Introduction to Probabilistic Methods in PDE
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Lecture 17
Brownian motion and its martingale property Part 02

Next, we discuss another important property of Brownian Motion.

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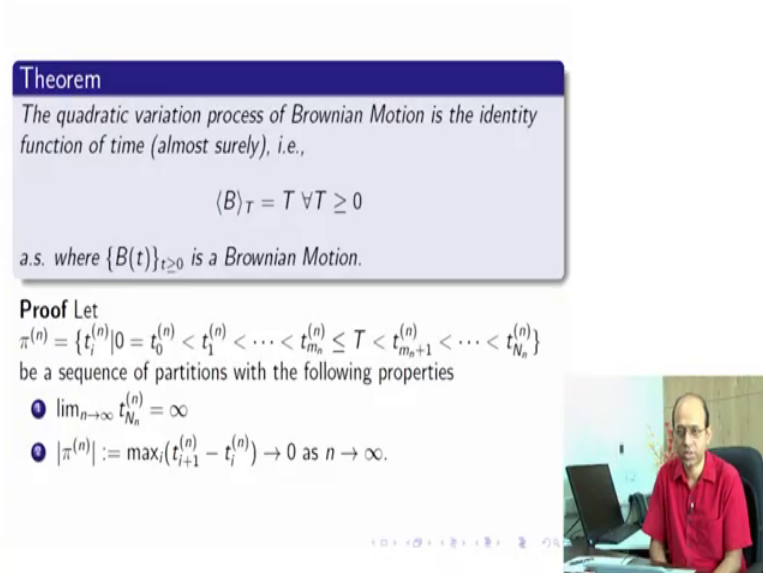

That its quadratic variation process is just the time itself. So, that is stated here as a theorem. The quadratic variation process of Brownian Motion is the identity function of time, almost surely that is, this is the notation that we are using since last 2-3 lectures. That the quadratic variation of B, at time capital T is exactly equals to T for all T, and that is, to the left hand side appears to be random variable right, because Brownian motion is random variable, but this is deterministic. So, that you see with probability 1.

So, this is theorem and we are going to prove this theorem. We are proving because this is very important theorem and we are going to use it repeatedly and we are going to use Brownian motion as building block of other martingales.

So, here we start. We first consider a sequence of partitions. A partition itself is a family of increasing time points on the real line, and then we are having several such partitions. We have sequence of partitions. So, every fixed partition for a given n we are going to fix one partition, that partition has many different time points.

The time points are like, you know, time t0, t1 etc. up to t capital N, n. The right hand point, is some time, and for nth one whatever the last time point, n plus 1th one, the time point would be little larger, so you may be right most side. And the number of points would also increase in a particular manner.

What is the manner, that limit n tends to infinity, tNn is going to infinity. So, right hand side of the partition, because partition always have finitely many points. But we are trying to partition towards the whole real line. So, this is the scheme that we are taking the right hand point goes to infinity as n tends to infinity. The superscript bracket n indicates that this is coming from the nth partition by n, and the subscript determines the order or index of the time inside that partition.

Another property we need, that is the mesh size of the partition. We denote it by modulus sign; so that is defined as maximum over all possible i of the increment ti plus 1 minus ti, and that is going to 0.

So, these two properties are actually specifying the type of sequence of partitions we are going to consider. We are going to consider only these type of sequence of partitions, where the partition mesh size is going to 0, shrinking to 0 but the end point is also going to infinity. So, we consider such partitions.

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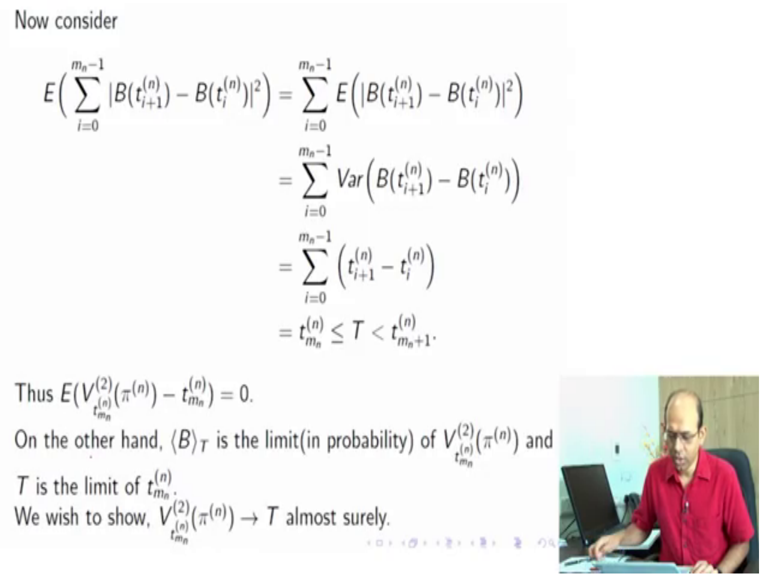

