Groups: Motion, Symmetry & Puzzles
Prof. Amit Kulshrestha
Department of Mathematical Sciences
Indian Institute of Science Education and Research, Mohali

Structure of groups
Lecture – 08
Cayley graphs of groups

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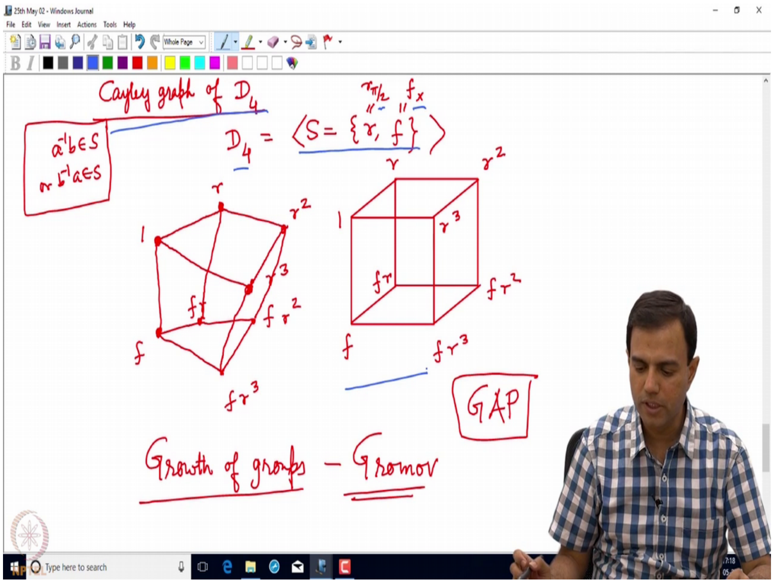

We are back now say as we would recalled, last time we were discussing Cayley graphs of groups. So, what has given to me is a group and generating set for this group. So, the case of r 4 we had taken in generating said to be r and f, r as we would remember, it was a rotation by pi by 2 and f was flipping about x axis.

And then we could see that the Cayley graph of D 4 is cubic. I had put two conditions, these two conditions. So, the vertices of the Cayley graph, they were points of the group, there were elements on the group and then two elements, two nodes, two vertices of the graph were connected by an edge; if either a minus b, but an S or b inverse a or in S. What is the use of expressing groups in this fashion? Many applications would come and before that, let us practice more.

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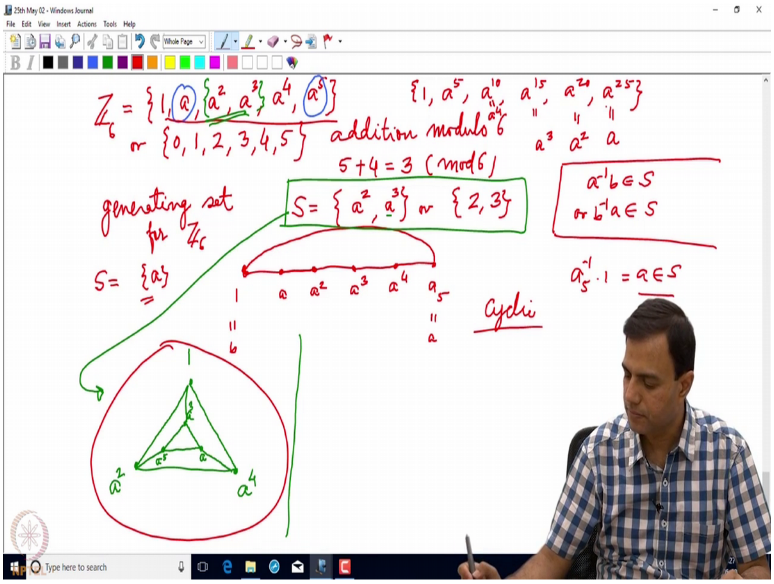

Let us see in few more examples, very simple example I would take Z 6 the cyclic group of order 6, what are the elements, I will just write them as 1, a, a square, a cube, a raise to the power of 4, a raise to the power 5, a raise to the power 6, then a raised to power 6 is identity.

So, better I will not write this only first 6 elements I am going to write this much. I would have also written this as zero, 1, 2, 3, 4, 5, these 5 symbols and these are forming a group under addition modulo 6. So, these are the remainders and the addition modulo 6. So, 5 plus 4 therefore, is 3, because that is what happens, mod 6. So, what about the generating set.

