Introduction to Probabilistic Methods in PDE
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Lecture 16
Brownian motion and its martingale property

Today we are going to see one particular example of stochastic process that is Brownian motion. In earlier lectures we have seen the general type of martingales, semi martingales. So, those are classes of processes, those include many examples of processes.

Brownian motion is one particular example of a stochastic process that I have actually used sometimes during my lectures just you know for, for referring to some particular properties or example, but I have not discussed the full definition and most important properties of Brownian motion.

So here, I am trying to present the definition of Brownian motion and its important path properties. So, the aim of this lecture is that we are going to see the definition of Brownian motion, its martingale property, its path property in the sense of quadratic variation process. So, these are the things we are going to see today.

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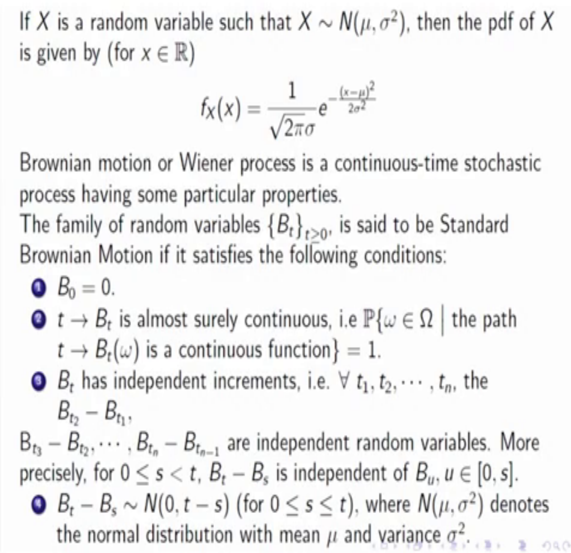

So, let us first recall what is normal random variable. Normal random variable is the building block for Brownian motion.

So, if X is a normal random variable such that, the PDF, Probability Density Function of X is given by, I mean this is written f small f subscript capital X of small x, this is a function of small x, where x is real number, if that is given by 1 over square root of 2 pi times sigma, see, sigma is outside of the square root. Times e to the power of minus x minus mu whole square by 2 sigma square.

Most of you have seen, you know, the normal random variable in your earlier courses in probability and statistics. So, people also call it in different names. Sometimes, you know we call it Gaussian random variable, because Gauss was the mathematician who actually analyzed such type of probability density functions.

Brownian motion or Wiener process, actually this is same process but having two different names, due to mainly historical reason. So, Robert Brown was a botanist who actually observed certain random motions in pollen grains suspended, you know, I mean, inside a liquid water, and then later on people realized that, that motion is not caused by the pollen grain itself but is caused by the water molecules bombarding different parts of the pollen grain which is giving unbalanced force and that is why the movement is exhibited. And later on, it was Albert Einstein who wrote a paper in 1985, where he gave mathematical formulation of Brownian motion.

Actually 5 years before the work of Albert Einstein, there was one work by one PhD student, mathematics PhD student called Louis Bachelier who also, I mean formulated mathematical definition of Brownian motion, as a limit of random walk. So, let us now come to the definition of Brownian motion.

Before that, let me also clarify why do you call it Wiener process, because Wiener was a mathematician who studied properties of Brownian motion in greater details and then, and it became more, you know, popular among mathematicians because we then got much richer properties, richer literature on Brownian motion. So, we call sometimes Brownian motion, sometimes we call Wiener process, during the course also I might use both the terms at different occasions.

So, Brownian motion is basically a stochastic process and we have already discussed, that a stochastic process is nothing but a family of random variables. And that family parameter here is, continuous time t from close, I mean 0 and onward, positive.

So, it is a continuous time process. So, time is not discrete, like 1, 2, 3 it is not discrete time process but continuous time process. That means that Brownian motion is a dynamics, which evolves continuously. And another thing is that its paths are also continuous, so that we are going to see.

So, we call a particular process, here so we are just having a family of random variables, Bt, t 0 or positive, is said to be a Standard Brownian motion if satisfies the following condition, that it starts with 0, from the origin, and then that, the map t to Bt, this path is almost surely continuous. That means, when you have real, when you observe realization of Brownian motion, you observe the whole path is a continuous function. And you realize this with probability 1. So, that is probability of the paths, I mean, such that this is continuous, this is 1.

