Introduction to Probabilistic Methods in PDE
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Lecture 14
Change of variable formula and proof Part 02

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Step 3:

$$J_3(\pi) = J_4(\pi) + J_5(\pi) + J_6(\pi) \text{ where}$$

$$J_4(\pi) = \sum_1^m f''(\eta_k)(A_{t_k} - A_{t_{k-1}})^2$$

$$J_5(\pi) = 2\sum_1^m f''(\eta_k)(A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$$

$$J_6(\pi) = \sum f''(\eta_k)(M_{t_k} - M_{t_{k-1}})^2$$

So, in step 3, we talk about J3 pi. So, here there is a problem because J3 pi is not evaluated at some particular time point tk minus 1. It is at some eta k, okay. So, and also here we do not have Mt difference or At, we have Xtk minus Xtk minus 1, whole square. So, lot of things has to be done here, okay to make it streamlined and straightforward so that we can analyze.

So, J3 pi is therefore rewritten as sum of three different objects, J4 plus J5 plus J6. What is J4? J4 is that, so whatever I have Xk so writing only A, A okay, instead of X, I am writing A, okay, Atk minus Atk minus 1 whole square. So, here square of this thing was there, here square of this thing.

However, we should have also cross terms, so A and M together, so the two times of you know, because Xtk minus Xtk minus 1 whole square would be this square plus 2 times this square plus this square. So, we are writing these three terms therefore okay, eta k, eta k, eta k everywhere. So, J4, J5 and J6. Later we would also, you know approximate J6 by replacing eta k by t k minus 1, okay so for the time being let us concentrate on J4 and J5.

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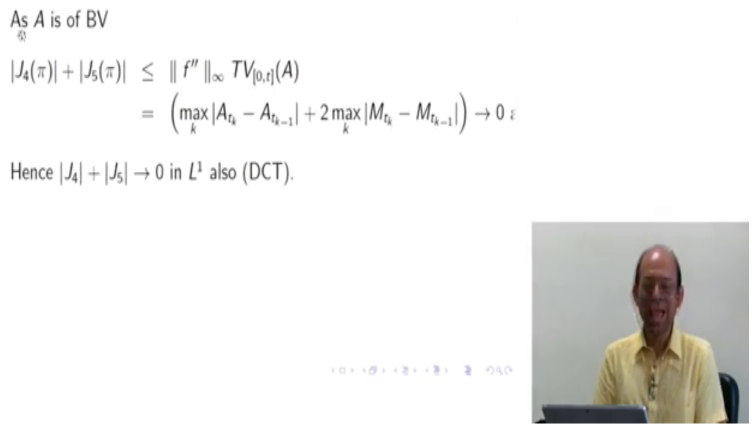

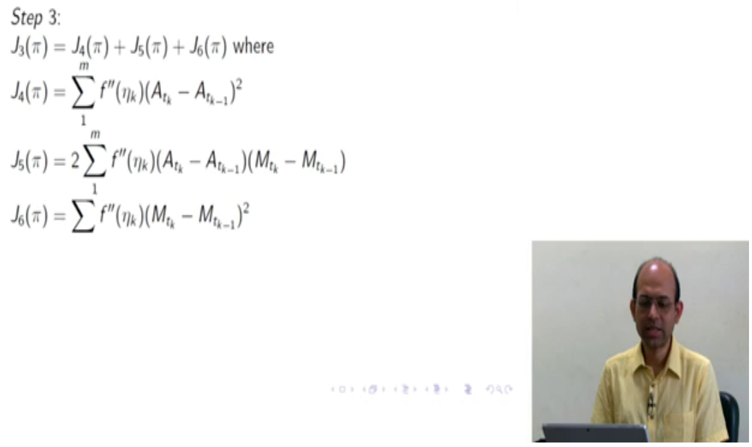

As A is a bounded variation, for J4 and J5 we have a nice bound. Why is it so? Because here f double prime is bounded, is bounded function, bounded function and Atk minus Atk minus 1, so this square I can write down as product of this with this, okay.

Now, in this product, one term I can take supremum of all possible tk and when I take supremum of all possible tk that particular term becomes independent of k and can come out of the integration and the remaining term, okay that I am going to take as a integration.

So, we are doing that, so this maximum of Atk minus Atk minus 1, mod of that, that we are taking out of that summation and then this also, this also is bounded. So, that also I am taking out of the summation and then what remains is mod of Atk minus Atk minus 1, then the summation that would be giving, that would give the bounded variation of A.

