Introduction to Probabilistic Methods in PDE
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Lecture 13
Change of variable formula and proof Part 01

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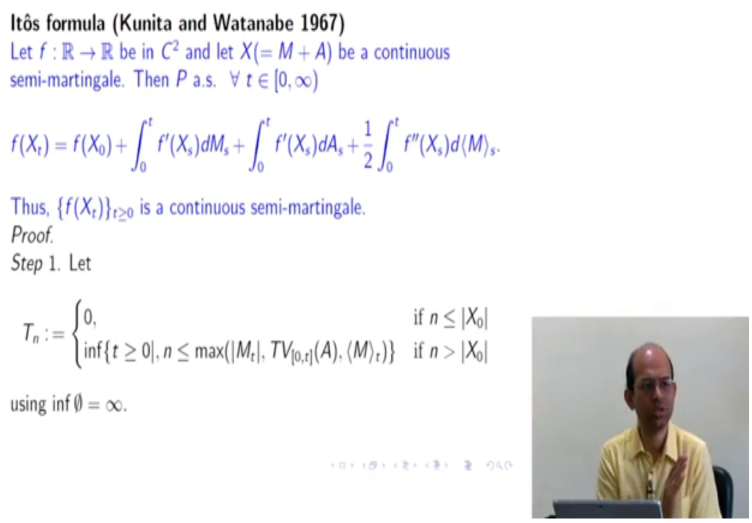

Okay, so let us come to the formula what we were talking about in the beginning of the slides that Itos formula is the one which is analogy to the fundamental theorem of calculus which will be useful to find out integration, okay without going into the first principle. So, let us read what is written here.

Let f is a function from R to R so only one dimensional, okay, real to real but it is twice continuously differentiable and assume that X is a semi martingale, okay which has the decomposition M plus A okay and this is a continuous semi martingale.

So, then, P almost surely for every time t from close 0 to infinity, f of Xt can be written as f of X0 plus integration 0 to t f prime Xs dMs so this much is like you know, what is coming like fundamental theorem of calculus that okay f of Xt is there, and you are taking derivative and you are also integrating, okay 0 to t.

However, it does not stop here. Plus integration 0 to t f prime, this is the derivative, f prime of Xs dAs plus, okay A and this together, because X is M plus A, so derivative of f and integration with respect to X is the addition of these two terms and then we add one more additional term here, that is that half of integration 0 to t, double derivative of f with respect to, integration with respect to the quadratic variation of M, okay. So, this is one additional term which arises in this case, okay.

So, now we discuss Itos formula, this version is by Kunita and Watanabe from their work in 1967, so here we consider a function, a real value function f from R to R, one dimensional and it is square, I mean it is continuously twice differentiable and X is a semi martingale, continuous semi martingale which has this following decomposition M plus A, M is a continuous local martingale and A is a bounded variation process adapted bounded variation and continuous process.

And then the function f of Xt is equal to f of X0 plus integration 0 to t derivative of f of Xs dMs integration 0 to t, derivative of f Xs dAs plus half of integration 0 to t double derivative of f evaluated at Xs and integration with respect to the quadratic variation of M, okay.

So, here we see that these two integration together stands for derivative of f integration of that with respect to dXs, okay. However, this term, okay is additional term which does not arise in the classical calculus. If we wish to ask the question that what happens for the special case where Xt is not a semi martingale but just a bounded variation process.

Okay, the kind of process, no, what we see in the classical calculus, I mean because in the classical calculus when we talk about integration we mean, you know Riemann Stieltjes integration, there we take integrator to be a function of bounded variation. If that is the case since this is continuous bounded variation process, its quadratic variation would be 0, this part would be 0, this part does not arise, okay.

However, so for our case quadratic variation would be non-trivial, okay in principle so we keep it, okay so that is the Itos formula. So, we are going to actually prove this Itos formula. It is, we are going to prove reasonably in full details, consequence of this Itos formula is that this f of Xt is also a continuous semi martingale, okay.

We started X as a semi martingale and then we have concluded that since f has this decomposition, because this is, you know from the theory of integration with respect to continuous local martingale, this is also a continuous local martingale and this part would be another continuous process because integration with respect to bounded variation process.

And here this would be also, like you know these are Stieltjes integrations, these two terms 0 to t these are continuous path process, okay so this is a local martingale and this part is the bounded variation process, continuous bounded variation process. So, this is also semi martingale decomposition of f of Xt. So, f of Xt is also continuous semi martingale.

