we have obtain result about this integration which is integration gsdWs 0 to tau minimum k. Now, we are looking at the L2 distance between these 2 random variables expectation of square of the difference. Now, we see that that is equal to this difference is 0 to tau and 0 tau minus that means tau minimum k to tau this integration of g square, so here are applying the Ito's isometry formula.

So, there we are getting that expectation of this gs square ds, less than or equal to, so Ito's isometry formula involves the first version involves only the deterministic time both the sides, but then for general stopping times also we know that we have the same Ito's isometry formula so we are using that here. And here we have gs square but g is a bounded process upper bound by M, so g square can be upper bounded by M square and then it is integration of this time.

So, these difference tau minus tau minimum of k expectation of that, however tau is assume to have finite expectation since this has finite expectation, so tau minimum minus tau minimum k that would go to 0 as k tends to infinity, why is it so? Because anyway as k tends to infinity this go these go converges to tau almost surely as tau is finite valued so this converges almost surely, but that tau have finite expectation so we can actually use dominated convergence theorem that this difference is dominated by 2 times of tau which has finite expectation, so we can actually take limit inside and the expectation of this thing goes to 0 as k tends to infinity, finite expectation of tau is used here.

Therefore, integration 0 to tau minimum k gsdWs this thing converges to this in L2, because we got it correct so this L2 norm distance goes to 0 converges to L2. So, L2 converges also implies L1 convergences for probability measure, so we get that this you know expectation is 0 all the time, so this is 0 all the time, so for limit also it is 0, however this limit is same as expectation of 0 to tau gsdWs since the converges in L1. So, at the end what we have obtain that 0 is equal to expectation of integration 0 to tau gsdWs. So, this is the end of the prove.