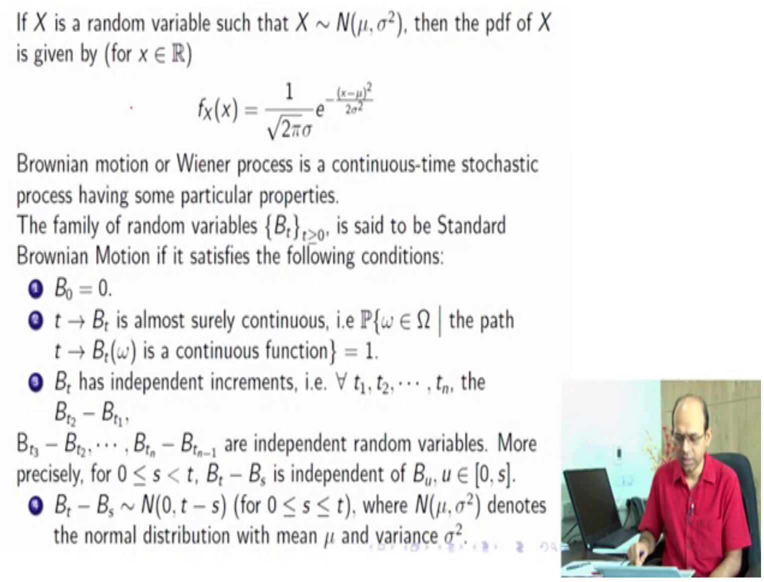

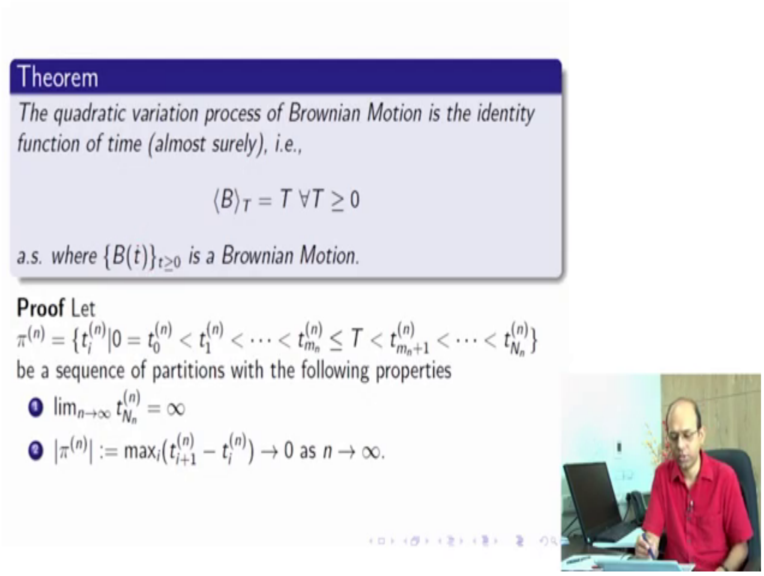

Then we consider, this summation of B, so here you note that earlier, in my earlier slide I have used B subscript t, as a notation of Brownian motion, as we have used in earlier lectures to denote any stochastic process.

To denote stochastic process as a function of time, we have written time as a subscript of the process. Here, we are writing B of t, correct, we are putting time as a function, I mean this is just a notational convenience.

Why this is so, because in the proof we are going to use lots of subscripts and superscript of this time, and if this time is in the subscript then it looks very, you know, clumsy and it is not much readable. So, for this particular slide, we deviated from our conventional notation. We are writing time as, you know, inside the bracket. So, it is having the exactly same meaning as B subscript.

