Introduction to Probabilistic Methods in PDE
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Lecture 10
Definition and Properties of Stochastic Integration Part 02

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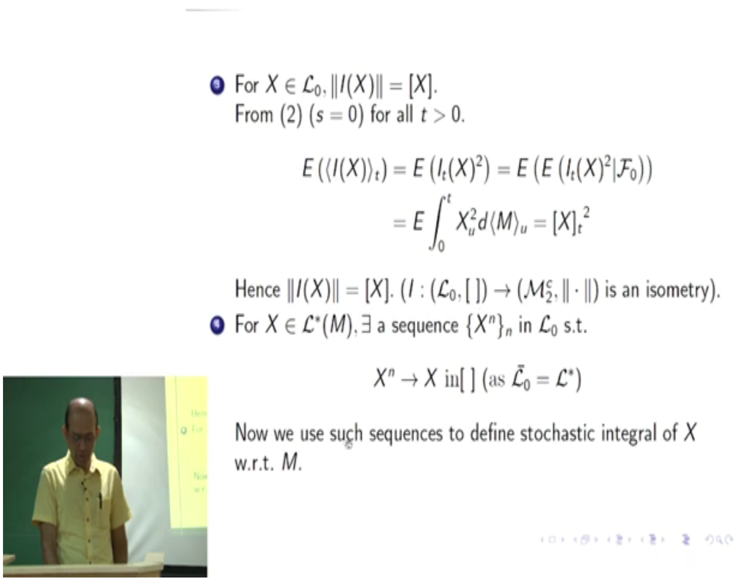

Now, we use such sequences to define stochastic integration of X with respect to M that is not yet done, till now what we have done? We have only defined what is integration of a simple processes with respect to a square integrable continuous martingale. And the results whatever Ito's isometry whatever we have obtained is also not general integrand, only for the integrand which is simple process.

Now, we are going to define stochastic integral for a progressively measurable process, okay, which we denote as you know space of this L star M and after defining that we are going to sorry, again as the same question whether that Ito's isometry still holds true there okay. So, we are going to prove that again there.

Of course the proof will be different because here in this proof we have heavily used that structure of the simple process that it can be retained as you know finite sum etcetera, but there we cannot, correct? So, we need to use the limit theorems etc. So, so, what is the idea? The idea behind defining this stochastic integral of X is that we so, we have X in one side and X n is the sequence which converging here and then for every X n I can derive one I X n and for I can change n and check whether this I X n converges anywhere if it converges wherever it converges that limit I would call that as I X, so that is the idea but for the as the first step I should either whether that converges at all.

Or in other words I would ask whether I X n Cauchy sequence if it is a Cauchy then it would converge because it is in the space of MC 2 which is a Banach space okay close subsets of M 2.

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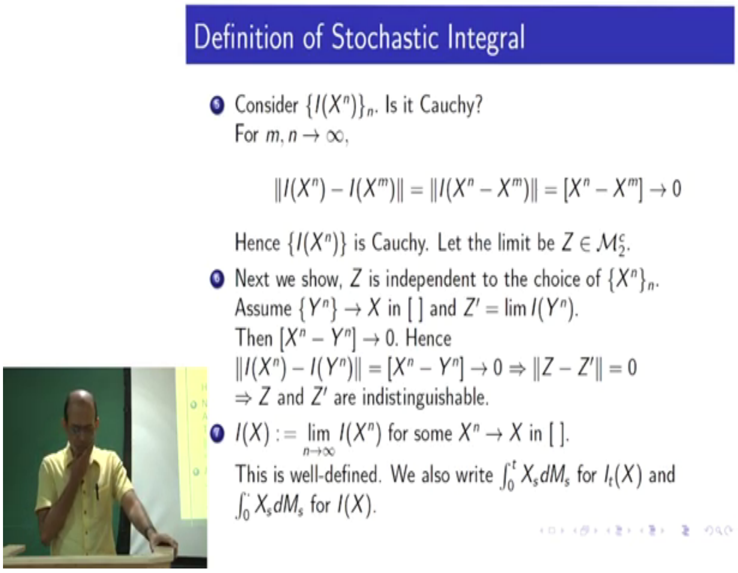