So, that total variation of, total variation of A is obtained here, okay and as we have seen for here also, if we take maximum over all these thing, okay the difference of martingale, increments over the martingales over all possible k, of course after taking modulus sign then that thing, that maximum is independent of k, that can come out of the summation and what remains there also as modulus and here, this is also bounded, this is also going to come out, the maximum of that, the upper bound comes out of the summation, and we are also going to get the total variation.

Now, we see that this does not depend on the choice of partition, correct? This is independent of partition but this depends upon partition. Now, if partition mesh size goes to 0, what happens? Okay so these are continuous functions, okay and then as mesh size going to 0, so here we have taken say, this partition you know 0 to t, time, closed interval 0 to t, and the continuous function is uniformly continuous, and then we are taking, you know, these difference goes to 0, the whole thing goes to 0. So, this maximum goes to 0, okay this goes to 0 as norm pi goes to 0.

So, we are going to obtain this, that J4 plus J5, okay that goes to 0 in L1 also, I mean this is almost sure convergence, but from there we are going to, because you know that boundedness we are going to use, the Dominated convergence theorem we can use to obtain the L1 convergence also.

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Define $J_6^*(\pi) = \Sigma f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2$

$|J_6(\pi) - J_6^*(\pi)| \leq \max |f''(\eta_k) - f''(X_{t_{k-1}})| V_t^{(2)}(\pi)$. (Defined earlier)

Using $E[V_t^{(2)}(\pi)]^2 \leq 48K^4$ where $|X_t| \leq K \ \forall \ t$, we get

$E|J_6(\pi) - J_6^*(\pi)| \leq \sqrt{E(\max |f''(\eta_k) - f''(X_{t_{k-1}})|^2)} \sqrt{48} K^2 \rightarrow 0$ as $|\pi|$ -

Hence $E|J_3(\pi) - J_6^*(\pi)| \rightarrow 0$ as $|\pi| \rightarrow 0$.

Step 3:

$J_3(\pi) = J_4(\pi) + J_5(\pi) + J_6(\pi)$ where

$$J_4(\pi) = \sum_1^m f''(\eta_k)(A_{t_k} - A_{t_{k-1}})^2$$

$$J_5(\pi) = 2\sum_1^m f''(\eta_k)(A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$$

$$J_6(\pi) = \sum f''(\eta_k)(M_{t_k} - M_{t_{k-1}})^2$$

As A is of BV

$$|J_4(\pi)| + |J_5(\pi)| \leq \| f'' \|_\infty TV_{[0,t]}(A)$$
$$= \left(\max_k |A_{t_k} - A_{t_{k-1}}| + 2\max_k |M_{t_k} - M_{t_{k-1}}|\right) \rightarrow 0$$

Hence $|J_4| + |J_5| \rightarrow 0$ in L^1 also (DCT).

Now, next we discuss what would be about J star 6, okay. So, J star 6 is, okay so J6 pi, actually I talked about J4 pi and J5 pi but J6 pi is further approximated by J star 6 pi, okay, J star 6 is nothing but the thing where this eta k is replaced by, replaced by Xtk minus 1 and summation is there. Here this summation is also from k is equal 1 to n.

Now, we consider this difference and in this difference, okay here we are going to get eta k and here we are going to get Xtk minus 1, okay so then maximum of that, if that maximum difference I take out of the summation, I would get one upper bound and then remaining terms square of this is nothing but the quadratic variation V2t pi, okay and we have seen already result earlier that if we have this V2t pi, that converges to the quadratic variation of M, correct? So, this is defined earlier.

So, here we are going to also use another result which I have not coated before but this is giving, you know, I mean instead of directly using this convergence okay we are also using another, you know estimate that this expectation of this random variable, okay is less than or equals to 48 times A to power 4, okay. It is really particular type of, you know upper bound.

One can get the details of derivation of this upper bound from the book of Karatzas and Shrieve, okay the book what I am following for this part, okay so where mod of Xt is less than or equal to capital K, okay so that is true for our case then the way we have defined, I mean whatever the K we first, if this is less than or equals to mod K, I mean K, then this would be less than or equals to 48 times K to the power 4.

From here, if we take expectation both sides, okay so then expectation of this, we can obtain that this is less than or equal to square root of expectation of maximum of f double prime eta k minus f double prime Xtk minus 1, this whole square okay, maximum of all these and square root of 48 times, instead of K to the 4, I have K square here, okay.

