Indian Institute of Science Education and Research, Pune.
Introduction To Probabilistic Methods In PDE
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Prerequisite Measure Theory Part 01

Today, I would introduce the basic terms and terminologies which would be used throughout the course. So, these are actually prerequisite for this course. Although, the students have already done some course on these I mean that is the prerequisite. However, I mean I would like to present those definitions because sometimes some terms and definitions differ from book to book.

So, to have the consistent notations and definitions so let me very quickly give you the basic theory of measures and integrations. And what do you mean by probability measure and the notion of convergences, what do you mean by integration with respect to a measure? And notion of convergence of integrals.

So, these are, these are the things which we are going to see today very quickly. In this one of our lecture. So, we start with question, what is the sigma algebra? So why did you call about sigma algebra? See for example, when we look into the application in probability so there we talk about events. What are events? Events are could be a very elementary event like you know one particular occurrence of some sample point or event could be a collection of sample points. For example, when toss the coin for five times then one can say okay, my event E is occurrence of head in the first two trials,etc.

So, event could be elementary or composite in manner. So, in principal event is collection of sample points. Now, one should also consider space of all possible events. Since events are itself a collection that means a set, so space of all possible events is a collection of collections. So, these collection should have certain particular structure, what are these? For example, if I can answer to the question what is the probability of non-occurrence of an event A if I know the probability of occurrence of event A. Okay,

Our basic intuitions says that if I know the probability of occurrence of event A then probability of non-occurrence of the event A should also be known. That would be just nothing but 1 minus of the probability of A. Therefore, if I say A is an event so the compliment of A that is A compliment should also be an event.Okay Because we can actually talk about its probability. Now if I say that, A, B, C, D etc.

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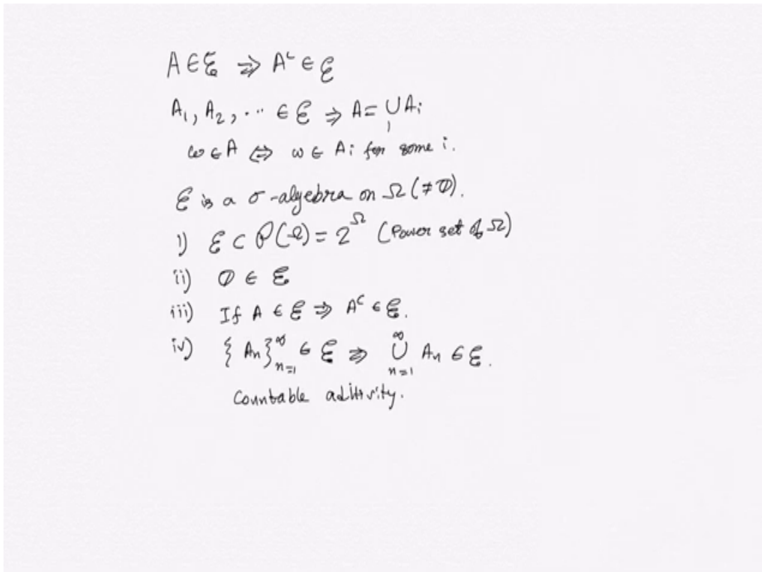

These are all say for example, I start writing here so say if A is in my event space okay that would imply that A compliment should also be in that event space. And again if I have A1, A2 etc, this countable in many events all are in event space, and then I ask that in occurrence of this is it an event. So is new event that A is occurrence of either of these.

So, that you can write down in mathematical notation union of Ai so union of all possible i so that is that I mean so if we say that omega belongs to A I mean that if and only if omega belongs to Ai for some i. So, we know that this is the rule of union. So, what is the bottom line the bottom line is that if A is in the space of all possible events then A compliment should also be in the space of all possible events and if I have a countable collection of events then there union should be in the collection the events.

So, based on this so let us define what we call as space of events? And that we should have this structure and there is a name for this structure we call this sigma algebra. So what is sigma algebra? So let us write down what is sigma algebra? So first thing is that it is the non-empty collection of sets so sigma algebra on some set Omega which is non empty that is a collection of subsets.