The initial question that why this formula gives the capability of finding integration? Thing is that if one is given this type of integration and asked to compute then one has to find out this function, one has to identify this as a derivative of some function. Or in other words, one should look at the anti-derivative of f prime okay that is f. And then this integration value can be obtained by calculating other integrations, okay.

So, one can obtain this integrations using the classical sense and these are these f of these values so therefore this integration value can be obtained by evaluating all other values. Okay, so that is also going to serve the purpose of calculating the integration.

So, the proof has many steps. So, in the step 1, what we do? We actually take a sequence of stopping times. That is important for our case because we would consider this stopping time so that this, we can localize the integrands and integrators both so that they are, for them we can actually make those as square integrable martingales, okay.

So, let us construct this Tn. What is this Tn? Tn is the first time when this maximum of these three things hits n, the threshold n okay. For whatever large n we choose that first time that mod of Mt, mod of Mt is the absolute value of the integrator and the total variation of A, okay if we have some A, non-trivial A here.

And the quadratic variation of M, okay so either of this, okay, if one of this maximum touches n then we call that time t for the first time when it touches as Tn and if never touches, if n is large and for infinite time you wait and the maximum never touches there then this is an empty set.

So, infimum of an empty set, by convention is infinity we are going to take Tn is equal to infinity for those occasions. And why this 0 is there? This is the case where the mod X0 itself is a larger than n, okay then we do not ask this question. We just put this 0, okay. So, mod X0 is equal to more than n, then we will put 0 and we wait for n to cross this mod X0. When X is more than mod X0 then we ask this question. Then we define Tn in this fashion. Before that we put Tn is equal to 0, 0, 0, 0, etc.

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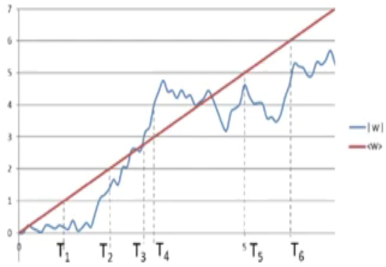

Define $X_t^{(n)} := X_{t \wedge T_n}$.

Itôs formula (Kunita and Watanabe 1967)

Let $f : \mathbb{R} \to \mathbb{R}$ be in C^2 and let $X(= M + A)$ be a continuous semi-martingale. Then P a.s. $\forall\, t \in [0, \infty)$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dA_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s.$$

Thus, $\{f(X_t)\}_{t\geq 0}$ is a continuous semi-martingale.

Proof.

Step 1. Let

$$T_n := \begin{cases} 0, & \text{if } n \leq |X_0| \\ \inf\{t \geq 0 \mid n \leq \max(|M_t|, TV_{[0,t]}(A), \langle M\rangle_t)\} & \text{if } n > |X_0| \end{cases}$$

using $\inf \emptyset = \infty$.

So, this is a graphical way of stating what is articulated here in this example. So, here consider the semi martingale to be just the Brownian motion W and then so W, you know Brownian motion is a, not only local martingale, actually it is a quadratic, you know square integrable continuous martingale and consider the quadratic variation of W. We would later see that okay that is nothing but the identity function t itself. So, we draw both. So, we draw mod of W. See here is mod of Mt, okay so for me this Mt is, this X is Brownian motion, A is 0 here for that example and M is Brownian motion.

So, this part I need to consider, this part I do not need to consider because A is trivially 0 and we need to consider these two things, okay. So, mod Mt is drawn here. This is a simulation of

Brownian motion of one path. Okay so when it is becoming negative but mod is pushing it, making it always positive, okay, the Brownian motion path, and then this is the quadratic variation of Brownian motion. And there n is equal to 1, 2, 3 etc. And X0 is equal to 0.

So, T0 is 0 here but T1 is the first time, blue or red okay which touches 1, so here red touches 1 so this is T1. Second time when either blue or red which touches the threshold 2, that is my T2. Then T3 is the first time blue or red whichever is touches 3. So, here blue touched 3 first, red touched little later so T3 is here.

Actually another time T4 actually, you know the red line touched 3 but okay, the blue line, the mod of, modulus of the integrator is way above there so this, it touched 4. So, now therefore it is, therefore, this time is T4.