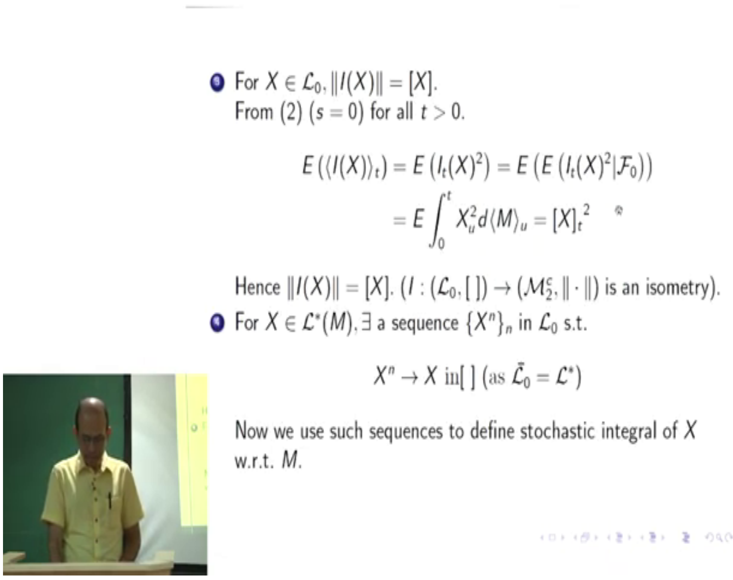

So, now we check we take any m and n okay and we would then let it go to infinity okay for any fixed m n first we look at I X n minus I X n we consider this difference okay and then this norm okay. So, due to the linearity of I we can write down I of X n minus X m and then we are going to use the isometry property which is established for the simple process, but here X n minus X m is also simple process so I can use it so this is equal to box norm of X n minus X m.

However, the main thing here is this X n is this sequence which comes X. So, this is itself a Cauchy sequence okay. So, I would get this difference also goes to 0, the difference goes to 0. Since this goes to 0 therefore, this goes to 0 and then what we have proved that I X n is a Cauchy sequence.

So, we call that limit be Z okay. So, Z is a limit. So, what is Z, Z is not random variable Z is a stochastic process because here all members of stochastic processes, correct. I have omitted So, whenever I have I do not write down any subscript t that means it is the whole stochastic process if I need to view only the stochastic process at a given time t then I write down the subscript t there, okay.

So, the limit is Z okay so Z is in the space of its continuous square martingale, so that is the limit Okay. Now, the question is that this Z is obtained via the sequence X n, a very natural question is that if I would have chosen another sequence perhaps Y n which also converges to X should I have got the same Z or some something different.

If I would have gotten something different then, then I cannot just then I am confused. Then I will be confused then which limit to be considered as the integration of X, okay, what should be I X, thankfully, we are going to show that we will be able to show that this Z is independent of the choice of the sequence.

It does not matter what sequence we choose which converts to X, always you are going to get the same Z okay. So, how to prove that I mean such type of theorem is generally proved by, you know, considering first as indeed okay let there be two even Z and Z prime and then show that Z and Z prime are exactly the same.

So, we do the same way. So, we show that Z is independent to the choice of X n okay. So, we assume another sequence of simple processes which converge to X in the box norm and Z prime is the limit of the integrals. So, limit of I Y n then since X n Y n both converge to same X okay that is our assumption.

So, box norm of X n minus Y n should go to 0 okay. So, this go to 0 okay. So, this much we have in our hand. So, now we consider the diff the norm of the difference of this two integrals I X n minus I Y n, again we are going to use it Ito's isometry what we proved earlier for simple processes. So, that is again X n minus Y n. But this goes to 0. So, this norm goes to 0 okay as n goes to infinity.

So, its limit and this limit should coincide so Z and Z prime the difference of that is 0 okay, norm of differences is 0 that implies that Z and Z prime are indistinguishable. Why do can we conclude it because already have proved that okay, if they are indistinguishable they are 0 and 0 that is indistinguishable okay.

Now, next when this so therefore, we are now good to use this Z as the definition of the stochastic integral of X with respect to m. So, this is well defined, okay, so we use this I mean so although you are using X n apparent apparently like you know we are choosing particular sequence, but this integral is a limit is independent of the choice of X n okay.

So, I X the integration of X is defined to be integration of X n with respect to M okay the whole process okay the process wise okay and then this limit is in the sense of the norm okay in that norm sense is not point wise or something in that that parallel norms for some X n converge X in box norm.

So, we write this integration okay this I X also in this manner, so, use this symbol integration 0 to t X s dMs we also denote this integration in this manner for It X and if I want to write down I X then for that as an integration, we often write integration 0 to dot okay Xs dMs to denote that okay I am not integrating from 0 to say a particular time but is like you know a stochastic process I just want to write down which is coming, which is arising from the integration okay. So, these are the notations we might use in the future also.