Another thing is that Bt has independent increments. That is that t1, t2 etc. if these are some time points you fix first, like partition time, and then you look at the increments of the Brownian motion during these subintervals, for example, Bt2 minus Bt1, Bt3 minus Bt2, etc. Btn minus Btn minus 1, these are all small, small increments. And these increments, should also be independent, independent random variables, so Bt minus Bs is independent of all the paths at time s or before.

So, if you consider s as your present time and t as your future, then Bt minus Bs is future increment and this statement is saying that the future increment is independent of the present and past. Because when time u is between 0 to s that is time present or past, so this increment, is a random variable but the random noise has no dependence on what happened during past and present.

There is another property, that this difference, the increment Bt minus Bs, this difference follows a normal random variable, so normal distribution with mean 0 and variance t minus s. Importantly that this is increment, s is present, t is future, so this is future increment Bt minus

Bs but that is a random variable with mean 0 and it has variance t minus s, the time difference, time difference is the variance.

So, here as I have mentioned here, normal mu sigma square, so here mu is the mean and sigma square is the variance. So, this is, t minus s is the variance, not the standard deviation, this is the variance. So, that is the definition of Brownian motion. When one defines, such object one can ask a very relevant and mathematical question, that does there exist any Brownian motion, why is it so, because we are putting so many conditions.

So, these conditions like you know, we are talking about that Brownian motion is a stochastic process, which has all these conditions. After putting all these conditions what is the guarantee that there exists one such process which satisfies all these conditions.

It is not immediate but one can prove that, actually that is a consequence of a theorem. One can use Kolmogorov Consistency Theorem which asserts that after only by, you know declaring or fixing, the finite dimensional distribution that means, you know you just take finite times and there you are determining what is the distributions of the process there and for all possible such kind of description can determine the process itself.

However, there are some more additional properties, for example that continuity of the path etc. So, that is not assured from the Kolmogorov's Consistency Theorem that needs additional work. So, that is actually consequence of one nice property of normal random variable, Gaussian here, since you know, this difference, the variance is becoming smaller and smaller as your t and s is becoming closer and closer, because t minus s is going to 0 then.

So, then that is say the variance is going to 0 and the mean is 0. That means you know, your Bt and Bs would be closer and closer, when t and s is closer and closer. So that gives, you know intuitive understanding that why should one expect that the Brownian path should be continuous. But of course, that is not the proof, the proof requires much more, detailed analysis.

So, we are not going into the details of the proof. So, we accept that, there exists a Brownian motion that means there is a stochastic process which satisfies all these properties. And then that process is unique because as I told, that Kolmogorov's Consistency Theorem says that, by fixing these, you know conditions we can actually pinpoint one single stochastic process.

So, uniquely one single law, so that stochastic process we are going to call as Brownian motion. So, that was a brief introduction of the definition of Brownian motion. Next, let us try to see that what do we mean by Brownian motion, how does the path look like.

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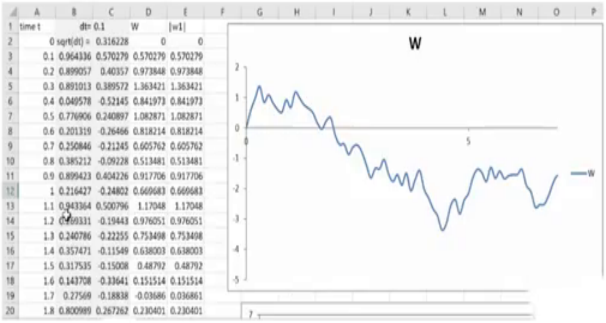