Similarly, therefore this time is T5, this time is T6. So, we see that okay, we are getting an increasing sequence of stopping times, T1, T2, T3, T4, T5, T6 it will increase. And then after obtaining this sequence of stopping times we construct a stochastic, stop stochastic process, the localized that X t minimum T n.

So, what does it mean? That okay, so after Tn, say for example, n is equal to 4. So, if stochastic process is here and then we are going to put this is equal to same thing that is constant, flat. That would be the sample path for that particular omega.

So, that means my localization of the integrator, so integrator is localized this way. So, here you remember that this localization I have not used integrands, I am not looking at it because you know that here just the, I mean, here integrand is very special thing. It is actually function of the integrator itself, correct? We do not need actually general, here the general integrand does not appear, right. Integrator is very special type, right? It is just the function of the integrator itself.

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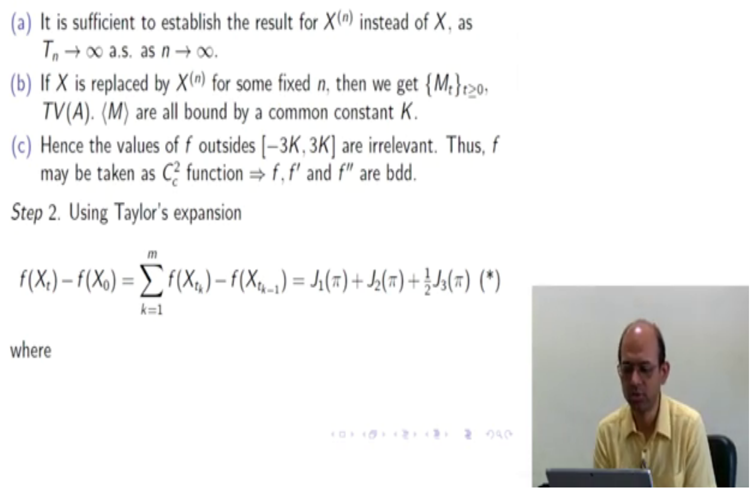

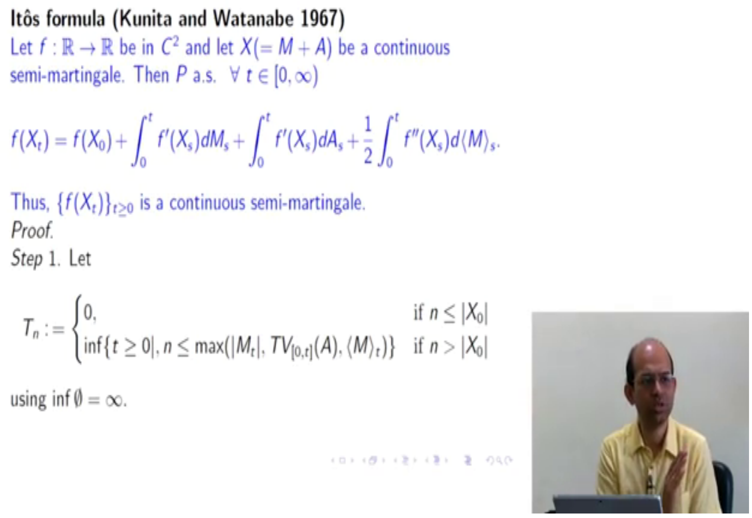

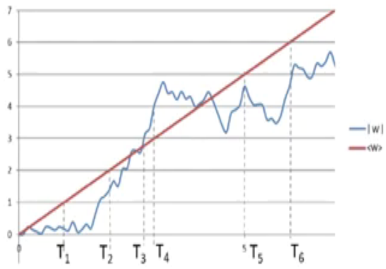

Define $X_t^{(n)} := X_{t\wedge T_n}$.

So, now we would consider that this Xn, okay so it is sufficient to establish the results for Xn instead of X as Tn tends to infinity as n tends to infinity, okay. So, whatever we would like to establish this, you know this equation we are going to establish only for this stopped process Xn, okay.

Why is it so? Because even if we do for Xn and then t will be replaced by t minimum T n everywhere, correct? Or everywhere t minimum T n and then as n tends to infinity, Tn goes to infinity, okay with probability 1. So, right hand side would converge to this almost surely. So, left hand side should also go to there, okay. So, we are going to consider only this, you know stopped process X n.

