Groups: Motion, Symmetry & Puzzles
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Symmetries and GAP exploration
Lecture – 09
Rotational symmetries of platonic solids

So, during last lecture we saw Cayley graphs. We saw directed Cayley graphs of various groups and we had some fun with that. And one question that they asked last time was what is a purpose, what are the applications of trying candy crabs like that.

And before I come to applications let me show few interesting objects I had shown you last time. This remember tetrahedron there are few more objects that I am going to show and. In later lectures we are going to talk more about these objects that is tetrahedron, this of course you know is cube, that is a dodecahedron it has 12 faces, dodecahedron, and this is a octahedron, it has 8 sides, 8 faces and then this is icosahedron it has 20 faces. These are some quite nice objects and you are going to have some fun with all these objects after sometime. Anyway let me start with this, this object, tetrahedron.

How do I write the group of rotations of this? And as I am moving this actually what am I doing is I am doing a random walk on the Cayley graph of the group associated to this. So, I can I can think of group of rotations a group of rotational symmetries of tetrahedron. I can write the Cayley graph of that and I can do random walk on those groups what does it mean? So, first let us see how can I write this group of symmetries of this object.

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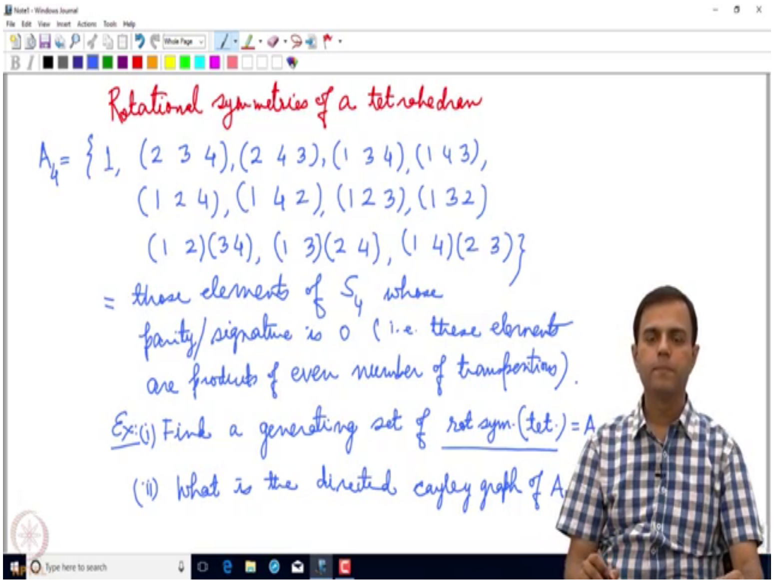

So, that is the purpose today rotational symmetries of a tetrahedron. How to put a notation to each vertex? So, here is the thing I put 1 to the top and 2 3 4 the bottom. Now, how can I get all the rotations of this? One ways I take one of the vertices and then I take center of the opposite face. So, vertex and center of the opposite face, I have like this maybe I will show you like this I am holding it let the axis. This thumb and this finger they are forming an axis. And then there are 3 options to rotate I can rotate by about this axis how much I can rotate by 120 degree and I can rotate by further 120 degree which is 240 degrees. So, this kind of rotations I can do.

So, how many choices for these vertices are there? There are 4 choices, 1 2 3 4 choices vertices are there. So, those are certain kind of rotations I have in this tetrahedron, these symmetries. So, first let me write all those. So, I have a identity which is doing nothing and suppose this top vertex is 1. So, when I am having this vertex 1 and when I am rotating this vertex is not moving at all what is happening is 2 is going to 3, 3 is going to 4, vertex 2 goes to 3, vertex 3 goes to 4. So, I will write it exactly in the same fashion as I take earlier in the permutation group, group of permutations s n. Here vertex 1 is fixed and then I do it once more. So, what I actually get? This 2 4 3. In fact, the square of 2 3 4 is equal to 2 4 3 that is easy that is easy to conflict, ok.