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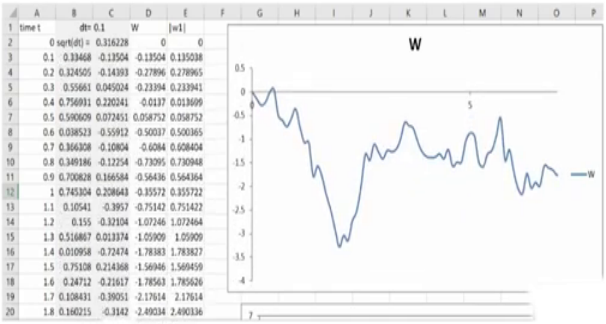
4	0.2	0.899057	0.40357	0.973848	0.973848
5	0.3	0.891013	0.389572	1.363421	1.363421
6	0.4	0.049578	-0.52145	0.841973	0.841973
7	0.5	0.776906	0.240897	1.082871	1.082871
8	0.6	0.201319	-0.26466	0.818214	0.818214
9	0.7	0.250846	-0.21245	0.605762	0.605762
10	0.8	0.385212	-0.09228	0.513481	0.513481
11	0.9	0.899423	0.404226	0.917706	0.917706
12	1	0.216427	-0.24802	0.669683	0.669683
13	1.1	0.943364	0.500796	1.17048	1.17048
14	1.2	[illegible]	-0.19443	0.976051	0.976051
15	1.3	0.240786	-0.22255	0.753498	0.753498
16	1.4	0.357471	-0.11549	0.638003	0.638003
17	1.5	0.317535	-0.15008	0.48792	0.48792
18	1.6	0.143708	-0.33641	0.151514	0.151514
19	1.7	0.27569	-0.18838	-0.03686	0.036861
20	1.8	0.800989	0.267262	0.230401	0.230401



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5	0.3	0.55661	0.045024	-0.23394	0.233941
6	0.4	0.756931	0.220241	-0.0137	0.013699
7	0.5	0.590609	0.072451	0.058752	0.058752
8	0.6	0.038523	-0.55912	-0.50037	0.500365
9	0.7	0.366308	-0.10804	-0.6084	0.608404
10	0.8	0.349186	-0.12254	-0.73095	0.730948
11	0.9	0.700828	0.166584	-0.56436	0.564364
12	1	0.745304	0.208643	-0.35572	0.355722
13	1.1	0.10541	-0.3957	-0.75142	0.751422
14	1.2	0.155	-0.32104	-1.07246	1.072464
15	1.3	0.516867	0.013374	-1.05909	1.05909
16	1.4	0.010958	-0.72474	-1.78383	1.783827
17	1.5	0.75108	0.214368	-1.56946	1.569459
18	1.6	0.24712	-0.21617	-1.78563	1.785626
19	1.7	0.108431	-0.39051	-2.17614	2.17614
20	1.8	0.160215	-0.3142	-2.49034	2.490336

	A	B	C	D	E
1	time t	dt=	0.1	W	|w1|
2	0	sqrt(dt) =	0.316228	0	0
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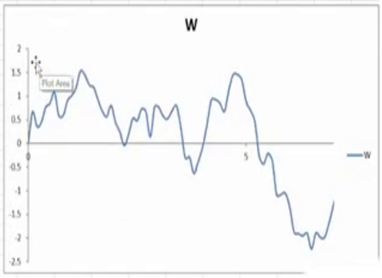
4	0.2	0.157514	-0.31772	0.360139	0.360139
5	0.3	0.608077	0.086744	0.446883	0.446883
6	0.4	0.845565	0.321792	0.768676	0.768676
7	0.5	0.605929	0.084978	0.853653	0.853653
8	0.6	0.763319	0.226742	1.080395	1.080395
9	0.7	0.06204	-0.48632	0.594078	0.594078
10	0.8	0.496458	-0.00281	0.59127	0.59127
11	0.9	0.838992	0.313168	0.904438	0.904438
12	1	0.636763	0.110623	1.015061	1.015061
13	1.1	0.705364	0.170729	1.18579	1.18579
14	1.2	0.867034	0.351797	1.537586	1.537586
15	1.3	0.383059	-0.09406	1.443522	1.443522
16	1.4	0.25716	-0.20622	1.237301	1.237301
17	1.5	0.433799	-0.05272	1.184583	1.184583
18	1.6	0.201891	-0.26401	0.920569	0.920569
19	1.7	0.226642	-0.23716	0.683413	0.683413
20	1.8	0.364797	-0.10931	0.574104	0.574104