Now, for the convenience of writing we replace X by Xn okay, next onwards. So, whenever we say in the next process X, actually I mean X superscript n. Whenever I say M, I mean that M superscript n etc., I mean that martingale part of Xn. Xn is a semi martingale but martingale part of this.

Okay X is replaced by Xn for some fixed n, then we get Mt okay and total variation of A, okay and this quadratic variation of M are all bounded by a common constant K because the way we have done. Okay, these are all bounded, and then since I have removed the notation of n, that means there is a fixed n, okay. So, everything is for a fixed n. I am not letting n tends to infinity anywhere here. Okay so for a fixed n, so there is a fixed constant K.

So, the values of f, the function f outside the interval, so values of function f outside the interval minus 3K and 3K are irrelevant. Why is it so? Because the process does not leave this thing, I mean this interval minus 3K to 3K. Why this 3K is coming? Because you know I mean these three terms I mean, three terms are coming, so mod Mt is also between minus K to K, this is also minus K to K, etc.

So, f prime of this, you know this inside X, okay does not leave that interval. So, for this integration, does not matter what the value of f and f prime, okay are there beyond that interval. Whatever I mean, even if it has, if I take two different function f1 and f2 which matches exactly the same in minus 3K to 3K both would have exactly the same, you know left hand side and right hand side. They would not, since the process does not go beyond this interval.

So, we can therefore, choose f to be twice differentiable and with compact support. Actually I can, minus 3K, beyond minus 3K to 3K I can extend the function such that it has a compact support and it remains twice continuously differentiable. So, what is the bottom-line? The bottom-line is that, that you know, that twice differentiable you know functions with compact support, they are all first, second derivatives are bounded, okay. So, they would be bounded.

So, now we are in a very good place because the integrand is now bounded therefore, it is bounded. Using Taylor's expansion we can write down f of Xt minus f of X0 is equal to, this is obtained by choosing a particular partition pi, that pi is partition of t1, t2, tk etc., tm, okay. So, this thing is not a telescopic sum, f of Xt minus f of X0 so that is f of Xtk minus f of Xtk minus 1, okay. So, here I have not used Taylor's expansion. I have used it here actually, okay.

So, this is just a telescopic sum so all these you know, first term is Xt1 minus Xt0. So, Xt0 is, t0 is taken as 0, etc. So, that sum, okay for each and every increment, so it has m number of increments. For each and every increment we are going to get, okay, we are going to use Taylor's formula there, okay. So, for that difference we are going to get expansion, okay. So, that expansion is also written in terms of J1, J2, and half J3. This is second derivative. This is the inverse second derivative. These two inverse, first derivative.

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$$J_1(\pi) = \sum_{k=1}^{m} f'(X_{t_{k-1}})(A_{t_k} - A_{t_{k-1}})$$
$$J_2(\pi) = \sum f'(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$$
$$J_3(\pi) = \sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2$$

where $\eta_k = X_{t_{k-1}} + \theta_k(X_{t_k} - X_{t_{k-1}})$ for some $\theta_k \in [0,1]$.

(a) $J_1(\pi) \to \int_0^t f'(X_s)dA_s$ a.s. as $|\pi| \to 0 \Rightarrow J_1(\pi) \to \int_0^t f'(X_s)dA_s$ in L^1.

(b) $Y = \{f'(X_s)\}_{s\geq 0}$ is in $\mathcal{L}^*$.

(c) $Y_s^\pi := f'(X_0)1_{\{0\}}(s) + \sum_1^m f'(X_{t_{k-1}})1_{(t_{k-1},t_k]}$.

(d) $[Y^\pi - Y]_t \to 0$ using DCT.

(e) So $J_2(\pi) \to I_t^M(Y)$ in $L^2(P)$.

So, where, let me write down what is J1. J1 is the f prime Xtk minus 1 times Atk minus Atk minus 1. And here second J2 is the same derivative, okay same integrand but Mtk minus Mtk minus 1, and J3 is the second derivative, correct, half of that, okay that appears. That f double prime but then derivative is evaluated at some point, okay, at some point which is between Xtk minus 1 and Xtk.