So, here we consider this summation, the summation i is equal to 0 to mn minus 1. What is mn, mn is actually a particular time point where it is, like capital T, between capital T. So, capital T is a fixed number, we are fixing for the time being. And then, that mn is the time point such that, mn and mn plus 1, these two partition points, they are enclosing capital T.

So, we are trying to approximate capital T using tmn. Capital T is any, say given fixed real number. And then when we take this summation i is equal to 0 to mn minus 1, B ti plus 1 minus B ti whole square, so this is what, this is just increment, square of the increment. And we are summing over square of successive increments. That is going to give me approximation of the quadratic variation, along that partition, till time, approximately till time capital T, tmn.

So, that we now evaluate expectation of this value. Expectation of this sum, is sum of the expectation because expectation is a linear operator. Expectation of B ti plus 1 minus B ti whole square. However, this B ti plus 1 minus B ti, so that is increment of Brownian motion and increment of Brownian motion is normal distribution with mean 0. So, expectation of square of a random variable which has mean 0 is nothing but the variance of that random variable, so we write down that this is same as variance of the increment B ti plus 1 minus B ti.

Now, we are going to use another property of Brownian motion that this increment is normal distribution with mean 0 and variance is the same as the time difference. That we have seen in earlier slide, that Bt minus Bs is normal distribution mean 0 and variance is t minus s.

So, here using that the variance we can write down as ti plus 1 minus ti. And that, we are summing over all possible i starting from 0 to mn minus 1. So, this is like telescopic sum. And then the sum total would be tmnn. Since, t0n is 0, so we are going to get exactly tmnn. But this, tmnn is the left approximation of capital T and is next partition point is more than capital T. So, we are now rewriting this whatever we have obtained here, thus this part is nothing but that, you know, quadratic variation approximation along the partition pi n. This notation we have used earlier, earlier when we have defined what is quadratic variation, after that we have stated one property of the quadratic variation which justifies why should we call this particular process as quadratic variation.

We have shown that, if we consider quadratic variation, that is nothing but the sum of square of increments, successive increments and that is like, you know quadratic variation, along the partition, that sequence converges to the quadratic variation process in probability, as mesh size of the partition goes to 0.

So, here this expression is nothing but actually this V2 pi n, till time tmn and this is equal to tmn, so if I subtract and take expectation that is 0. So, we get it 0 for all possible n. So, here as I am recollecting, that this sequence converges to the quadratic variation in probability. So, this is the limit of this thing.

On the other hand, T is the limit of this. So therefore, this converges to here in probability, this converges to T in probability. And we are actually aiming to associate relation between Bt and T. So, this is going to this, this is going to T. So, we wish to show that, this, you know quadratic variation along the partition goes to capital T almost surely.

I mean, this is synonymous because to show that this quadratic variation is equal to T almost surely, is the same as showing that this converges to T almost surely. Why, because anyway this converges to this, you know in probability. So, if this converges to T almost surely, so this converges to T in probability also then this becomes the quadratic variation.

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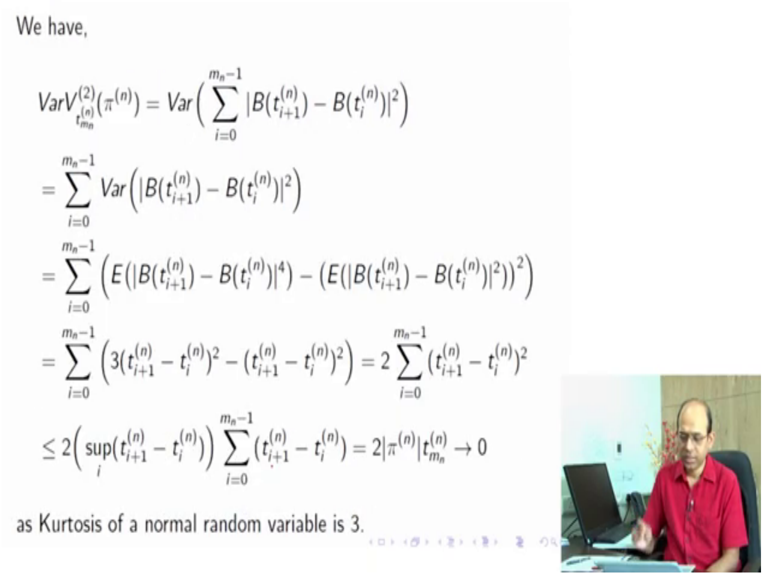

Next to prove this; we should actually need more finer result. So, just expectation is matching is not sufficient to obtain it using some other convergence notion. Next to prove this; we should actually need more finer result.