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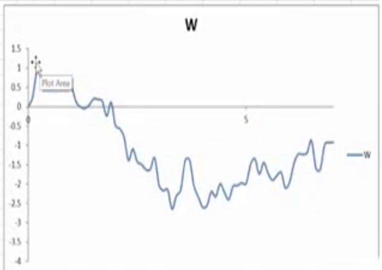
4	0.2	0.98186	0.662108	0.908584	0.908584
5	0.3	0.312708	-0.15438	0.754205	0.754205
6	0.4	0.515784	0.012515	0.76672	0.76672
7	0.5	0.405082	-0.07596	0.690758	0.690758
8	0.6	0.561491	0.048936	0.739694	0.739694
9	0.7	0.485418	-0.01156	0.728133	0.728133
10	0.8	0.289281	-0.17566	0.552472	0.552472
11	0.9	0.524071	0.019092	0.571564	0.571564
12	1	0.393524	-0.08543	0.486136	0.486136
13	1.1	0.122682	-0.36736	0.11878	0.11878
14	1.2	0.34647	-0.12487	-0.00609	0.006088
15	1.3	0.452583	-0.03767	-0.04376	0.043763
16	1.4	0.613906	0.091552	0.047789	0.047789
17	1.5	0.685146	0.152466	0.200255	0.200255
18	1.6	0.478004	-0.01744	0.182811	0.182811
19	1.7	0.444934	-0.04379	0.139023	0.139023
20	1.8	0.117178	-0.37606	-0.23704	0.237038



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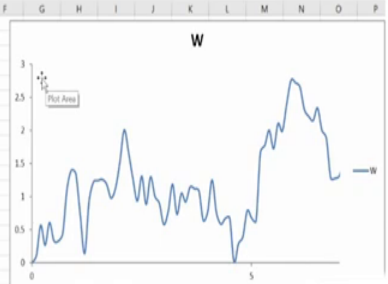
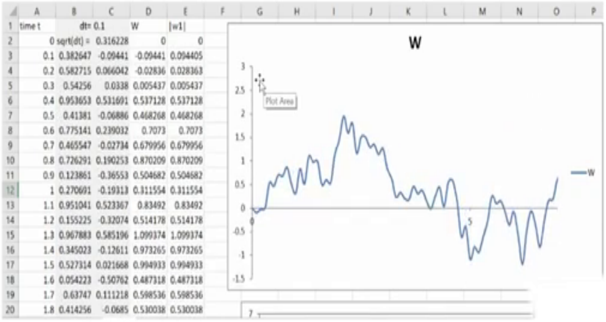
4	0.2	0.927578	0.461057	0.568445	0.568445
5	0.3	0.170059	-0.30166	0.266785	0.266785
6	0.4	0.859528	0.340957	0.607742	0.607742
7	0.5	0.191462	-0.27592	0.331826	0.331826
8	0.6	0.495941	-0.00322	0.328609	0.328609
9	0.7	0.63932	0.11278	0.441388	0.441388
10	0.8	0.98418	0.67955	1.120939	1.120939
11	0.9	0.812546	0.280594	1.401533	1.401533
12	1	0.425606	-0.05924	1.34229	1.34229
13	1.1	0.024524	-0.62239	0.7199	0.7199
14	1.2	0.033619	-0.57872	0.141177	0.141177
15	1.3	0.99363	0.787714	0.928891	0.928891
16	1.4	0.820299	0.289825	1.218716	1.218716
17	1.5	0.52835	0.022491	1.241207	1.241207
18	1.6	0.521931	0.017393	1.2586	1.2586
19	1.7	0.411868	-0.07044	1.188162	1.188162
20	1.8	0.251939	-0.21137	0.976795	0.976795





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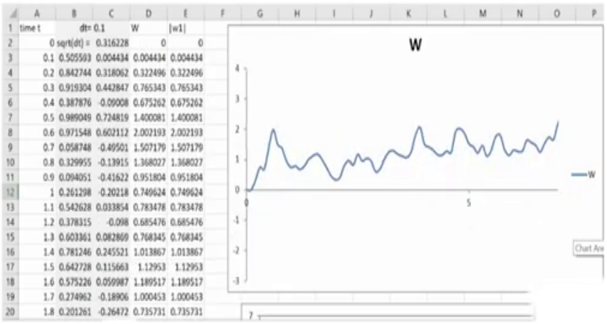
5	0.3	0.54256	0.0338	0.005437	0.005437
6	0.4	0.953653	0.531691	0.537128	0.537128
7	0.5	0.41381	-0.06886	0.468268	0.468268
8	0.6	0.775141	0.239032	0.7073	0.7073
9	0.7	0.465547	-0.02734	0.679956	0.679956
10	0.8	0.726291	0.190253	0.870209	0.870209
11	0.9	0.123861	-0.36553	0.504682	0.504682
12	1	0.270691	-0.19313	0.311554	0.311554
13	1.1	0.951041	0.523367	0.83492	0.83492
14	1.2	0.155225	-0.32074	0.514178	0.514178
15	1.3	0.967883	0.585196	1.099374	1.099374
16	1.4	0.345023	-0.12611	0.973265	0.973265
17	1.5	0.527314	0.021668	0.994933	0.994933
18	1.6	0.054223	-0.50762	0.487318	0.487318
19	1.7	0.63747	0.111218	0.598536	0.598536
20	1.8	0.414256	-0.0685	0.530038	0.530038