So sigma algebra I denote say E is a sigma algebra, E is a sigma algebra on Omega which is non empty. What does it mean? It means that E is a subset of power set of Omega often we write down power set as 2 to the power omega also, set of all subsets of omega and we need it to be non-empty, so it is non-empty I write down separately. So, if I just say empty set belongs to this collection this is done this is non-empty.

Because empty set is here in this collection and we require that if A is in this collection script E then we must have A compliment is also in script E. And if I have a collection of events An which is in E, every An is in E for all An n is equal to 1 to infinity. Then I must also get that union of An, n is equal to 1 to infinity is also in script E. So, that is the definition of Sigma algebra. And I have explained why do we need this structure?

Because you know, if you do not have this structure then you would not be able to answer to the question that if I know probability of a set A, what is the probability of A compliment? I cannot ask that question if that set A compliment is not an event at all. So, these are the, so this is the structure that the (())(8:23) sigma algebra.

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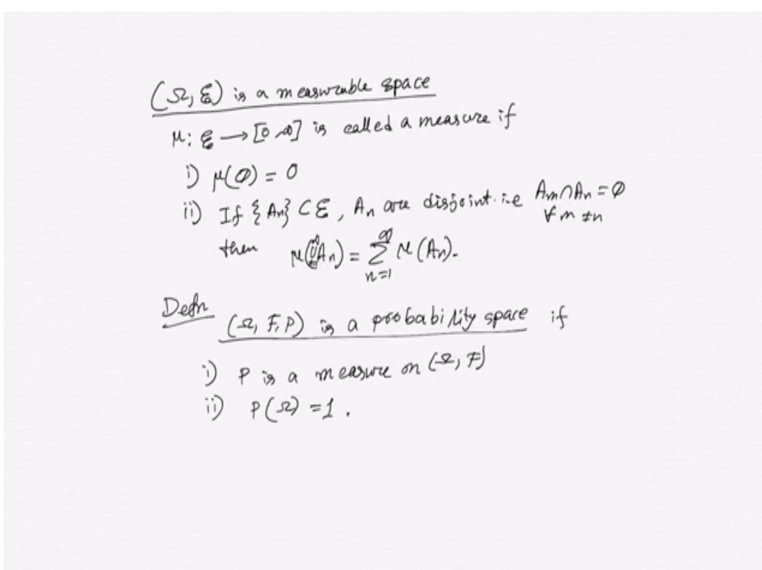

Next we define, what do you mean by a measure? So, get to next slide. So, here we have omega and then E, so E is sigma algebra. So, if omega is an non-empty set E is a sigma algebra on Omega then you call Omega is a measurable space, measurable space and then mu which is defined on this script E, that means define on a sigma algebra and taking value 0 to infinity closed. What does it mean? That all positive real number including the plus infinity.

So, that is called a measure if the following is a true. First thing is that, mu of empty set is 0. Second thing is that if I have countably many disjoint events, so if An are you know if a sequence in sigma algebra E where An are disjoints. Then mu of union of An n greater than equals to 1 is equal to sum of mu of An, n is equal to 1 to infinity. So, what does it mean? The right hand side can be given a proper meaning why is it so?

Because although this is an infinite sum but these elements are all non-negative, why is it so? Because look, the mu is taking the value 0 to plus infinity. So, this is, this is, this is you know sum of non-negative terms. So if you look at the partial sum sequence that is always increasing need not be always bounded but is always increasing. So if it is bounded it converges it value would be this minimum this sum.

If it is this not bounded that means it is unbounded than this sum would be plus infinity. So this are when a map satisfies these conditions then we call that as a measure. So, each gives a measures of a set. So, this second property is called the additivity property, additivity property of a measure. What is probability measure? So, omega In my successive lectures I would use script if mostly to denote sigma algebra.

What I was talking about script E till now I would switch to this notation script F. Script E sounds like you know space of events but however, F stands for field. So, in the literature or textbooks of probability theory you would see that sigma algebras are often called as sigma fields. So, that is also one reason that this script if notation is so common. So, we write down Omega F,P, is a probability space if, first thing is that P is a measure on the measurable space Omega, F.

