Introduction to Probabilistic Methods in PDE
Professor Doctor Anindya Goswami
Department of Mathematics
Indian Institute of Science Education and Research, Pune
Lecture 12
Extension of stochastic integral

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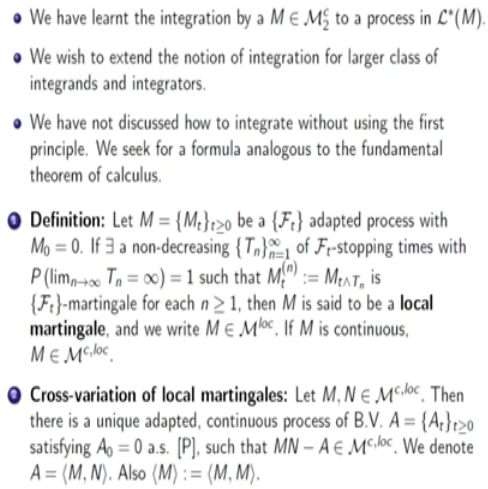

In the last lecture we have seen stochastic integration of integrands with respect to square integrable martingale, okay. So, we have learnt when integrand is M which is member of Mc2, square integrable continuous martingale and the integrand is the process from L star M which is progressively measurable and also square integrable with respect to M so that process. We wish to extend this notion of integration for larger class of integrands and integrators.

We have not discussed also how to integrate without using the first principle. What does it mean? I mean that the definition of integration is via the convergence, okay and there we have used the sequence of simple processes, okay and that is the way we have obtained this integration. So, or in other words we just know only the first principle.

If you compare this integration theory with the Newtonian integration theory, that also had on first principle integration, correct because you take the integrand and then approximate using step function and there integrand is just a continuous, bounded continuous function and then you take the, using the partition you get the step function and then for that you obtain the area under the curve of that specific function and that converges to the actual integration. But that is the first principle approach to find an integration, okay.

However, for integration we know there is a second approach also for classical Newtonian integration. How? That is like anti-derivative, okay. So, what is anti-derivative? That is obtained by using fundamental theorem of calculus. What does it say? It says that if we integrate the derivative of a function, then we get the function back. So, here also one can ask the same question, okay. I understand the definition of integration but how can you calculate easily using some sort of theorem which is like fundamental theorem of calculus here? So, we have not discussed that yet.

So, we seek for a formula analogous to the fundamental theorem of calculus, okay and that formula we call Itos formula. So, that gives the ability to find out integration from a formula, okay without going to the first principle, okay.

Now, let us start definition, okay. So, this is a definition of a larger class of processes which need not be a martingale. However, a martingale is also a subclass of this. Okay, so this subsumes the class of square integrable martingales. Here we call this as a local martingale. So, let us start.

Let M be a process, which is written as M subscript t, t is 0 or more than 0, positive, so this is Ft adapted process with starting 0, M0 is equal to 0. If you remember that was also the constraint we put for denoting script M2 and M2c.

So, if there exists a sequence of non-decreasing family of random variables Tn, n is equal to 1 to infinity of Ft stopping times such that it grows to infinity with probability 1, okay. Now, if you look at the limit, so limit is infinity with probability 1, and with the help of this Tn if we construct this stopped process M subscript small t minimum capital Tn what is the meaning of this?

The meaning of this is that, the value of this process at times small t would be exactly as Mt if capital Tn is more than t. However, for some realization when small t is more than capital Tn, that means there would be capital Tn is smaller than small t then we are going to choose only M at capital Tn. So, minimum of t and capital Tn we are going to choose, okay and that would be my M superscript (n) t.

Now, for every n okay I am going to get one such stochastic process. Now, my stochastic process M is such that there exists a sequence of increasing stopping times Tn so that this derived new sequence of processes for every member n, we are going to get a martingale, okay. M is a Ft martingale, okay. So, this is an Ft martingale for each and every n.

So, we have actually sequence of processes but for every n you get a process and that process is a martingale. Then we say that this M is a local martingale and we denote the class of all local martingales as script M loc. And if M is continuous in addition to be a local martingale we call that M is a continuous local martingale and that subclass we denote by script Mc comma loc, okay so Mc comma loc is a subclass of M loc.

Now, for this we would like to define what is quadratic variation, okay, covariation of such processes. Okay, so now we will recall that what was a definition of covariation and quadratic variation. For quadratic variation we actually used the Doob-Meyer decomposition that given M is going to be a martingale, there with M square we look at the Doob-Meyer decomposition then the natural adapted increasing process was taken as a quadratic variation. But here when M and N are two continuous local martingales their square integrability is not assured, okay.