So, here we are taking the notion of L2 convergence. So, we are considering variance, variance of this random variable. So, V2 pi n, tmnn so I am not reading it every time because it is, you know very lengthy to read but we understand this is just approximation of capital T, left approximation of capital T.

So, variance of this random variable is equal to variance of this, this is just redirecting the full expression of this and then variance of sum is there. However, variance of sum is, sum of the variance only if that random variables are independent inside. Is it, yes it is because the increments are independent. That is also the property of Brownian motion, the increments are independent of the past, so whatever increments you take, the disjoint increments, so they would be all independent to each other.

Since, it is sum of independent random variables the variance of the sum is, the sum of the variance. Next line what we are doing is that we are rewriting formula of variance. Variance of X is, expectation of X square minus whole square of expectation of X. So, we are going to use that.

So, variance of this, inside thing already we have a square, now expectation of whole square of this would be like to the power of 4. So, here we are getting expectation of the increment to the power of 4 minus the whole square of the, so here the square is appearing outside of the expectation of this term, the square of the increment.

And now, we need to evaluate this term. So, this expression is what, this is like you know Brownian motion, the increment of the Brownian motion that is normal random variable with mean 0 and variance the ti plus 1 minus ti, the time difference. And we know, that for normal random variable, does not matter what are its parameters, mu or sigma, its kurtosis is 3.

What does it mean, it means that, the fourth order moment is same as 3 times of the variance square. So, variance is ti plus 1 minus ti and square is appearing here, so this expression is 3 times of ti plus 1 minus ti whole square.

Now, we look at the remaining terms. Here, I have square outside but inside I have expectation of this whole square. And then again, this is nothing but variance of the increment of Brownian motion and that is ti plus 1 minus ti, inside, but square is outside, so this square also comes here.

So, I have 3 times of the square of the difference in time minus square of difference of time. So, I have 3 minus 1, means 2 time, 2 times square of the difference in time. And then, sum over i is equal to 0 to mn minus 1, so this summation is already there.

So, what is this value, this value turns out to be very, very small as n tends to infinity. Why is it so, because that I am also, as m tends to infinity this mn is also increasing, I understand because you know more and more data, I mean, partition points are appearing inside and then if we actually rewrite this square as product of the same term twice, then one product term, that we are going to replace by the maximum distance, maximum increment of the time. So, then I am going to get less than or equal to sign.

If you replace, one of the this, one of this, you know term by maximum of that, so then I am going to get supremum of this thing, then I am going to get less than or equal to sign, 2 is already here. But re square, I have another term, same identical term ti plus 1 minus ti. So, that I retain, that I do not change and that stays here, and then this summation i is equal to 0 to mn minus 1. Since, supremum of the term over all possible i is independent of i, so that comes out of summation, it is outside.

However, this summation is a telescopic sum. Its summation is exactly tnmn, which is approximation capital T. So, that means this is a sequence of n which converges to capital T therefore it is a bounded sequence.

On the other hand, this supremum over all possible i of the increment is actually the, mesh size of the partition and mesh size of the partition by your choice goes to 0. So, this you know, this mesh size goes to 0. So, this part goes to 0. This is bounded; so therefore, the product goes to 0. So, what we have obtained is just the variance of the quadratic variation, approximation along the partition goes to 0.

What does it mean? We know that variance of a random variable is nothing but say, X minus expectation of X whole square and then expectation. So, that is the L2 norm between, L2 norm of the difference of X and expectation of X

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Now consider

$$E\Big(\sum_{i=0}^{m_n-1}|B(t_{i+1}^{(n)})-B(t_i^{(n)})|^2\Big)=\sum_{i=0}^{m_n-1}E\Big(|B(t_{i+1}^{(n)})-B(t_i^{(n)})|^2\Big)$$
$$=\sum_{i=0}^{m_n-1}Var\Big(B(t_{i+1}^{(n)})-B(t_i^{(n)})\Big)$$
$$=\sum_{i=0}^{m_n-1}\Big(t_{i+1}^{(n)}-t_i^{(n)}\Big)$$
$$=t_{m_n}^{(n)}\le T<t_{m_n+1}^{(n)}.$$

Thus $E(V^{(2)}_{t^{(n)}_{m_n}}(\pi^{(n)})-t^{(n)}_{m_n})=0$.