Second thing is that the total measure P of Omega the total set has measure 1. The measure of the total set is 1. One can ask that how am I sure that this omega is also in F? Thing is that I have assumed that empty set is in F. Is in that event space. So, compliment should also be there. Since that is the structure of the sigma algebra. So, if empty set is there so the whole omega is also there.

So there is a definition of probability. Now we discussed about what is a definition of Lebesgue measure. Because Lebsgue measure would be visited a time in again and this is the, the measure which we encounter in day to day life. Like length, area, volume etc. So, we go to the next slide for introducing what is Lebesgue measure?

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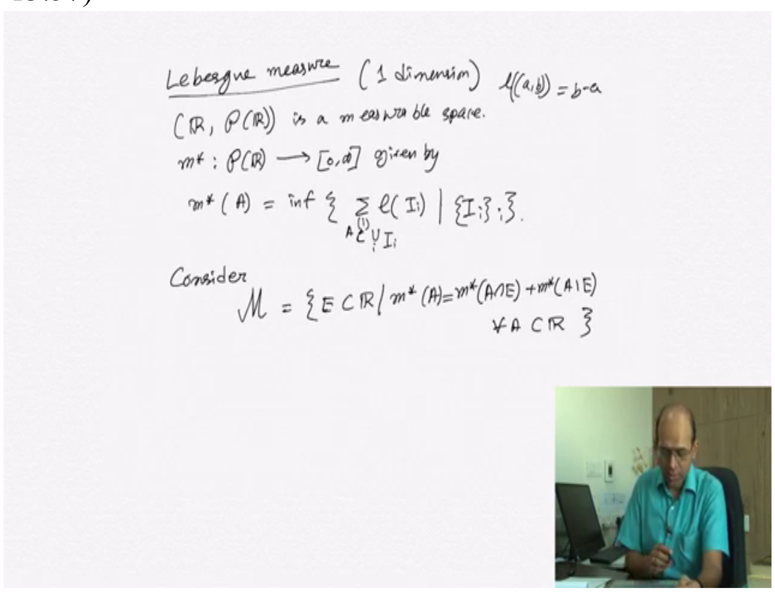

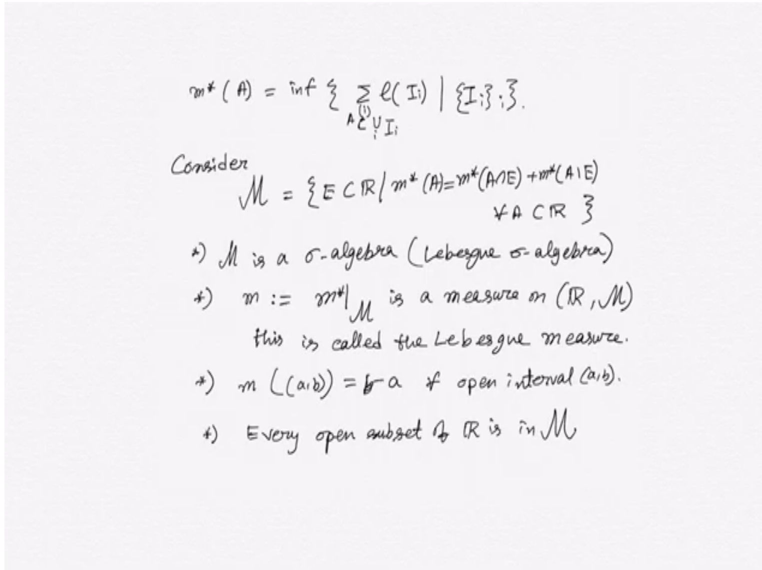

So, imagine that you have straight real line 1 dimension so we here we are just talking about 1 dimensional, 1D. So assume that you have you know so we, we consider first set of all real numbers on that we want to introduce 1 sigma algebra and on the sigma algebra we are going to introduce the measure which gives length. So, measure of an open set, open interval should be the length of the interval.