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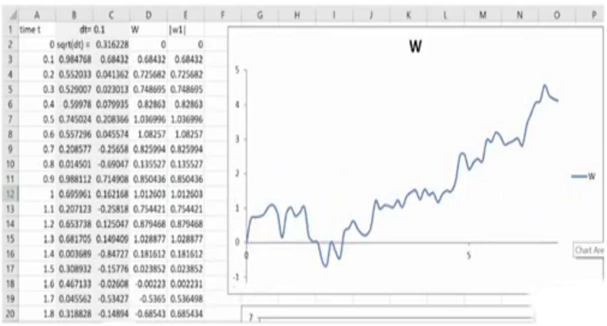
5	0.3	0.919304	0.442847	0.765343	0.765343
6	0.4	0.387876	-0.09008	0.675262	0.675262
7	0.5	0.989049	0.724819	1.400081	1.400081
8	0.6	0.971548	0.602112	2.002193	2.002193
9	0.7	0.058748	-0.49501	1.507179	1.507179
10	0.8	0.329955	-0.13915	1.368027	1.368027
11	0.9	0.094051	-0.41622	0.951804	0.951804
12	1	0.261298	-0.20218	0.749624	0.749624
13	1.1	0.542628	0.033854	0.783478	0.783478
14	1.2	0.378315	-0.098	0.685476	0.685476
15	1.3	0.603361	0.082869	0.768345	0.768345
16	1.4	0.781246	0.245521	1.013867	1.013867
17	1.5	0.642728	0.115663	1.12953	1.12953
18	1.6	0.575226	0.059987	1.189517	1.189517
19	1.7	0.274962	-0.18906	1.000453	1.000453
20	1.8	0.201261	-0.26472	0.735731	0.735731



	A	B	C	D	E
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5	0.3	0.529007	0.023013	0.748695	0.748695
6	0.4	0.59978	0.079935	0.82863	0.82863
7	0.5	0.745024	0.208366	1.036996	1.036996
8	0.6	0.557296	0.045574	1.08257	1.08257
9	0.7	0.208577	-0.25658	0.825994	0.825994
10	0.8	0.014501	-0.69047	0.135527	0.135527
11	0.9	0.988112	0.714908	0.850436	0.850436
12	1	0.695961	0.162168	1.012603	1.012603
13	1.1	0.207123	-0.25818	0.754421	0.754421
14	1.2	0.653738	0.125047	0.879468	0.879468
15	1.3	0.681705	0.149409	1.028877	1.028877
16	1.4	0.003689	-0.84727	0.181612	0.181612
17	1.5	0.308932	-0.15776	0.023852	0.023852
18	1.6	0.467133	-0.02608	-0.00223	0.002231
19	1.7	0.045562	-0.53427	-0.5365	0.536498
20	1.8	0.318828	-0.14894	-0.68543	0.685434