So, now what is the behavior of the right hand side? So this is fixed but this, as k, as mesh size goes to 0 then eta k comes close and closer to Xtk minus 1 and then this maximum of all these, that also goes to 0 as we have explained earlier. So, but this is actually bounded function. So, inside, this thing converges to 0, so after even taking expectation that also goes to 0 okay using Dominated convergence theorem. So, you get that this J star 6 is approximating J6 pi in the sense that the difference goes to 0 in L1.

So, from here, how can you get J3 minus J6 star pi also goes to 0? Thing is that J3 is J4 plus J5 plus J6, this total sum, the modulus goes to 0 and J6 minus J star 6 plus J star 6, I can add and subtract, and that difference goes to 0. So, J star 6 remains here and this J star 6 if I put this side, then I would J3 minus J star 6, okay so this right hand side goes to 0, left hand side also goes to 0.

So, what do we get is that expectation of this goes to 0, okay. You remember, earlier L1 convergence we have obtained, so all, in the same notion, this is important, correct I mean because of this L1 convergence here, we get this, goes to 0. Okay so next what we obtain, we see that we would like to further take another sum J7 pi, okay.

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Define $J_7(\pi) = \sum_{k=1}^{m} f''(X_{t_{k-1}})(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})$.

Since M and $\{M_t^2 - \langle M\rangle_t\}$ are martingales

$$E\left[(M_t - M_s)^2 - \langle M\rangle_t + \langle M\rangle_s | \mathcal{F}_s\right] = 0$$

Therefore,

$$\begin{aligned} E\,|J_6^*(\pi) - J_7(\pi)|^2 &= E\left|\sum_1^m f''(X_{t_{k-1}})\,(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\right|^2 \\ &= E\sum_1^m f''(X_{t_{k-1}})^2\left\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\right\}^2 \\ &\le 2\,\|f''\|_\infty^2\, E\left[\sum_1^m (M_{t_k} - M_{t_{k-1}})^4 + \sum_1^m (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2\right] \\ &\le 2\,\|f''\|_\infty^2\, E(V_t^{(4)}(\pi) + (\langle M\rangle_t - \langle M\rangle_0)\max(\Delta\langle M\rangle_{t_k})) \end{aligned}$$

Define $J_6^*(\pi) = \Sigma f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2$

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$E|J_6(\pi) - J_6^*(\pi)| \le \sqrt{E(\max|f''(\eta_k) - f''(X_{t_{k-1}})|^2)}\sqrt{48}K^2 \to 0$ as $|\pi| -$

Hence $E|J_3(\pi) - J_6^*(\pi)| \to 0$ as $|\pi| \to 0$.

So, in the J7 pi what we have, we have like discretization of the Riemann Stieltjes type of integration with respect to the quadratic variation of M, okay. So, we have to relate this J7 with this J3.

Otherwise we are not yet done because you remember that when we have stated this, here integration was with respect to quadratic variation of M, okay. Till now J star 6 which is, you know approximating J3 is like Mtk minus Mtk minus 1 whole square, okay so this, this integration, okay. But this is not in the form of integration, correct because this is like a square of an increment.

So, this we can put only in terms of Stieltjes integration when it is, like an increment of some process, okay. So, here we take, it is increment of quadratic variation process J7. Since M and Mt square minus quadratic variation Mt both are martingales, this is the definition of quadratic variation, both are martingales, so Mt minus Ms this term, and then difference of this thing, that Mt, if I put back then there will be Mt minus Ms okay but if I open the bracket there will be minus Mt plus Ms, okay so this given Fs is 0, correct? Because you know, this is martingale. Since this is martingale so its difference in. So, we now consider J6, J star 6 and J7 okay and we are taking L2 norm here. We are going to show L2 norm goes to 0 first and then we are going to get L1 norm convergence also.

So, now J star 6 pi minus J7 pi, this mod square, okay this is L2 norm we are taking, expectation of summation, so now we are just writing down the definition, okay, just exactly, so J star pi is like this, Xtk minus 1 M whole square and for J7 we are getting exactly this thing but now integrator term is like this, okay.

I think some bracket is missing here so here there should be a bracket here, and, and a bracket here, okay. So, is equal to expectation of summation 1 is equal to 1 to m, okay then we are writing okay, so this is f double prime square times Mtk minus Mtk minus whole square okay, minus this, I mean the whole thing whatever I had, this whole square.

How did you get it? Because you know we know that for cross terms their expectation is 0 because of the martingale property of this thing, okay, and basically we are going to use this thing for here okay.