So, the cyclic group; Cyclic group meaning there is one element in this group which is capable of generating it entirely. So, the way I have written like this, it is very clear that is a one of them. Well I could have taken a to the power 5 as then, a to the power 5 would have been as good as a, because a to the power of 5 and then a to the power 5 to the power 2, which is a power 10, a to the power 15, a to the power 20 and a to the power 25; that actually is same as this, because a to the power 5, a to the power 6 as a we call a is 1.

So, it your 25 is therefore, just a, because a to the power 24 is identity and now this is a square, this is a cube a to the power 10 is a to the power 4 is a. So, all the elements are there. So, a as well as a to the power 5, both of both the, both elements are generators. Is there any other generating set? Well I would say a 2 together with a 3 also generates the same thing.

So, if I take S to be equal to a square a 3. Oh I just take it to be 2 3; that is a generating set for cyclic group as the cyclic group Z 6. So, there are two options let me take as to be a, then how does the Cayley graph look like, I have 1, I have a, I have a cubed, I have a raised to the power, sorry this was a square.

So, this was a square a cube a raise to the power 4 a raise to the power 5 like this and then do I connect a raise to the power 5 with 1. So, what is the condition for connecting edges. So, either a inverse b should belong to S or b inverse a should belong to S. So, if I take this to be a, this to be b, a inverse b, a is a which actually belongs to, was a to the power 5 inverse 1; that is a and that belongs to S right.

So, I will connect it back. So, the name is quite justified cyclic group. I have connected say a to a square, because a inverse times a square is a which belongs to this set. So, what about some other generating set; So, I take this to be generating side, this time this one, this generating set and. So, let us see what happens. So, I have identity and then I have a square, I have a cubed and then back to this and then I have oh. Sorry a square I have a to the power 4 and then back to this, I am connecting it, because a square belongs to.

So, here I am doing a inverse b. Let us a square which belongs to S generating set and now I have a cube here and then a raise to the power 5 and then a raise to your 7 and back to this, and then I also connect a square with a 5, because a to the power minus 2 times a to the power 5 which is a cube, actually belongs to S and similarly I do like this. So, it turns out to be the, Cayley graph turns out to be a triangle in another triangle and all these vertices getting connected by an edge to the outer void x, do the some other group of order 6 Z sis is 1 another group is S 3 which is the group of permutations of three elements.

So, with that group also, let us do an experiment. Remember this, remember this picture and do the experiment with S 3. So, what is S 3 or call it D 3.

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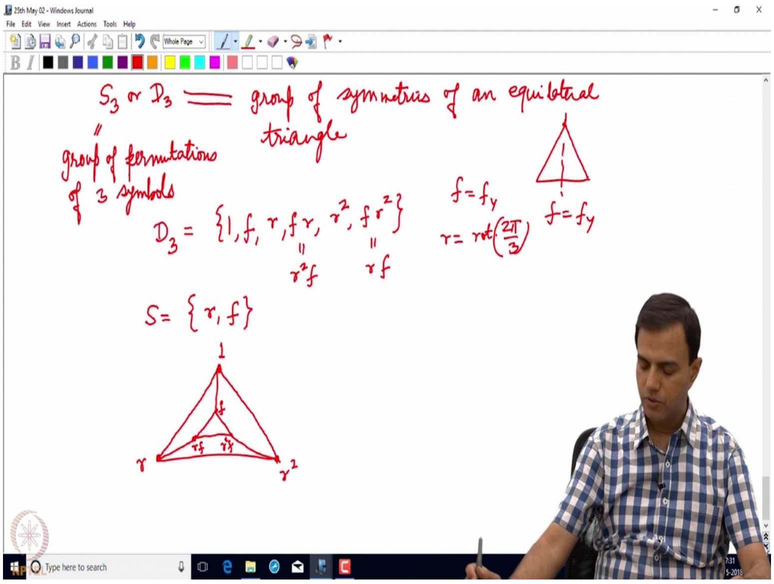

You want to understand it as understand it as S 3, then this is group of permutations of 3 symbols, will not understand it as D 3, then this is group of symmetries of 3 gon an equilateral triangle. So, we called what is D 3 D n we had seen. So, we have identity, we have flipping, we have rotation by this case 2 pi by 3 1 20 degree and then we have f r we have r squared and we have f r square and it is not very difficult to see that f r is actually same as r square f and f r square is same as r f.