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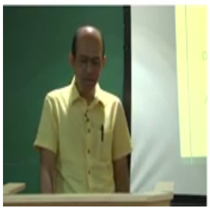
- Let $\mathcal{L}_0 \ni X^n \to X \in \mathcal{L}^*$ in [ ], then using the Jensen's inequality for conditional expectation, $I_t(X_n) \to I_t(X)$ in $L^2(P)\ \forall\ t > 0$. For $A \in \mathcal{F}_s$, (using (2)), we get

$$E\left[1_A(I_t(X) - I_s(X))^2\right] = \lim_{n\to\infty} E\left[1_A(I_t(X^n) - I_s(X^n))^2\right]$$
$$= \lim_{n\to\infty} E(1_A E\left[(I_t(X^n) - I_s(X^n))^2|\mathcal{F}_s\right])$$
$$= \lim_{n\to\infty} E(1_A E\left[\int_s^t (X_u^n)^2 d\langle M\rangle_u|\mathcal{F}_s\right])$$

Denote $(X^n \odot 1_A)_u = \begin{cases} X_u^n\ \forall & u < s \\ 1_A X_u^n & u \geq s \end{cases}$. Thus



$$\int_A E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right] dP$$
$$= \lim_{n\to\infty}([X^n \odot 1_A]_t - [X^n \odot 1_A]_s) = [X \odot 1_A] - [X \odot 1_A]_s$$
$$= E\left[1_A \int_s^t X_u^2 d\langle M\rangle_u\right] = \int_A E\left[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] dP\ \forall\ A \in .$$

$E[I_t(X)|\mathcal{F}_s] = I_s(X) + \sum_{i=m+1}^{n-1} E[\xi_i 0|\mathcal{F}_s] + E[\xi_n 0|\mathcal{F}_s] = I_s(X)$.
Thus if $X \in \mathcal{L}_0$, $I(X) \in \mathcal{M}^c$. Now consider

$$E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right]$$
$$= E\left[\left\{\xi_m(M_{t_{m+1}} - M_s) + \sum_{i=m+1}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})\right\}^2 \Big| \mathcal{F}_s\right]$$
$$= E\left[\xi_m^2(M_{t_{m+1}} - M_s)^2 + \sum_{i=m+1}^{n-1} \xi_i^2(M_{t_{i+1}} - M_{t_i})^2 + \xi_n^2(M_t - M_{t_n})^2 \Big| \mathcal{F}_s\right]$$
$$= E\left[\xi_m^2\left(\langle M\rangle_{t_{m+1}} - \langle M\rangle_s\right) + \sum_{i=m+1}^{n-1} \xi_i^2\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}\right)\right.$$
$$\left. + \xi_n^2\left(\langle M\rangle_t - \langle M\rangle_{t_n}\right) \Big| \mathcal{F}_s\right]$$
$$= E\left[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] < \infty.$$

Now, we consider I mean what I am going to prove is that the kind of Ito's isometry have seen we have seen for the simple processes. Now, for this integration what is newly defined here we are trying to prove that okay this also is an, is an isometry.

So, for that we again choose sequence X n from space of simple processes which converge to X in L star, okay because our X is in L star okay and L star is a closer, is the closer of L0 okay. Then what we do we use Jensen's inequality for conditional expectation to argue that this integration of X n I t X n converges to I t X in L 2 P. Okay L 2 P means you know that expectation of I t X n minus I t X, whole square, that goes to 0, okay.

And then for any A which is in the sigma algebra of sth time sigma algebra so Fs. We consider this conditional expectation, this expectation, the expectation of I t X minus I X this is the increment of the stochastic integral square of that, square the increment on the here with integrate A function of A that means that we are integrating only on the domain A okay expectation the integrating in the sense not time integration the space integration that probability the probability when you integrate on the set A.

Why we are doing that? Because we are trying to prove certain measurable functions to be something else okay. So, like you know we are going to identify. So, there is a, this is a standard way to do that. So, we have non-negative measurable functions F and G and if you can show that integration of f with respect to mu on a set A is same as integration of G or with the same mu on the same set A is equal does not matter what is the measurable set A you choose.

Then we can actually obtain the F and G are equal almost everywhere. Okay, so we are going to use that. So, here we are therefore using A to be Fs measurable and we are taking indicator function of A here. So, now this expectation of this square is equal to limit n tends to infinity expectation of 1 A multiplied I t X n minus I s X n whole square, how did I get it? Because this I t X and I s X are the limits and then this limit I have taken outside okay.