However, we generalize the notion of quadratic variation here. So, this theorem, this statement says that, there is a theorem which asserts that existence of unique adapted continuous process of bounded variation, okay A such that A starts from 0 and product of the process M and N minus, subtracted A, so that is a continuous local martingale, right.

So, since M and N, neither of them could be square integrable, so I cannot assume that this difference would also be square integrable, okay. However, since this is M c loc, I can look for M N minus A is in Mc loc, okay, continuous local martingale. And if that is the case then we denote this A as quadratic covariation of M and N, okay and if I have M and N both identical then the quadratic variation, covariation of M and N that is called quadratic variation of M, we also denote it by only writing M here.

This condition is coming straight from the generalization of the Doob-Meyer decomposition. There we had M N, okay is equal to a martingale and plus some bounded variation process, square integrable martingale and bounded variation process. Here, and then if we subtract we get a square integrable martingale. But here we cannot expect, so we are just asking the difference is just continuous local martingale.

Okay so here let us also clarify this, you know, issue that whether this notation is generalization? This coincides with the notation and definition what you have done before, yes, because square integrable continuous martingale is also continuous local martingale. Why is it local martingale? Because you can choose Tn is equal to be exactly n, or deterministic and then you are going to get that, okay, that satisfies this conditions, and you are going to get that this is a subset of that. Next, what we do is that we try to develop the notion of integration with respect to local martingale.

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- **Result:** If $M \in \mathcal{M}^{c,loc}$ is of bounded variation, then $M \equiv 0$.

If M is local martingale, continuous local martingale and is of bounded variation then M is identically 0. So, this is also extension of the result what we have actually proved in earlier class, okay so where M was Mc 2, square integrable continuous martingale, so that is a special class of this class. So, for the special class we have obtained this result but this result is also true, this conclusion is also true for a general class; that if we have a continuous local martingale and it is of bounded variation it is actually a trivial martingale.

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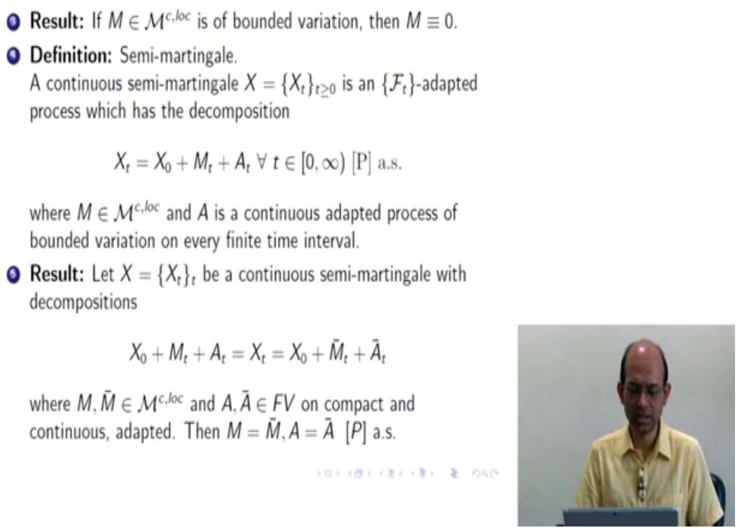

Now, we define a even larger class of stochastic processes which we call as a semi martingale. So, what is semi martingale? Semi martingale is an Ft adapted process. Why I am saying always Ft adapted? That means even without specifying each and every time I always assume that there is a filtered probability space in our hand already, so omega F P is the probability space and Ft is the filtration there.

So, given, and fixed that filtration okay, and X is Ft adapted process and has the following decomposition. What is this decomposition? Xt is written X is 0 which is just a random variable, F0 measurable and we have Mt, Mt is a continuous local martingale and At is a continuous adapted process of bounded variation, okay on every finite time interval.

Why finite time interval? Because even if you take a very nice, you know increasing process say At is equal to t, but its total bounded variation will be infinity if you take whole closed 0 to infinity, okay. So, when you make a finite interval, on that time interval if we compute the total variation then we are going to get finite, okay, so boundary variation on finite interval.

So, this is Xt, it is a semi martingale. We are going to use this term quite frequently for this lecture and one or two next because this is the more general process we are going to discuss. Okay so next we talk about the uniqueness of the decomposition if X is a continuous semi martingale with a decomposition, two different decompositions, for example X naught plus Mt plus At, another is X naught plus M bar t plus A bar t, then where this M and M bar both are local martingale, continuous local martingales and A and A bar are both finite variation processes on compacts.