Then when we have obtained this, you know square of the sum is sum of the squares, okay so then this boundedness property of a double prime we are going to use to obtain the f double prime square, so we are going to estimate this by, you know upper bound of this, and that is independent of k, that we take out of the summation, and then expectation of summation 1 to m, then inside term.

Inside term, you know that we have difference and then whole square so if I write down fully, this to the power of 4 appears, and then this to the power of square also appears, okay so this is properly written in the brackets, and the cross terms etc, okay. Then there also we are again use the earlier thing to remove that, okay.

So, we have this upper limit. Now, here what is this? This is like, you know quadruple variation, okay, like to the power 4, Vt to the power 4 pi, okay, expression of that plus, and then this, you know here I have a square. So, I can think that okay, 2 terms, I mean this term is multiplied twice.

For one term we are going to take the, replace that by supremum of all possible k that will be independent of k, that I take out, so that I call the maximum of increment of, you know quadratic variation of M and the remaining term would be just telescopic sum. Why is it so? Because you know Mtk is increasing, right, increasing process.

So, when I take out of this maximum I do not need to change the sign, okay? So, this is delta Mtk and then remaining term is like, this sum, we are going to Mt minus M0. So, we have seen that J star 6 and J7, the difference between J star and J7 square and expectation, this is nothing but the square of the L2 norm of J star 6 and J7. We have seen that as a norm of pi, the partition, goes to 0, the difference, I mean the L2 norms difference, okay that also goes to 0 of J star 6 and J7.

From here we can conclude that, okay J star 6 and J7 both have same L2 limit, okay. They have the same L2 limit. Here we have seen that okay, this norm is less than or equals to this finite number and then this part where maximum of delta quadratic variation M tk that goes to 0 as the partition mesh size goes to 0 due to the continuity of the martingale.

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Thus $E|J_6^*(\pi) - J_7(\pi)| \to 0$ (as L^2 conv $\Rightarrow L^1$ conv).
As $J_7(\pi) \to \int_0^t f''(X_s)d\langle M\rangle_s$ a.s. and also in L^1 (using DCT),
$J_3(\pi) \to \int_0^t f''(X_s)d\langle M\rangle_s$ in L^1.
Step 4: If $\{\pi^{(m)}\}$ is a seq of partition s.t. $|\pi^{(m)}| \to 0$ as $m \to \infty$ the above-mentioned J_1, J_2 and J_3 converges to the respective limits in L^1 and hence in probability too. Therefore, there exists a subsequence of $\{\pi^{(m)}\}$, say $\{\pi^{(m_k)}\}_k$ s.t. $[P]$ a.s.

$$\lim_{n\to\infty} J_1(\pi^{(m_k)}) = \int_0^t f'(X_s)dA_s$$
$$\lim_{n\to\infty} J_2(\pi^{(m_k)}) = \int_0^t f'(X_s)dM_s$$
$$\lim_{n\to\infty} J_3(\pi^{(m_k)}) = \int_0^t f''(X_s)d\langle M\rangle_s.$$

Define $J_7(\pi) = \sum_{k=1}^{m} f''(X_{t_{k-1}})(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})$.

Since M and $\{M_t^2 - \langle M\rangle_t\}$ are martingales

$$E\left[(M_t - M_s)^2 - \langle M\rangle_t + \langle M\rangle_s | \mathcal{F}_s\right] = 0$$

Therefore,

$$\begin{aligned}
&E\,|J_6^*(\pi) - J_7(\pi)|^2 \\
&= E\left|\sum_{1}^{m} f''(X_{t_{k-1}})\left(M_{t_k} - M_{t_{k-1}}\right)^2 - \left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)\right|^2 \\
&= E\sum_{1}^{m} f''(X_{t_{k-1}})^2\left\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\right\}^2 \\
&\leq 2\,\| f''\|_\infty^2\, E\left[\sum_{1}^{m}(M_{t_k} - M_{t_{k-1}})^4 + \sum_{1}^{m}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2\right] \\
&\leq 2\,\| f''\|_\infty^2\, E(V_t^{(4)}(\pi) + (\langle M\rangle_t - \langle M\rangle_0)\max(\Delta\langle M\rangle_{t_k}))
\end{aligned}$$

$$\begin{aligned}
J_1(\pi) &= \textstyle\sum_{k=1}^{m} f'(X_{t_{k-1}})(A_{t_k} - A_{t_{k-1}}) \\
J_2(\pi) &= \textstyle\sum f'(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) \\
J_3(\pi) &= \textstyle\sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2
\end{aligned}$$

where $\eta_k = X_{t_{k-1}} + \theta_k(X_{t_k} - X_{t_{k-1}})$ for some $\theta_k \in [0,1]$.