So, how could I do that? We can do because that the very definition of the convergence the definition of convergence is that I t X n converges to I t X in the parallel norm in the parallel norm is like you know after taking expectation and that thing converges there. So, so from the definition of the parallel norm. So, we are going to get directly. So, it is just coming from the direct definition, okay?

Because this you know it is not limit inside because what I told in the beginning is not true because this is exactly close to this that's it. Because the I t X is not the point was almost sure limit of I t X n okay. So, It X is the L2 limit or the parallel I mean that norm limit of I t X n so this is coming directly and now from here what we are going to do is that we are going to condition it okay, so what are the condition I mean so here time t and s appears and A is also Fs measurable. I want to get rid of A okay.

So, for that what I am going to do, I am going to take the sigma algebra Fs okay. So, we condition that I t X n minus I s X n whole square given Fs and then this indicator function of

A comes out of this first integration because A is Fs measurable and then this I mean remaining term using the result what is obtained earlier Ito's isometry for simple processes.

I can write down this one okay I take these different square, the square of this difference is integrations s to t X n u square dMu okay. So, this part is obtained from the from the earlier results so, this thing okay from we are using this result here Okay,

So, after using this result we have A here again so now this A I can put again inside because this is Fs measurable, okay, so just to use that result I took 1, 1A outside, but now I am going to put it again inside. So, when I put it inside it is not time dependent u dependent. So, it goes inside this integration also and then square of integrator function.

is integrator function itself even goes inside a square. So, 1A into X n u okay that process we are going to discuss okay. So, for the time being we actually. So, we define this process I mean this notation we are using this notation I mean this is exactly what we would get here integrator function of A 1A in X n u okay and u is more than s okay.

So, like this and when u is less than s which is usually even for all case anyway. So, for that we are just putting it X n u. Okay so this is this is another process we are going to consider now we are considering this process okay. This X n into 1 A this is this again remains to be simple process. Why?

Because, now I am just multiplying and 0 or 1 okay at places okay. So, nothing would be affected only Xi i value would be affected sometimes it would be Xi i as it is sometimes it will become 0 say okay if say omega is not in A it would becomes 0 otherwise is exactly the same.

So, that is the simple process we are constructing here okay. So, why we are doing that? We are doing it so, that to emphasize that we are still having simple process a simple process multiplied with integrator function is still a simple process. And now we are rewriting this left hand side again okay. So, this is just a re-rewriting because this I can write down as again conditional expectation given Fs. And then expectation outside but then that expectation there taken for say that conditional expectation what I am going to get multiplied with integral function of A.

So, that means integration of A with this with the probability measure. The outside expectation would become this thing. And inside expectation, conditional expectation what I am going to get is I t X minus I X s the whole square given Fs. So, left hand side what is written here would be this thing.

Then right hand side, so I am writing right-hand side now, using this notation, using this notation I am writing right hand side the limit n tends to infinity, Xn, so now this we understand that this is nothing but the box norm of this process, right because this 1 A is here inside and then 0 to s to t this thing.

So, it is the box norm of the I mean the difference of the box norm because 0 to t and 0 to s and the difference so this box norm of this thing, so limit n tends to infinity of this difference okay so that thing we have obtain and now we are again using that X n converges to X okay in that box norm X n multiplied by 1 A would also converge to X okay in the same box norm why because you know while multiplying 1 A you have not changed anything fundamental just some of Xi i You know multiply with this 1 same thing or 0. And that same thing now remains also you multiply to X also. So, so here it converges to X into 1 A.

Student: How it is an indicator function of A.

Professor: Which one?

Student: You denoted that.

Professor: This one?

Student: whenever u is less than s.

Professor: Yes X n u exactly. So, then they are same correct. So, X n u. So, now, if you take the box norm definition the box norm.

Student: that indicator function went inside.

Professor: Yes exactly. So, this integrated function went inside here.

Student: We can only push when A is measurable Fs.

Professor: Fs of course A is measurable Fs, A is in Fs that is the whole point.

Student: Then it should be in both case.

Professor: Exactly, so, A was here and then I have taken out here just to apply that Ito's isometry formula for simple processes. And then after obtaining this using that formula, again my work is job is done. Then I again I would put 1 A inside, correct? So, when I put it inside then the new simple process is 1 A dot X n and that is here. Denoted, the notation is properly explained here I do not care what is what happens before s, since I do not care much I just put it same as X n u.