So, basically we say there are two different decompositions, okay, semi martingale decompositions then this M and M bar would be equal with probability 1 and A and A bar would also be equal with probability 1, that means they are indistinguishable.

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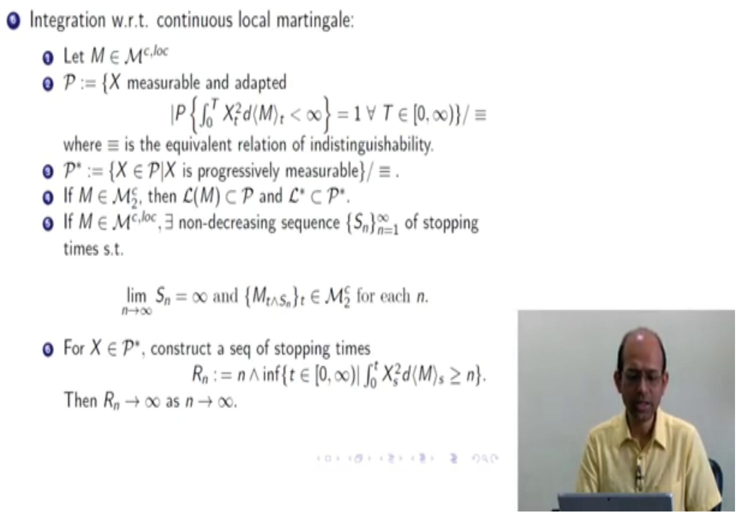

Now, we describe how do we actually go for integrating with respect to continuous local martingale. If M is in M c loc, continuous local martingale we would consider then for the class of integrands, this class, X is measurable and adapted, okay and the probability that this integration Xt square dMt, 0 to T is finite with probability 1.

Okay, remember that when we have obtained the class L of M then we have taken expectation of this random variable and we have required that expectation to be finite. However, here that is very strong condition to impose.

Why is it so? Because my M is such that, okay, M is just a continuous local martingale, so it is, I mean I might not have that expectation to be finite of this thing, okay. However, I can just require this to be finite valued random variable, okay. If some random variable has expectation finite that implies that the random variable has, is finite valued, okay.

But the reverse is not true. I just have finite valued random variable that means real valued random variable, okay. It could be anything, it need not be L1, L2 okay. So, the probability of this random variable is finite, is 1.

So, that type of processes we just gather and make this class script P okay and then in that class we would of course take the quotient with respect to the equivalence relation of indistinguishability so that two different processes which are indistinguishable should be identified as a one single object.

So, that is the class P. And P star, P star is actually, you know, resembling with L M and L star M as we have introduced in the earlier lecture for the case when we consider the integrand for square integrable continuous square integrable martingale.

Here for continuous local martingale, we have P and P star. Here X should be in P however, X is progressively measurable. And then you take the quotient. If M is a continuous square integrable martingale and then we have that L M as I have described earlier. Here you had expectation of this thing was finite, okay. So, that implies that, okay this is also finite. So, therefore, L M is a subset of P and L star is a subset of P star. So, this is a generalization we understand, okay.

So, if M is in M c loc, okay, continuous local martingale, so then there exists a non-decreasing sequence of Sn okay, of stopping times such that this following result holds. What is that? That the stopped process is square integrable continuous martingale.

So, now I mean, let me clarify. Given a continuous local martingale we know that there exists of course a sequence of stopping time, okay for which this stopped process becomes a continuous martingale. But this is telling something more, okay. This result is saying that I can choose the sequence of stopping time, increasing stopping time such that the stopped process becomes square integrable continuous martingale, okay.

For definition of local martingale, you just need that stopped process becomes a martingale, right. Here we are asserting that we can further choose a sequence of stopping time such that this becomes square integrable continuous martingale, okay.

Next result is that, for X is in P star, P star as it is defined in the third point, construct a sequence of stopping times. So, what is this stopping time? We are considering, say X square is d <M>s, <M> is here is quadratic variation, 0 to T and when it is more than n. We understand that we are looking at this criteria actually that finite should be probability 1, okay. Now if it is crossing a large number n okay, and then we are checking that at which time it is crossing that n, okay. So, wherever it is crossing that n, that time that I am recording Rn, okay.

However, if this time is too large, okay then we might run into some problem, technical problem so we are taking minimum with n also. So, if you know this integration becomes larger than n before time t is equal to n, then that time would be Rn. If it takes longer time than n then I am going to take n as this, okay.

So, Rn has this property that as n tends to infinity because if you take larger and larger N, we know that for t is equal to 0, it is 0, left hand side so it would take more longer and longer time to cross the threshold n and then this Rn would increase to infinity okay so then the Rn goes to infinity as n tends to infinity, okay. So, actually these are results, so these all, these results are not very immediate or obvious. It needs proof. However, we are just stating these results here. Rn tends to infinity as n tends to infinity.