So, I have this and then I can fix say vertex number 2. So, I have 1 3 4 and 1 4 3 and then I fix other vertices vertex number 3. So, I have 1 2 4 and 1 4 2 and when I fix vertex number 4, I have 1 2 3 and 1 3. Is that all? Well, there are few more. You pick an edge consider the midpoint of this and then there is a opposite edge which is perpendicular to this. So, here is an edge, I take the midpoint and there is opposite edge which is actually perpendicular to this. So, I pick the midpoint of that edge as well. So, I pick like this and then I can rotate by 90 degree, I can rotate by 90 degree.

So, what is happening? In the process suppose this is vertex number one this vertex number 2 and these are 3 and 4. So, position is the position of the vertex 1 is being changed with position of vertex 2 and position of vertex 3 is being changed with the position of vertex 4. So, what is happening? 1 goes to 2 and 3 goes to 4 and this may happen. There are other choices of vertices I can pick the vertex 1 and 3 and go to the edge which is joining those 2 vertices. So, I have 1 in 3, 1 goes to 3 and the opposite 1 is 2, 4. And there is one more 1 4, I pick vertices 1 and 4 and then I pick an another one opposite side. So, there I am swapping 2 and 3. So, these are all the elements here in this the symmetry in the group of rotational symmetries of tetrahedron

How many of these are? There are 3 of these, 4 of these, 7 and 5 of these eleven 5 of these twelve and this group has a name A 4. Why do we give it this name? Because these are, so happens that these are precisely those elements of A 4 whose parity I use the word parity couple of lectures ago or I also use the word signature is 0, 0 was for the even. So, that is these are, these are product of, these elements are products of even number of transpositions.

And now what about the Cayley graph of this? So, maybe that you should do in the assignment. So, exercise for you would be first find generating set of this group is rotational symmetries of tetrahedron, right, rotational symmetries of tetrahedron which is A4 and second exercise would be finding Cayley graph. What is the directed Cayley graph of this group? So, well I am going to tell you what exactly the Cayley graph is. How does it look like I am not going to label it, but just for you for some choice of generators for some generating said the Cayley graph looks like this.

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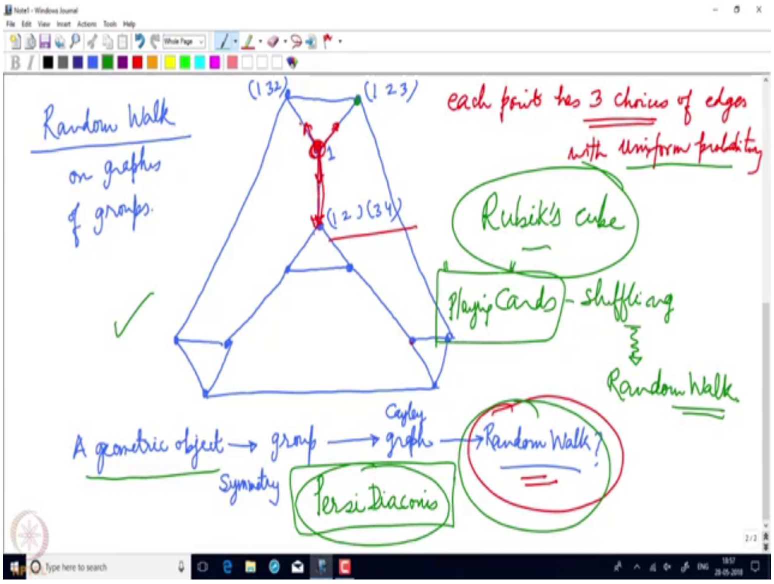

I am just drawing it, just as a picture.

So, Cayley graph say this is identity element this could be 1 2 3, this could be 1 3 2 likewise this is 1 2 3 4. This is identity, this is 1 2 3, 1 3 2 and so on other things I am not labeling, so 1 2 3 4 5 6 7 8 9 10 11 12 and this. So, here in this graph let me talk off random walk. So, what I am talking about is random walk on graphs of groups those graphs which are those Cayley graphs which are coming out as graphs of groups and groups are coming out of automorphisms of certain objects. So, that is quite interesting.