And for u more than s early this is relevant here so far that I defined this this multiplication whatever. I mean the reason that I have here I have kept it here because that I want to retain the measurability issue. You understand? If I multiply still 1 A here. They does not remain simple process because this is Fs measurable there, this is u and then the definition of simple process is violated.

So, so, that is the reason so, now it is simple process. Now, Xn 1 A is a simple process and I can therefore use this you know use this use this convergence that X n converges X n box norm, so, Xn into 1 A so this process also converges to X into 1 A okay. So, now after this convergence is done.

So, it is now time to rewriting you know this. I mean here I think small t is missing, so subscripts small t is missing and then is equal to expectation of this 1 A of X square u dMu. So, how did I get it because again and I am in writing down this as integration of the above and then again we have taken 1 A outside and then you obtain that okay. So, 1 A X square u d<M>u with respect to this quadratic variation okay X square u.

So, now here this thing I can write down as that this outside I mean I can put one conditional expectation here condition with respect, condition given the Fs okay sigma algebra. So, when we do that, then outside expectation becomes just integration or with respect to probability measure on set A and inside the conditional expectation what I am going to insert would be expectation s to t X u square d<M>u given Fs.

So, using a tower property so here we have inserted one conditional expectation here and that is survived here and the outside expectation becomes this integration with respect to P on the set A. So, at the end what did I get? I got this equality for all A which is Fs measurable.

So, integration of this non negative measurable function Fs measurable function correct Fs measurable random variable with respect to probability measure on set A is same as this is non negative square is there expectation conditional expectation of this thing with this is this whole thing is Fs measurable. So, expect integration of this non negative random variable on the same set A with respect to the same measure P so, they are equal for all A in Fs, what can you conclude now? We can conclude that these integrands are same.

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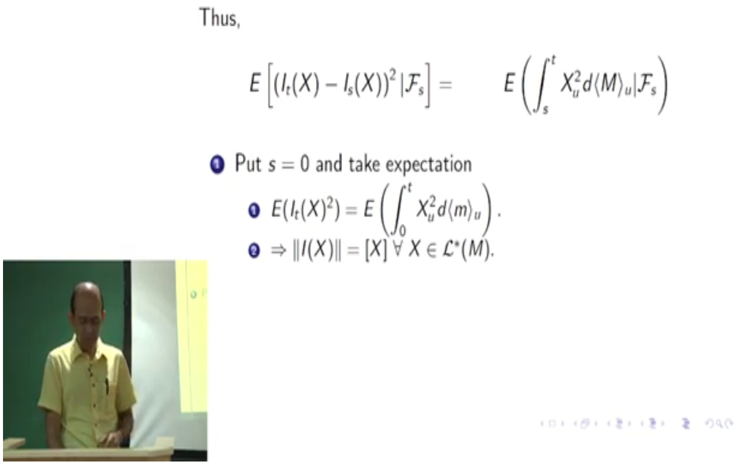

$$
\begin{aligned}
E(I_t(X)^2|\mathcal{F}_s) &= E\left[(I_t(X) - I_s(X) + I_s(X))^2|\mathcal{F}_s\right] \\
&= E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right] \\
&\quad + 2I_s(X)E\left[I_t(X) - I_s(X)|\mathcal{F}_s\right] + I_s(X)^2 \\
&= E\left[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] + I_s(X)^2 \\
&\quad \text{(as } I(X) \text{ is a martingale)} \\
\text{or, } E\left[I_t(X)^2 - \int_0^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] \\
&= \left(I_s(X)^2 - \int_0^s X_u^2 d\langle M\rangle_u\right)
\end{aligned}
$$

Okay, so in the next slide, we write down that expectation of I t X minus I s X whole square so square of the difference of the integrand is equal to expectation of the integration of the square of the integrand with respect to this quadratic variation given Fs. Now, we can put s equal to 0 and we take expectation and then we are also this part vanishes, we are going to get this thing okay.

Again this is saying that that norm of I X okay it does not directly say that so we have to put t is equal to n and it is to just you know, write down this for all possible n. This is true right hand side is just you know, this small m, it should be capital M, it is a typo.

And this right hand side of the box norm at t okay at n so t is equal to n we are going to put and then they are equal for all n. So, now you take 2 to divide by 2 to the n and sum over all possible n to obtain that okay norm of I X is equal to box norm of X okay.

So, that is true you have obtained this for all possible X in L star M. So, what did we prove we have proved that isometry that I is an isometry also from the space L star of M to M 2 c, Okay L star of M is on the one side on one side, where X is coming from and I X is going to Mc 2 the square integral continuous martingales and this I map is an isometry. It preserves the norm.