	A	B	C	D	E
1	time t	dt= 0.1		W	|w1|
2	0	sqrt(dt) =	0.316228	0	0
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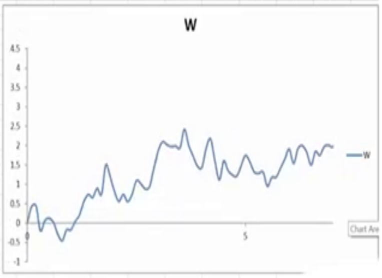
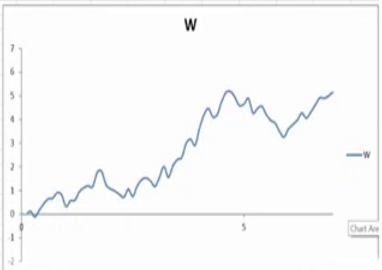
5	0.3	0.019793	-0.65081	-0.20314	0.203143
6	0.4	0.797366	0.26318	0.060037	0.060037
7	0.5	0.576343	0.060889	0.120926	0.120926
8	0.6	0.376617	-0.09942	0.02151	0.02151
9	0.7	0.15866	-0.31622	-0.29471	0.294712
10	0.8	0.299889	-0.16593	-0.46064	0.460642
11	0.9	0.816378	0.285126	-0.17552	0.175516
12	1	0.498177	-0.00144	-0.17696	0.176961
13	1.1	0.760421	0.223781	0.04682	0.04682
14	1.2	0.70192	0.167579	0.2144	0.2144
15	1.3	0.867579	0.3526	0.567	0.567
16	1.4	0.708853	0.173937	0.740937	0.740937
17	1.5	0.399868	-0.08022	0.660714	0.660714
18	1.6	0.772238	0.23598	0.896694	0.896694
19	1.7	0.303443	-0.16271	0.733987	0.733987
20	1.8	0.992232	0.765151	1.499138	1.499138



	A	B	C	D	E
1	time t	dt= 0.1		W	|w1|
2	0	sqrt(dt) =	0.316228	0	0
3	0.1	0.131287	-0.35428	-0.35428	0.354278
4	0.2	0.604093	0.08347	-0.27081	0.270808
5	0.3	0.603827	0.083252	-0.18756	0.187556
6	0.4	0.147605	-0.33101	-0.51857	0.518571
7	0.5	0.892821	0.39265	-0.12592	0.125921
8	0.6	0.124666	-0.36429	-0.49021	0.490207
9	0.7	0.430137	-0.05566	-0.54587	0.545871
10	0.8	0.944202	0.503136	-0.04273	0.042734
11	0.9	0.690414	0.157173	0.114439	0.114439
12	1	0.212318	-0.25248	-0.13804	0.138038
13	1.1	0.321488	-0.14659	-0.28462	0.284623
14	1.2	0.130833	-0.35495	-0.63958	0.639577
15	1.3	0.349045	-0.12266	-0.76224	0.762242
16	1.4	0.161515	-0.31251	-1.07475	1.074754
17	1.5	0.894476	0.395502	-0.67925	0.679252
18	1.6	0.355234	-0.11739	-0.79664	0.796644
19	1.7	0.694277	0.160647	-0.636	0.635997
20	1.8	0.903968	0.412518	-0.22348	0.223479



	A	B	C	D	E
1	time t	dt= 0.1		W	|w1|
2	0	sqrt(dt) =	0.316228	0	0
3	0.1	0.131287	-0.35428	-0.35428	0.354278
4	0.2	0.604093	0.08347	-0.27081	0.270808
5	0.3	0.603827	0.083252	-0.18756	0.187556
6	0.4	0.147605	-0.33101	-0.51857	0.518571
7	0.5	0.892821	0.39265	-0.12592	0.125921
8	0.6	0.124666	-0.36429	-0.49021	0.490207
9	0.7	0.430137	-0.05566	-0.54587	0.545871
10	0.8	0.944202	0.503136	-0.04273	0.042734
11	0.9	0.690414	0.157173	0.114439	0.114439
12	1	0.212318	-0.25248	-0.13804	0.138038
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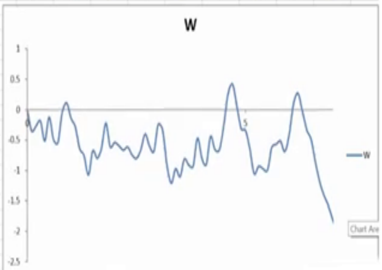
18	1.6	0.355234	-0.11739	-0.79664	0.796644
19	1.7	0.694277	0.160647	-0.636	0.635997
20	1.8	0.903968	0.412518	-0.22348	0.223479



So, here I have made one random walk, random walk plot. So, this random walk here, the time step is very small, so as I told that the Louis Bachelier's approach was, that Brownian motion as a limit of random walks, so I took that approach to approximate a Brownian motion.