(a) $J_1(\pi) \to \int_0^t f'(X_s)dA_s$ a.s. as $|\pi| \to 0 \Rightarrow J_1(\pi) \to \int_0^t f'(X_s)dA_s$ in L^1.

(b) $Y = \{f'(X_s)\}_{s\geq 0}$ is in $\mathcal{L}^*$.

(c) $Y_s^\pi := f'(X_0)1_{\{0\}}(s) + \sum_1^m f'(X_{t_{k-1}})1_{(t_{k-1},t_k]}$.

(d) $[Y^\pi - Y]_t \to 0$ using DCT.

Next, and this since we know that L2 converges implies L1 convergence. So, we can state that, okay J star 6 and J7, okay so this expectation of modulus of this difference goes to 0. Okay, so from here what did we conclude? We conclude that this J3 and J7 would have the same L1 limit, okay. Where does J7 converge? J7 converges to 0 to t, double derivative of f and integrate the double derivative of f at Xs with respect to quadratic variation of M from 0 to t, so that would be the limit of J7.

Why is it so? Because of construction of J7. J7 is constructed this way where this is the integrand tk minus 1 at the interval tk minus 1 to tk so it is taking the value at the left point, and then the integrator, the increment of the integrator is here, okay. So, that is the reason that this is like nothing but Riemann Stieltjes integration sense, okay. So, J3 converges to this integration in L1. And this is the integration what we want in the expression of Itos formula.

So, now what do we do? We have obtained only L1 convergence. Actually at some places we have obtained almost sure convergence. For example for J1 and J2, correct? So I mean here for J1 we have obtained that almost sure convergence, correct?

But then weaker form is L1 convergence, okay, however we actually want to have one single setting for each and every term's convergence, because some terms we did not have almost sure convergence, we have only, you know in the sense of some mean or some square mean L1 L2.

So, we used the boundedness of the integrands and integrators and using that we got all terms convergence in terms of L1 convergence, in the sense of L1 convergence and then if a sequence of measurable functions, you know converges in L1 sense, okay then that would also imply that it converges almost in measure, okay. So, here measure is probability so that will imply that the convergence is in probability, okay.

However, if a sequence of measurable function converges to, say for example 0 in probability, okay then there is a subsequence which converges almost surely, okay. So, here what is the sequence we are getting? We are getting sequence due to the partitions, okay, finer and finer partitions we are getting new and newer sequences, okay, new and newer terms of the sequence.

So, if we talk about a subsequence then you have to take the subsequence of the partition sequence, okay as that partition mesh size goes to 0. So, we take these subsequences of the partitions, right? So, initially I have not figured you know fixed any particular sequence of partitions, okay but here if we have such convergence, for some sequence of partitions in L1, then of course for that particular sequence there is a subsequence.

We call this you know pi superscript m, so that, that converges to 0 as m tends to infinity and along this sequence, subsequence of partitions J1, J2, J3 all converges to the respective limits, okay. So, this is actually the sequence m but mk is the subsequence, along the subsequence mk that would converge almost surely.

So, let us write down that thing whatever we have obtained, so these three convergence, okay, 1, 2, 3, so limit n tends to, actually here, there is a typo, I should have written k, correct, limit k tends to infinity, J1 pi mk, okay so that limit is integration 0 to t f prime Xs dAs okay, limit n tends to infinity, actually here for these two things one could have taken the subsequence, I mean the sequence itself as the subsequence, correct, because we already had almost sure convergence along with that pi, okay.

However, we wanted to talk everything in the single setting so we are considering this, and limit k tends to infinity J3 pi mk that is equal to integration 0 to t double derivative of f Xs and integrated with respect to the quadratic variation of M. Okay so t is here fixed okay and we remember that the stopping time what you have introduced, that also we have fixed. We have not changed the stopping times here.