To start with what is there say a geometric object like this one, and then you have a a group which is being cooked out of it and how does it come some kind of symmetry considerations you have you consider certain symmetries of that object. And then you choose some set of generators and make graph Cayley graph of this and on this Cayley graph you are considering random walk. So, what is random walk and what is the advantage of considering certain things the language of random walks?

So, I am trying to I am trying to explain it through an example. We take a point say here, that is the identity position say that is the identity position of this and then at each node I have options to traverse either to this or through this or through this. So, there are 3 options. At each node I have 3 options I can go like this, I can go like this, I can go like this in 3 different directions. So, what is assigned to me? What is prescribed to me is at each point you see there are how many at each point there are 1 2 3, 1 2 3 there are 3 edges which are coming out of it then that is how the graph looks like here.

So, I am just as an example I am mentioning. So, each point has 3 choices of edges. So, from here you can either go here or you can go here or you can go here. What if we assign probability? Let us say that there are probability of taking this direction taking that direction taking that direction equal probability. So, 3 choices say with uniform probability. So, uniform probability is 1 by 3 in this case whether 3 choices and accordingly. So, suppose with 1 by 3 probability it comes here again there are 3 possibilities. So, take your system take the configuration which is initially say at the identity position, and through a number of steps it can go anywhere it can come here you choose this with one type probability, chooses this with one type probability, chooses this with one type probability.

So, when I am actually moving this all the motions that I am doing here rotational motions I am actually doing random walk on this, and I have to be careful in deciding what are the actions, however moving because those the basic moves here should correspond to an element in the generating side there. So, when I go from certain position to certain other position I will be doing an action here which corresponds to a generating element here. So, for example, when I go from here to here what am I doing? I am taking 1 and 2, suppose they are 1 and 2 vertices and they are 3 and 4 vertices and they are just moving like this. So, this move which I take for one end to and vertex 3 and 4 which are below that corresponds to going from 1 to this point. So, our actions are being converted into random walks.

Now, I can do this thing not just for these kind of objects, but for more complicated objects as well. Did I mention some complicated object in these lectures? Yes, Rubik's cube, Rubik's cube there are so many configurations and in forthcoming lectures we are actually going to see how many configurations are there. So, now, you imagine a huge graph, what are the nodes? What are the vertices of that graph? Those are all possible configurations. So, all possible configurations of Rubik's cube.

And what are the edges of that graph? Those edges are determined by the actions that you do on the Rubik's cube. What kind of actions you can do? Say for example, you can rotate by left or the left face by say 90 degree or the top face by 270 degree all those options are there. So, those will form generators of Rubik's cube and using that you can create huge extremely large graph and then walking along that graph is essentially making lots of moves on the Rubik's cube.

So, it is just a random play random scrambling of the Rubik's cube, you can do using these Cayley graphs, right. We are going to talk we are going to eventually talk of Rubik's cube, but not through graphs, but through relations and generators using the software called gap that will come later on.

So, you can have lots of random walks. So, start to start with you have a geometric object and whenever actually play with the geometric object you have random walk. And now this object need not be geometric all the time, you have for example, cards playing cards. When you are shuffling when you are shuffling playing cards very random fashion what is happening? Card shuffling is actually a random walk. One can certainly ask various questions for example, one question could be you start moving on a graph, on graph of the group say at this point and you are falling certain probability distribution need not be reform probability distribution, but it would be some other probability distribution.

And after n number of steps where would you be nothing is deterministic here. So, in terms of probability you can say, one interesting statement would be that after how many steps you are likely to be they are equally likely to be there on all the steps. So, after how many steps the probability of finding this point after doing all this random walk would be a uniform probability. So, those are quite interesting questions.

And one name that I would like to mention is this mathematician Persi Diaconis, he is not just a mathematician he is also a magician. He has invented lots of card tricks, not just card tricks beyond that some really professional magic and he has some theorems in shuffling of cards which are concerning for example, in how many steps, how many steps are required to make a pack of card so to say random. So, quite interesting things, right. Groups are there, symmetry is there, geometric objects are there, graphs are there playing cards are there, Rubik's cube so many interesting things are there just in one book.