This is simulation of approximate Brownian motion. Here, the time step is taken as 0.1. So, then we can actually see that, various different sample path. It started from 0 and then if we run it again, we get various different paths of Brownian motion. So, it is a stochastic process, each and every time we are going to get different, different realization.

So, I am here not seeing the dynamics, you know over time, as time is increasing but I am looking at here the whole, whole evolution of Brownian motion between time 0 to time 7 perhaps, and then various different realization of that. And we see that, these are the paths what is obtained from this.

And as I told that this is just an approximation, so that the granularity is 0.1, so we cannot see zigzag pattern smaller than 0.1. It looks like smooth, but Brownian motion is not this smooth, it is, I mean, does not matter how small interval you zoom in, you would always find, you know fluctuations in the process, it is just an approximation. From far you can possibly see, I mean, it looks like a Brownian motion.

Now, we are going to see one more nice property of Brownian motion that Brownian motion is a martingale. We have seen in earlier lectures, the properties of martingale. We have seen the definition of martingales and we have actually discussed how to integrate with respect to a martingale, square integrable continuous martingale, and also you have seen that how to integrate with continuous local martingale. So, Brownian motion happens to be a square integrable continuous martingale.

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$$\mathcal{F}_t := \sigma\{B_u : u \in [0,t]\}$$

$$E[B_t \mid \mathcal{F}_s] = E[B_t - B_s + B_s \mid \mathcal{F}_s]$$
$$= E[B_t - B_s \mid \mathcal{F}_s] + E(B_s \mid \mathcal{F}_s)$$
$$= E(B_t - B_s) + B_s$$
$$= B_s$$

$\{B_t\}_{t \geq 0}$ is a $\mathcal{F}_t$- martingale.

So, let us first show that, Brownian motion is a martingale, with respect to a filtration. So, what filtration we are choosing, so we are choosing the filtration Fs, or say Ft is equal to the sigma algebra generated by the Brownian motion Bu, where u is from time 0 to t. So, the smallest sigma algebra generated by Bu. And we, this you know, family of sigma algebra because if we change t we are going to get various different sigma algebras. This family of sigma algebras gives you a filtration.

However, this is not the usual, might not satisfy the usual hypothesis. One might need to augment, you know, some more negligible sets to make it complete. So, we assume that, we have done that, so this is the filtration which is generated by the Brownian motion itself. Now, our question is that, is Brownian motion a martingale with respect to this filtration? For answering to this question, we look for the conditional expectation.

So, conditional expectation of Bt given Fs, we know that we need to prove that this conditional expectation of the future of Brownian motion value, given the filtration generated by present and past, so s is the present time s, is equal to Bs. If we prove that then we would be able to confirm that B is a martingale, the Brownian motion is a martingale.

So, let us write it down. So, what we do first, we subtract and add the present value of the Brownian motion, we get this equality and then here what we do is that this part we take together, Bt minus Bs given Fs and the remaining term just Bs we write down separately, Bs given Fs.

Now, here we have Bt minus Bs given Fs. What is Bt minus Bs, this is increment; the future increment of the Brownian motion given Fs, Fs is the filtration generated by the Brownian path from 0 to time s, s is present.

We have seen, as a property of Brownian motion that future increment, that random noise or random variable is independent of the path of the Brownian motion from time 0 to s. It is independent. So, this is the situation when we are talking about conditional expectation, of a random variable given a sigma algebra, such that the random variable is independent to the sigma algebra. Then we know, that this conditional expectation would be the same as just the expectation of the random variable, Bt minus Bs.

On the other hand, here what we have is that Bs, so this Bs is Fs measurable. Why, because Ft is actually the signal generated by time 0 to t. So, that means Fs includes Bs itself, so Fs, Bs is Fs measurable. Since, we know that conditional expectation of a random variable given sigma algebra is the random variable itself, when this is measurable with respect to sigma algebra.

So, these are two extreme cases when this random variable is independent to the sigma algebra, then we find out the conditional expectation is expectation of the random variable, where the random variable is measurable with respect to sigma algebra then conditional expectation just the random variable.

However, we know the increment is normally distributed in mean 0; mean 0 so expectation is 0, so this part is 0, only Bs remains. So, here we have proved that conditional expectation of Bt given Fs is equal to Bs. Hence, the process Bt is a Ft martingale.