On the other hand, $\langle B\rangle_T$ is the limit(in probability) of $V^{(2)}_{t^{(n)}_{m_n}}(\pi^{(n)})$ and T is the limit of $t^{(n)}_{m_n}$.

We wish to show, $V^{(2)}_{t^{(n)}_{m_n}}(\pi^{(n)})\to T$ almost surely.

Thus $V^{(2)}_{t^{(n)}_{m_n}}(\pi^{(n)})-t^{(n)}_{m_n}\xrightarrow{L^2}0$. Or, $V^{(2)}_{t^{(n)}_{m_n}}(\pi^{(n)})\xrightarrow{L^2}T$.

Hence, $V^{(2)}_{t^{(n)}_{m_n}}(\pi^{(n)})\xrightarrow{P}T$. This implies that there is a subsequence $\pi^{(n_k)}$ such that $V^{(2)}_{t^{(n_k)}_{m_{n_k}}}(\pi^{(n_k)})\to T$ almost surely as $k\to\infty$. Hence,

$$P(\{\omega|\langle B\rangle_T(\omega)=T\})=1\forall\ T>0.$$

Next we would like to prove a stronger statement

$$P(\{\omega|\langle B\rangle_t(\omega)=t\quad\forall\ t>0\})=1$$

i.e., if $N_t=\{\omega|\langle B\rangle_t(\omega)\neq t\}$ then

$$P(\cup_{0<t}N_t)=0.$$

What was the expectation of the quadratic variation, it was tmnn. That we have seen in the last slide. This expectation is tmnn. So, here what we have obtained is that, the random variable here minus expectation. So, this is going to 0 in L2 norm. Since, the variance is going to 0 that means the L2 norm of this is, you know, difference is going to 0.

So, and since, tmnn converges to capital T point wise, so we get immediately that this sequence of random variable converges to capital T in L2. So, L2 convergence gives convergence in probability. So, that means that this sequence of random variable converges to capital T in probability.

So, this implies that there is a subsequence, because you know we actually are aiming for obtaining almost sure convergence, but we have only obtained here convergence in probability. So, this also says that, from that result what I have shown earlier, that this is quadratic variation approximation and this convergence to the quadratic variation of the process in probability, so that also says that T is actually the quadratic variation.

However, we want that convergence almost surely, but here we just have converge in probability. And we know, that although this is a weaker notion of convergence but there is a subsequence of this sequence for which we can establish a stronger notion of convergence that is almost sure convergence.

So, this implies that there is a subsequence pi nk, such that when I replace n by nk, so tmnk and the partition is pi nk, so this sequence of k converges to T almost surely as k tends to infinity. So, that would give us that, that this T is quadratic variation anyway but that is true with probability 1, that probability that <B>t is equal to t is 1, for all t positive.

We are not yet done. Why, because here we have obtained that this as a stochastic process of t is matching with this, t is another dynamics, if you vary t and they are matching, for, if you fix time they are matching, with probability 1 at that given t. But as a whole path, are they going to match, with probability 1, that is still not answered.

For that, we need to put this for all t inside this. So, at present what we have obtained is that this process and this process, the identity process are modification to each other. But we have to achieve indistinguishability. For that we are going to use the continuity property, as we have used earlier. But for Brownian motion, we are actually giving more precise steps, how we achieve it. Because it is very simple for this situation.

So, next we would like to prove a stronger statement that, probability of observing that this quadratic variation <B>t is equal to t for all t, that means path is exactly this, is 1. So, for that, that is synonymous of saying that we are going to observe complementary part of that, that means it is not equal, that Nt has, you know probability 0.

So, that means that all possible t, so I mean it is true for all t, its negation is that it is not equal for some t. So, union of all such Nt, Nt is what; Nt is where it fails, where the quadratic variation fails to be t. So, that if we take all possible t union, so then probability of that is 0. So, this condition is synonymous of this. So, if you prove this thing then we are done.