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Thus passing to the limit in $(\star)$,

$$f(X_t) - f(X_0) = \sum_{k=1}^{m} f(X_{t_k}) - f(X_{t_{k-1}}) = J_1(\pi) + J_2(\pi) + \tfrac{1}{2} J_3(\pi) \quad (\star)$$

we get the desired equality almost surely.

$$f(X_t^{(n)}) = f(X_0^{(n)}) + \int_0^t f'(X_s^{(n)})dM_s^{(n)} + \int_0^t f'(X_s^{(n)})dA_s^{(n)} + \frac{1}{2}\int_0^t f''(X_s^{(n)})d\langle M^{(n)}$$

$$f(X_t^{(n)}) = f(X_0^{(n)}) + \int_0^{t\wedge T_n} f'(X_s)dM_s + \int_0^{t\wedge T_n} f'(X_s)dA_s + \frac{1}{2}\int_0^{t\wedge T_n} f''(X_s)d\langle M\rangle_s$$

$$\downarrow \quad \downarrow \text{ as } n \to \infty$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dA_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s.$$

Or in other words, the LHS and RHS are modifications of each other. However, both are continuous thus they are indistinguishable.

So, thus passing to the limit in star, so what is star? So, it was in earlier, few slides earlier, so I have written down again, so where this, you know, f of Xt minus f of X0, the increment of this, you know function of the stochastic process, that is written as the, you know, as the telescopic sum and that is written as these three terms.

So, now at this place if we take limit, in which sense, now almost sure sense, correct and then what are the sequence? Instead of pi I am going to take pi mk, along the subsequence we take the limit, okay.

Then we know the right hand side convergence. So, let me write down that, so where it converges? J1 pi would converge to, you know this term and J2 pi would converge to this term and J3, half J3 pi would converge to half of this terms. This thing is just obtained newly, and then f of Xt minus f of X0 is like that, okay.

Now, here I am going back to the earlier notation, because here I mean, because as I have mentioned that okay I have taken only the localized version of the process and then pretended that okay that is my original process. Now, when for that we have obtained this, you know Itos formula for this, you know localized version of the process so then we go back to the earlier version.

Now, this X is the original process, okay, original semi martingale and then what we have obtained actually in terms of original semi martingale, only this thing that okay so where it is only the localized version of the original semi martingale, okay, f of Xt n is equal to f of X0 plus integration 0 to t f prime Xs dMs plus 0 to t f prime Xs dAs plus half integration 0 to t double derivative of f Xs and integration with respect with to the quadratic variation of M, M superscript n.

Okay, so now the right hand side let us recall the definition of the stopped semi martingale. There we just stopped at time Tn, okay Tn and then onward it was exactly the same value, it was just constant there. So, when my s is between 0 to t okay, and if Tn is more than small t, then this integrand and integral everything is as same as the original one. However, if T, capital Tn is smaller than small t then only you have that problem that okay sometimes it stops and it deviates from the original given semi martingale.

So, this, therefore, this whole integration is exactly nothing but 0 to t minimum Tn of f prime Xs dMs, okay. So, now each term is now rewritten in this way. So, right hand side is nothing but the integration of the original process with respect to the martingales.

Only thing is that the upper limit is instead of time, you have time minimum with the stopping time Tn for each and every n, you have this equality. And here we have obtained this equality. This equality is true almost surely, probability 1, okay. When small t is fixed and this is true, for you know probability 1 for each and every n, okay.

Now, we know if you fix omega and you vary n tends to infinity the Tn would cross every finite time as it goes to infinity with probability 1, right, for every omega we are going to see, every omega means with probability 1, okay, almost every omega we are going to see that Tn go to infinity. So, if we now let n tends to infinity, okay here on this right hand side, this small t minimum capital T would become small t only, okay because capital Tn would become larger than t after a finite n onward.

So, that would take place on each and every term on the right hand side. So, right hand side we are going to get exactly this thing as n tends to infinity, and left hand side would also be, since you know right hand side converges almost surely, left hand side would also converge almost surely and you know that okay, that f is a continuous function and if X Tn converges to XT you know and therefore this also converges here. Okay, so we obtain this final equation which is not yet exactly Itos formula but it looks like Itos formula, correct? It is true for all t, okay.

What is the difference, okay what is the gap? Now, that we have obtained for each and every, after fixing every t, we have obtained this with probability 1, okay but Itos formula says that, left hand side as a process and right hand side another process, these two process are equal, okay, or indistinguishable with probability 1, okay.

So, here what we have obtained is just a modification. We have obtained the left hand side is a process, right hand side another process and they are modification to each other because they are equal with probability 1 for each and every t. But we need to show that for all t, they are equal; that, probability of that is 1.

However, since both the sides are continuous, okay so since both are, both sides are continuous processes so modifications are therefore indistinguishable, okay. So, here since they are continuous they are indistinguishable so we know that, okay that is the proof of the Itos formula. Itos formula is complete here.