I mean that if I have a open Interval a, b then the Lebsgue measure of that interval should be b minus a. So that is the, goal for that first we need to define what is the sigma algebra and it turns out that to define sigma algebra one has to do little more work. I mean this is not way immediate way there is no particular immediate way to define that particular sigma algebra. Why is it so? Because the main reason is that we can consider taking say spot set of R.

So all set of all subsets but then one cannot define you know Lebsgue measure the way we want. What are the properties of Lebsgue measure? We know that it should be translation invariant that means if the measure of the interval a, b is b minus a does not matter if I translate that interval so I add 1 million to a and b, so 1 million plus a, 1 million plus b that interval should also have the same measure.

So that is the underlined goal. So, it is not easy to define such measure it is not possible to define such measure on set of all subsets of real numbers. So we need to exclude some subsets of real number but still construct one sigma algebra on which one can define the Lebsgue measure. So that construction has many steps so the first step is that introducing what is called outer measure. So we first consider this order pair that R P(R) that means R and the power set R that is also a measurable space because set of all subsets is of-course a sigma algebra, why is it so?

Because sigma algebra has 2 conditions having those conditions of-course satisfies because it has a all subsets. There is no absence. However, so on that we are constructing m star, m star, m star is taking a member of power set of R and giving values 0 to infinity closed given by mu star of A is equal to infimum of the sum of length of open intervals Ii where Ii are intervals which cover the set A. So given a set A we consider a countable collection of intervals such that there union covers A. Union covers A. So given a set A all is you can cover it by some you know countable union of intervals is not a big deal.

For example you can just first take all open intervals all the size little minus n to n and n tends to infinity so all possible n. So you would cover you know that union of that would be whole R, so that covers all the sets. So of-course this collection is non empty the collection of such covered by open interval is of-course non empty but we choose any of those and find out this sum, this sum of L of Ii what is the L of so let me write down what do I mean by that.

So, length of, length of interval a, b is b minus a. So that is the straight forward definition that we are going to use here and here we consider this number but this number is arbitrary it could be you know very bad number it would be you know over estimation of the size of A then we take consider all possible such cover and then we take infimum of that and when you finally going get some positive real number and that we are going to sometimes we may not get the real number, it would be plus infinity and that we associate to be m star of A.

Next we see that this m star the way it is defined need not be measure however this has some good properties it has some activity and m star of empty set is 0. So, this m star is not a measurable because it is not additive as I have emphasized here the definition that measure of union of disjoint sets should be sum of the measure of the sets. So that is not true for this m star. So, what we do is that we find out collection of subsets of R for which m star you know has a very good property and that subset that, that class of subset turns out to be a sigma algebra and on that if you restrict m star we get the Lebsgue measure.

So, let us see, let us consider, consider the set of I mean consider the collection of subsets of R, such that you have the following property. M star of A is equal to m star of A intersection E plus m star of A set minus E, for all A which is subset of R. What we consider? So here this is a collection of subsets of R we denote it script m. So how is this collection define it is defined as a collection of E which has a particular property which property that no matter what subset of A you choose, subset of R you choose the outer measure of A is equal to the outer measure of its two parts.

What are the parts? One is the part which is inside E and another part which is outside E. The way E cuts A. So in that case the part which is inside E and which is outside E the outer measure of the both if you add you are going to get the outer measure of the full set A. Of course you understand that this is one kind of additivity property. So if you consider such kind of collection and call that m and it turns out that so I am not proving it, it needs a rigorous (())(22:39) proof that m is forms a sigma algebra.

And not only that if I restrict m star on script m and then whatever I am going to get we call that m is we are going get a measure because m scripted on sigma algebra where restricting this thing on script m so then m is a measure on R, script m and furthermore this m has I mean this needs a proof actually that I have a sigma algebra and I have restrict m star on script m.

What I would get is not only outer measure but that is also a measure. So, this is a result and then this measure has the following property that m of any interval a, b is equal to b minus a. So the property what is a intended for a Lebsgue measure and also that it is translation invariant. For this actually when I write down this I intrinsically assume that this interval is in this sigma algebra. So that is also one result that every open set, so every open sets is in M.