So, we are going to prove this. But here, we understand that this is not easy to prove. Why, because it is uncountable union. Even if we know, that the first statement is saying that basically N capital T has probability 0, NT is 0 probability.

However, this uncountable union of null sets, null sets means 0 measure sets, uncountable union of 0 measure sets need not be 0 measure set, so we cannot conclude here just directly from here. We need to work little more.

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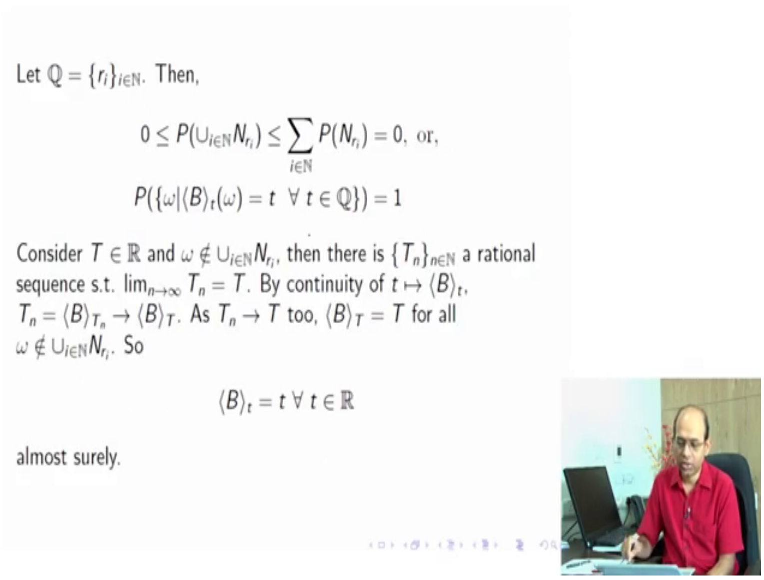

So, consider the set of rational numbers. That we denote by Q. Since, Q is countable, we can actually write down ri, ri, i belongs to N. We can actually write down Q as a sequence of rational numbers, as sequence.

Then we consider, instead of all union we just consider only those Nri. So, only at the rational number points, only that time, and then take all unions. And we know that, Nri, so these are each and every Nri is, you know of 0 measure and this countable union of 0 measure, this is 0. So, this is less than or equal to sum over P of Nri and these are all 0s, so this probability is 0.

So, we have shown that, that not the full thing yet, but the union over only you know, countable time points which is, you know rational numbers, on rational times, there if we see, then that union is 0. However, rational numbers are very useful here.

Why did you choose, we chose rational number because that is dense in R. So, every real number can be approximate by rational numbers. So, here we are rewriting what we have obtained here, that Bt is equal to t for all t in Q, that probability is 1.

We have obtained here, and now we fix a capital T in R. And now, if we choose omega which is not in this set, I mean the 0 measure set, then there is a Tn, a sequence of, you know rational numbers, such that limit n tends to infinity Tn is equal to T, this is coming from the density of rational number. Rational number is dense enough, so we can always find out a sequence of rational numbers converging to the real number.

Now, by continuity of this mapped t to B, quadratic variation of Bt. Why is it continuous, because Brownian motion is a continuous process, and continuous martingale and its quadratic variation is also therefore continuous. So, using the continuity of this function we can now use this result, that Tn is < B>Tn anyway, because Tn is rational number.

We have established that when omega is not in this, then we know that, that these two do match. So, Tn is equal to BTn. However, as Tn goes to T, <B>Tn would go to <B>T, using the continuity of the quadratic variation process. So, as Tn goes to T too, therefore we are going to get <B>T is equal to T for all omega not in Nri.

So, here this Tn goes to capital T, and this <B>Tn goes to <B>T, we have obtained this, so for all Nri. But this T is any arbitrary real number. So, we have obtained that <B>t is equal to t, for any arbitrary real number, for every omega not in this set. That means, when omega is not in this set, I am going to get <B>t is equal to t for all t in R.

However, this set has measure 0. So, that means this is true almost surely. So, this might fail only on a set of measure 0, so this is true almost surely. So, that is the proof of the fact that quadratic variation process of Brownian motion is just the time, just the identity function of time. Thank you.