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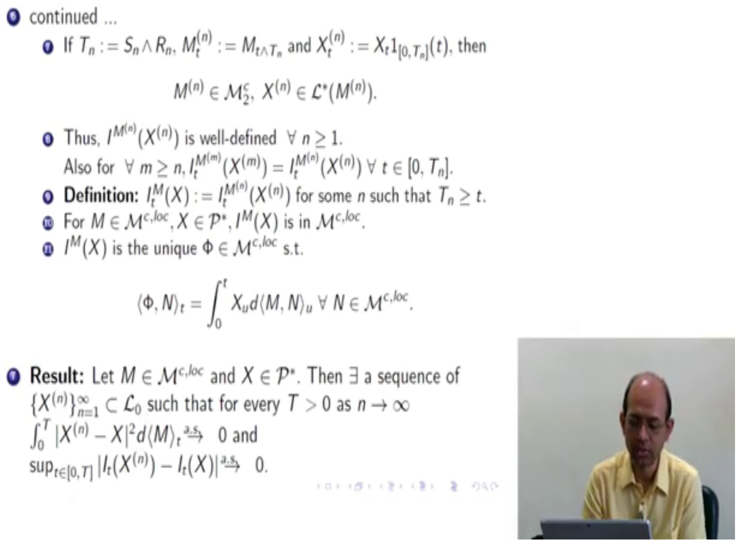

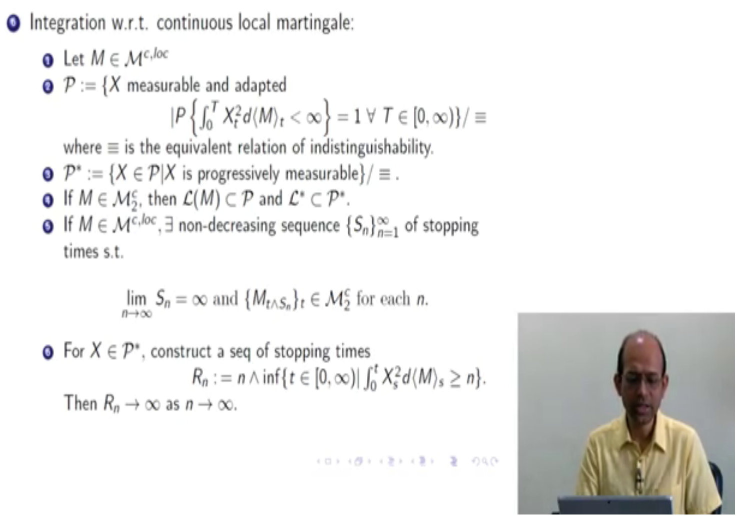

Next, we consider the minimum of Sn and Rn. So, let us see again what is Sn. Sn was localization of the martingale, see. We call localization, correct because there we are stopping it so that we are preventing M to fly away and then that means we are localizing that, okay.

So, that is the reason of the name of localization and this Sn is the sequence which is localizing the integrator. Here Rn is on the other hand, is actually helping to localize the integrand, okay in the sense that this, you know, with respect to this martingale integration so that this thing remains bounded.

So, with Rn and Sn together we want to retain both the properties we take minimum of Sn and Rn, okay, that we call as Tn. So, Tn is such that both the properties do hold, that integration of Xs square with respect to quadratic variation, that also remains bounded and also that Mn t, so this also becomes a square integrable continuous martingale, okay and the integrand we choose as Xt, okay for all time till T n, okay. After Tn we are going to kill that.

That would be just 0, okay so that is the kind of localization we are doing. Now, from previous construction Sn and Rn we know that Xn is square integrable continuous martingale and Xn, the way it is constructed, is also in L star of M (n).

Why L star? Because you know, this process you know started with this adapted and continuous for this and then, I mean then you get that L star of M (n), progressively measurable. This is actually not continuous, this is actually left continuous, correct? So, if it is adapted either left or right continuous, okay so then you get that it is progressively measurable.

So, thus, now we can actually integrate this process with respect to this martingale, okay. Both are actually different from our original integrand and integrator. However, with these approximations, okay we can talk about integration exactly the notion what we have defined earlier, okay.

So, we construct this sequence of integrands. So, for every n we are going to get one integration, okay. That integration itself is also function of t, correct, that itself is also a stochastic process. So, this is well defined for each and every m more than or equals to n. So, also for, this is another nice property. When m is more than n, okay so we have It Mn Xn but now I am having m which is more than small n, okay and we are constructing Xn.