So, the rest of the lecture let me talk about siblings of this, what are those? I told you this is dodecahedron, this is cube, this is icosahedrons, and this is octahedron. So, it would be fun for us to write symmetry groups of all these things. Let me just tell you and that is also the exercise for you to find what are symmetry groups for so called platonic solids and that is a key word, platonic solids.

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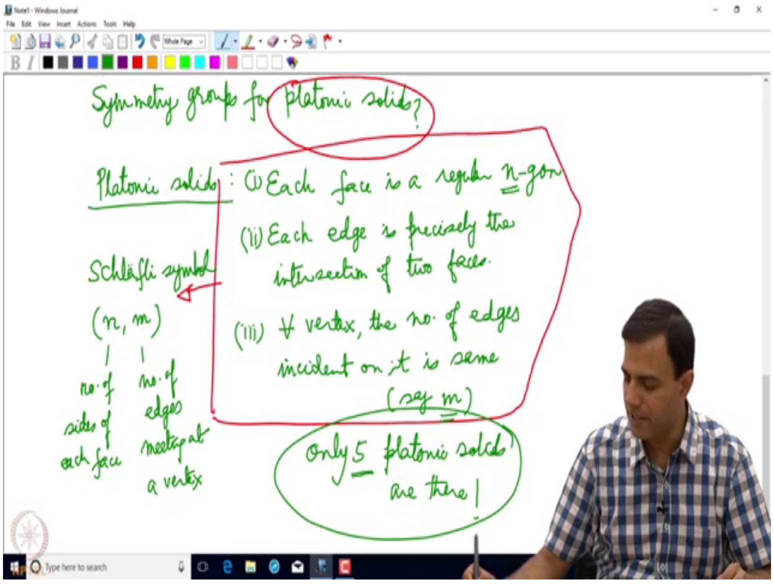

So, what are platonic solids? So, these are some interesting objects which are quite symmetric. So, there are some interesting groups associated to them and one can have lot of fun with those groups. So, let me just explain what are all these platonic solids and later on in the later part of the course we are going to understand the groups of symmetries of these rotational symmetries of these as subgroups of what is called so 3, the group of rotations of sphere. But nevertheless first let us understand what are platonic solids.

So, the order you understand them let me pick one of them. See, I am just say I am picking cube there is some symmetry right, in what sense. Each face is a square and the square of the same size, right. The square of the same size this is square each face is a square, right and then each if you take an edge each edge is actually it intersection of 2 faces. So, 2 faces are meeting exactly on the edge. What else? We take this vertex 1 2 3, there are exactly 3 edges which are adjacent on to this vertex and this property is same for all the vertices. So, if I pick some other vertex say this one again there are 3 edges which are edges side to this.

So, I will just write this I would say that each side or rather each face, each face is a regular n-gon, in this case n was 4 everything was regular 4 gon which is a square. And then each edge is precisely the intersection of 2 faces, it is point 1, point 2 and then for each vertex the number of edges you see which are incident on this vertex the same and the same number we denoted by m. So, this is what defines a platonic solid. Let me take some other see this v octahedron in octahedron you have this triangle, so everywhere is the same triangle of the same size and if I pick this thing one vertex. Now, how many edges are incident on this? 1 2 3 and 4, ok.

So, for octahedron n is 3 and m is 4. So, do each platonic sonic therefore, you can associate what is called Schlafli symbol. Schlafli was mathematician from Germany. So, what is the Schlafli symbol? n comma m n is that regular n-gon and m is number of edges which are incident on a vertex. So, this is a number of sides of each face and this is number of edges meeting at the vortex. And it is very interesting thing, how many platonic solids are there? Only 5. That is quite interesting statement and how does one see because having only 5 platonic solids has something to do with the subgroups of rotational symmetries of the sphere.

So, I am quickly going to connect it with graph theory here is a very famous statement in graph theory which is attributed to Euler.