So, next we go to the question what is borel sigma algebra. So here we started with the set of real numbers with powers of set of R which is a very big sigma algebra. There we discussed that we cannot define a measure like you know length measure which is translation invariant so for that and also measure of unit interval 1 finite so that type of measure I cannot define here. So, we have removed some sets to obtain a sigma algebra which is smaller than the power set of R we call this script m is the Lebsgue sigma algebra and then we restrict the m star outer measure on this sigma algebra to obtain a measure which you call as the Lebesgue measure.

And then we are now going to discuss that even smaller sigma algebra than script m. As we have seen earlier that this Sigma algebra m includes all open sets now see that, what is the smallest?

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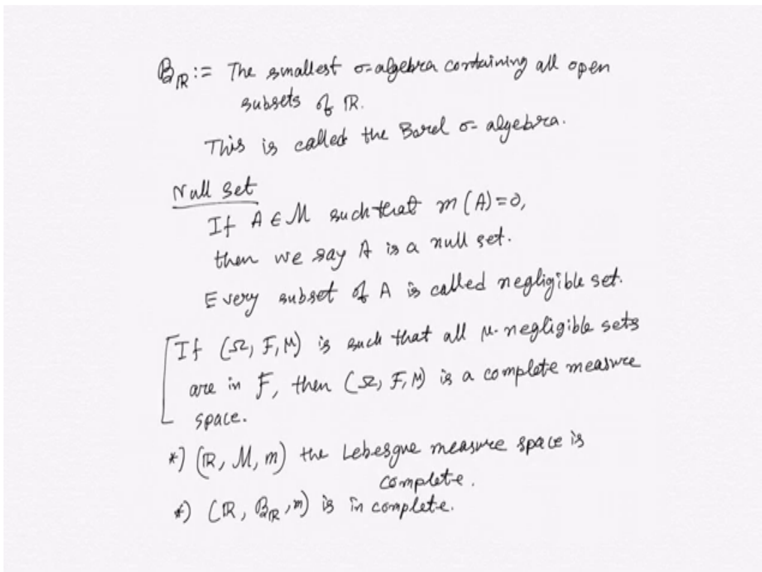

The smallest sigma algebra containing all open sets, all open subsets of R. So smallest sigma algebra is smaller than the Lebesgue sigma algebra. We give a name we denote it by B(R) and we call that borel sigma algebra. So, this borel name comes from a mathematician's name so borel was a mathematician.

Now we talk about what are the zero measure sets on this. So, one before that let me also clarify one very important difference between these two sigma algebra. So that is the set of measure zero so let us talk about what is null set. So null set, so if A is in script m such that Lebesgue measure of A is 0, then we say A is a null set, well then A is a null set. And then every subset of A is called negligible set.

Every subset of A is called negligible set. Now question is that if I have a null set that is a measure zero what should be the measure of subset of that? To answer this question I must have the subset of A that is in the sigma algebra otherwise I cannot talk about this measure. So anyway, so if that is there in the sigma algebra then you can ask this question and answer should be zero, why? Because if I have cut this whole set A into this, this you know this smaller subset and is compliment inside A then both should have non negative measure and then addition of that should be same as measure of A, but measure of A is zero

So sum of two non-negative number is zero that means both are zero. So, that actually clarifies that all subset of a zero measure set if those are measurable should have measure 0. But the question is, are they measurable? In principle when we talk about borel sigma algebra that often does not include all negligible sets.

So, if we consider a particular measure space mu such that all mu negligible sets are in F, then we say (Omega, F, mu) this measure space is a complete, complete measure space, complete measure space. So it turns out that Lebsgue measure space omega and then F is replaced by script m and mu is replaced by m, so that is a complete measure space. So here I write down (R, M, m) is complete. However, R so M is replaced by the smaller subset sigma algebra that is B(R) and m here again on this sigma algebra restricted to the restricted to the sigma algebra is incomplete.