What would happen? So, here this Tn would be replaced with Tm, that would be, you know, that will survive for longer time and then become 0. But that, if I look only time t is equal to 0 to Tn, even I have killed later but I am looking only for the beginning time that it coincides, the beginning time would coincide with the exactly Xn, Okay and therefore, when small t is between 0 to capital Tn, then these two processes has identical values.

On the other hand, if you look at M m and M n they would also have identical values for every t between 0 to Tn, okay. So, we are going to get both the integrals are exactly same for every t between 0 to Tn. So, this is very useful observation.

Why is it so? Because that would give us a notion of convergence, correct? Because you know when I am observing on n and then n plus 1 onward that value of the stochastic integral is not changing on the interval 0 to Tn. However, as n is larger, this Tn is also going to infinity, correct.

So, for a given small t I can look for one n onwards so where Tn is more than t and then the stochastic value is not going to change further. Okay, so let us define from this observation that I M t x is defined as I t M n X n for some n such that Tn is more than t.

So. what does that mean? So, let me clarify again. So, here one should remember that Tn is a random variable. But t is deterministic. So, what does it mean? So, given a small t fixed, I am looking at some particular omega, for that particular omega I know that sequence of Tn omega is going to infinity, so after large n, it should cross small t, okay so whenever it will cross okay, that I am going to put that, okay, that n, and then with that n, we can actually construct this, you know stochastic integration, and whatever it is that I am going to declare like this, okay. So, for every omega we can do that way, okay.

So, this n is also, you know random variable, correct because you know, the n for which Tn is exceeding t, that is also a random variable. So, I mean it is not that I am saying that I mean one can find out deterministic n and then it remains, no it is not. n is also a random variable satisfying this.

However, that random variable is finite, is not infinity. Why? Because Tn goes to infinity almost surely, so for every small t, for every omega we are going to get a finite n and that we are going to plug in here, okay and then we do not need to worry. Then that n onward every time, okay stochastic integral would be the same value.

Next, we consider M is a continuous local martingale and X is in P star, okay then I M X as it is defined here, okay so here this is stochastic process with respect to time t, so this stochastic process is also in M c loc. It is also a local martingale, continuous local martingale.

So, remember what we have done earlier, that we have considered square integrable martingale here and then we have defined the stochastic integration to be a square integral martingale satisfying certain things, correct? But here we are not doing that way. We are cooking up this limit, okay, stochastic integration and we need to therefore, cross-check that whether this is really in this class, okay.

So, this is the result that yes, of course after obtaining this stochastic integration of I M t x, this process is also a local martingale, okay with continuous path almost surely, okay. So, what did we get? We got that, okay, I M, okay this functional, I is functional okay, for X in P star okay for a given local continuous martingale we are getting another continuous local martingale, okay.

Now, there are some other properties. So, this property is also true for a special class which we have already seen earlier that if I have quadratic covariation of this stochastic integral with respect to another local martingale, okay, in earlier discussion we had all the places square integrable martingale but now we have continuous local martingale.

So, if we have a continuous local martingale here and phi is integration of X with respect to another continuous local martingale M then the quadratic variation of that is X this integrand and the integrator is quadratic variation of M, this M and this N, okay, M and this N. And this is true for all N in M c loc, okay.

So, this is also another characterization. I mean no other process satisfies this condition, okay. So, the stochastic integration of X with respect to M is the only process for which you get this identity. So, this is also another result that let M is a continuous local martingale and X is in P star.

Then there exists a sequence of Xn from the simple processes such that for every time capital T positive okay, as n tends to infinity this approximation, this difference okay X n minus X whole square and with respect to the quadratic covariation d M t that goes to 0 almost surely, okay.

So, earlier what did we have? We had that this used to go to 0 in L1, correct because expectation of this used to go to 0, okay. So, that was the box norm. Here this result is more stronger, it is stronger than the earlier result.

It is saying that, okay if I have X from P star and then I can choose a sequence of simple processes so that, without taking expectation okay, so just this random variable itself, which is the integration 0 to capital T, for a fixed capital T and this integration is with respect to quadratic variation of M, so this random variable goes to 0 almost surely, and then for integration, corresponding integration that I t X n and I t X we take the difference.

So, earlier what we had shown is that expectation of this square, okay that goes to 0, okay but here we have stronger result that okay if we consider the supremum of that, that also converges to 0 almost surely, okay. So, it converges uniformly almost surely, right, supremum is going to 0. So, this is a function of t and also another function of t we take the supremum, the difference and that goes to 0.