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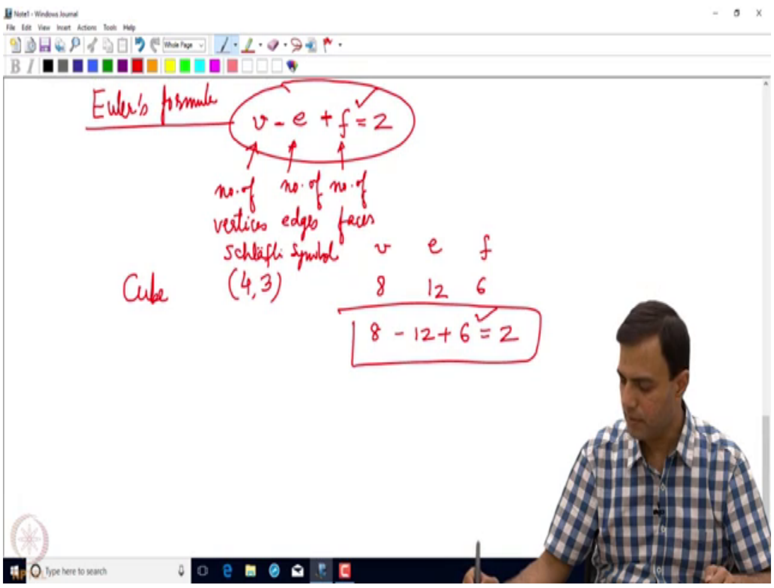

Call it Euler's formula which is you take any closed shape like this not necessarily uniform not necessarily a platonic solid you take any closed shape then number of vertices minus number of edges plus number of faces is a constant its actually 2. So, this is number of vertices, this is number of edges and this is number of faces. They make this computation always going to get the number 2 that is quite curious. Let us see this one. I have taken cube for cube the Schlafli symbol is n comma m n is 4 square every side is a square 3 because each vertex is having 3 edges. So, this is Schlafli symbol.

And then cube how many vertices are there? Top 4 bottom 4 8 vertices 8 vertices are there. How many edges are there? 4 up 4 down and 4 vertical, so 12. And how many faces are there? 6 faces are there f 6. So, 8 minus 1, 2 plus 6 is indeed 2, we have verifying this. Same is case with other platonic solids in fact, not just platonic solids, but as I said any solid any solid graph any closed craft we have this property. So, how to see that there are only 5 platonic solids? As I said in the previous slide proof is very easy and very interesting it is just very nice interpretation of Euler's formula.

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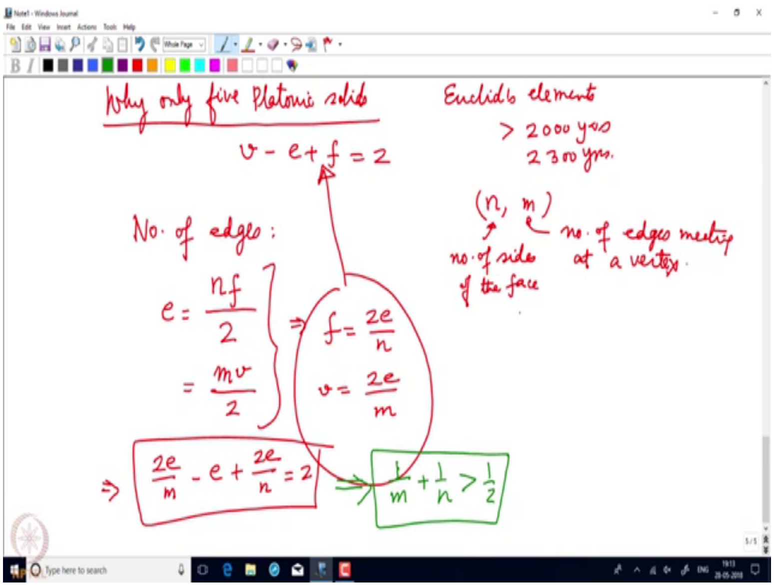

So, let me just do this. So, why only 5 platonic solids? If you know this book by Euclid which is called elements, why story book which is 2000 years, may be even more maybe 2300 years story book which was the first we can document available for geometry for the formal geometry that we have Euclid actually mentions catalytic solids quite interesting, ok. So, why only 5 cationic solids? So, what do I do? I keep this at the back of my mind and then I count in this number of edges in 2 different face number of edges. How can I count? You remember Schlafli symbol n comma m what does it denote, n was number of sides of the face and m was number of edges per vertex.