So, how do you write? So eta k is a number which is like Xtk minus 1 plus theta k times Xtk minus Xtk minus 1 where theta k is between 0 and 1, okay. So, this is a consequence of Taylor's expansion, okay. So, if we expand Taylor's, do expansion of each and every term then we are going to get. However, we have sum over k, k is equal to 1 to m, so we are getting this sum also, k is equal to 1 to m, this sum is also there.

Now, let us look at each and every term separately, okay. So, it is a little long proof, okay. So, J1 pi, J1 pi is, this is like you know f prime Xtk minus 1 times Atk minus Atk minus 1. So, Atk can be viewed as integrator here which is a bounded variation process and here this integrand is evaluated at Tk minus 1, okay. So, since it is, you know continuous function of time, why is it continuous? This X is a continuous function of time, f prime is also continuous. So, composition of continuous functions.

Continuous function of time, it does not matter what we choose, okay so, and then which you are choosing the left point, and then as, you know, the mesh size goes to 0 and then this would give me just the, Riemann Stieltjes integration of f prime of X with respect to integrator A. Okay so this J1 pi would converge to this, Riemann Stieltjes integration as mesh size goes to 0. And that would happen with probability 1; almost surely we are going to see this, okay.

Since f prime is bounded, the total variation of A on 0 to t is also bounded so therefore, this is the bounded random variable, okay, bounded random variable and that measure is, probability measure is also 1, okay so finite measure.

On finite measure the bounded, the sequence of bounded random variable, if that converges almost surely, that converges in L1 also, correct? We can use actually Dominated convergence theorem okay, bounded convergence theorem actually, bounded convergence theorem.

J1 pi converges to 0 to T f prime Xs ds in L1 as mesh, this partition, mesh size goes to 0. Remember that, here we are not taking limit as n tends to infinity, correct because n I have fixed earlier and that I am not changing. I am just changing the mesh, the partition here, time partitions, okay.

Next is Y, what is Y, Y is that integrand f prime Xs. Why are you looking at this? Because in J2 pi we have this, you know integrand f prime Xtk minus 1 and here the candidate of integrator is the martingale, okay. So, we need to justify that, you know, convergence of the integration.

So, for that what do we need to do? We need to actually justify where is it, okay, which process? So Y is in L star. Why is it so? Because it is continuous and then it is also adapted, so it is progressively measurable. So, it is in L star, it is actually bound, this is bounded so all these, you know box norm, finite, expectation box, okay, that is also true because it is bounded, okay. So, all these conditions we can satisfy. So, Y is in also L star.

Now, for this Y, let us look at this, this is like discretization of this Y, okay. So, let us write down this discretization precisely. So, this is, Y pi s for this particular partition, f prime Xtk minus 1, integrator function of Tk minus 1 to Tk, close Tk. So, this approximation of this Y, if you integrate with respect to martingale, you are going to get exactly this thing, correct, exactly this thing, okay? So this integration.

So we are, to understand J2 pi, we can write down, J2 pi as therefore integration of this with respect to the, with respect to Mtk okay. So, here M due to the localization is a martingale, correct, it is no more local martingale; it is a martingale and square integrable martingale.

So, we are properly in the setting that this is now a sequence of simple processes which is converging to Y, okay so we can also cross verify this thing, easy to cross-verify, it is not difficult actually, that using the Dominated convergence theorem, that this Y pi and Y, this box, metric box metric, that goes to 0, okay for each and every time t and then we know that how to compute the box metrics overall, that for every n, and then you take minimum of 1 and divide it by 2 power n and sum over all possible n. So, that will also go to 0 therefore, okay, for every point it is going to 0.

So, J2 pi, because J2 pi is nothing but integration if thing with respect to martingale, and then this integral, integrand is converging to Y in box norm so we know that okay, this integration of this Y with respect to the square integrable martingale M is just nothing but the limit, okay, L2 limit of J2 pi. So, J2 pi converges to this, L2P. Okay we have this, we save this result, okay, J2 pi.

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Step 3:

$J_3(\pi) = J_4(\pi) + J_5(\pi) + J_6(\pi)$ where

$$J_4(\pi) = \sum_1^m f''(\eta_k)(A_{t_k} - A_{t_{k-1}})^2$$

$$J_5(\pi) = 2\sum_1^m f''(\eta_k)(A_{t_k} - A_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$$

$$J_6(\pi) = \sum f''(\eta_k)(M_{t_k} - M_{t_{k-1}})^2$$

Now, we should start talking about J3 pi, okay.