So, if I am going to count number of edges then maybe I should not say the number of edges per vertex, but number of edges meeting at a vertex, yeah. So, this is a Schlafli symbol and his number of sides of the face and n is number of edges which are meeting at a vertex. So, if I consider edges per face, edges per face how many? m times, so the n times number of faces end times number of faces, but then this edge is being counted twice it is counted for this as well as this one its being counted twice.

So, I have to divide by 2 that is e number of edges. But similarly I could also count this is actually total number of edges yeah. So, that is e number of edges e is n f by 2. But then I could have also counted it in different way because I have a vertex and at that vertex I have these many edges which are coming out of it. So, maybe m v and each edge is shared by 2 vertices.

So, each edge is having this vertex and that vertex. So, (Refer Time: 34:44) m u by 2. So, from this what do I have? I have that f is 2 e by m and v is sorry f is 2 e by m f is 2 e by and v is 2 e by m, now all this information I put here and then what do I get 2 e by m minus e plus 2, e by n is 2. So, from all this you can conclude that at 1 by m plus 1 by n is actually strictly greater than half.

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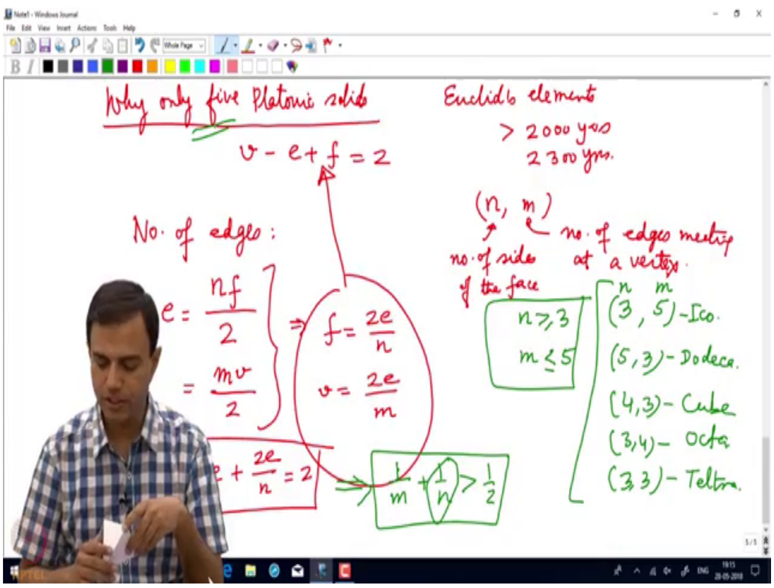

And certainly number of sides each phase has to be greater than equal to 3. So, this quantity is greater than equal to 3 and in order to maintain this inequality one has to have that n is less than equal to 5. So, this is what puts a restriction and when you have all the possibilities. So, when you have say 3 comma 5 that is a possibility which is icosahedron each side is a triangle and each vertex is having 5 edges which are coming out of it.

5 comma 3 could be possibility this is n this is m, 5 comma 3 would be a dodecahedron, and then icosahedrons, dodecahedron, and then this would be 4 coma 3 which is a cube, and then 3 comma 4 would be octahedron, this is 3 4 and then and have this 3 comma 3 and that would be tetrahedral and these are the only possibilities.

So, there are only 5 platonic solids. In fact, everything can be understood in terms of the Schlafli symbol that I had mentioned. So, you can actually have fun with all this you can try to find generators of all these groups, you can write Cayley graphs of all these groups, and in coming lectures we are going to understand the groups of symmetries of these objects in quite interesting fashion and some relations of these with some interesting statements in group theory are also going to be made. So, keep watching.

Thank you.