So, this is, this 6 elements are there f if you want. So, here is a triangle f, if you want is flipping about y axis and r is rotation by 2 pi by 3. So, what would be the Cayley graph. So, I take generating set as two elements r and f, r and f action during the whole, whole D 3. So, I have identity element and then I have r, then I have r square and then again r square will be connected to 1, because a inverse b. So, r square inverse 1 which is just r r belongs to S.

So, I make these connections and then similarly I have smaller triangle here inside. So, I have r and this is f. So, this is r f and this is f. So, this is f here, and here I have r square f. So, the graph turns out to be like this. So, you carefully look at this graph and carefully look at the graph that was their f or the case of Z 3 Z 6, what the graphs look same are these two groups same. No these two groups are not same; one is a billion group, another is non-billion group. So, these are two different groups what their graphs look quite.

Similar you see there is a triangle line that is trying it outside and then these vertices are inner vertex is connected to the outer one and similarly f or the group D 3, you have same thing coming out. Is there a way to distinguish these two. Can you read from the graph somewhat like this is a graph of D 3 and the other one is the graph of graph of Z 6. So, here is a very nice way, which is you, you give it direction, you consider directed graphs.

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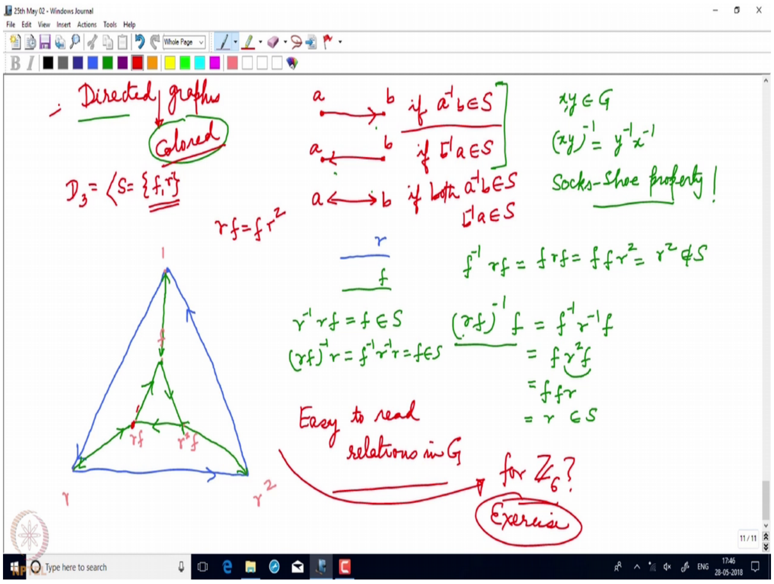

So, you can consider directed graphs. In fact, you can also have directed colored graphs. So, you can give some direction in that picture, you can also put some colors in that picture, so that you have a way to understand which one is as 3 and which one is for a billion group of 6 order; So, how do I put the graph, how to, how do you put the direction to this graph. So, I first do it for D 3 for which I know generating set is f and r what are the elements 1.

So, first maybe I will, I will put some. So, maybe I will put some vertices first and then I am going to connect them with edges. So, 1 r r square and here is my scheme, I would put this thing. So, this is a, this is b. I would put arrow like this if a inverse b belongs to S that is it. So, if b inverse a belongs to S, in that case I would put. So, it b inverse a belongs to S, then I would put a picture like this and if both a inverse b then b inverse a, if both of them belong to S and then I would put picture like this, then I will give direction to this what I can also do is, I can actually color my, I can color code my generators. So, for each generator you have a color.

So, here let me just fix one color; for r and we do have this color. For f, I am going to have this color. So, I am going to have colored and directed, directed by. So, direction I would good, I would put using these conditions and color I anyway put to all the generators. So, how many colors are there, say as many as number of generators and remember each choice of generating set gives you a different Cayley graph ok.

So, first I am going to just put the vertices 1 r r square and f r f r square. So, these are vertices and how to put color in the direction. So, first I am going to do it for r. So, one I would connect to r 1 inverse r is belonging to S, S, it is belonging to S and the direction is therefore, this.

And now I am going to put the direction like this, because r inverse times r squared is r which belongs to S and here I am going to put a direction like this. So, you have 1 r r square r cube is same as 1 what about f f, we have chosen green color. So, the color the direction would be both sides, because f is same as f inverse. So, f as well as f inverse both belong to S.

Now, f is here, what about this f is here r f is there, do I connect them when let us see. So, f inverse r f so, here its f it is r f. So, should I join them? So, in order to realize whether I should join the invert to with direction, I should consider r, I should consider f inverse r f and r f inverse f. So, what is our f inverse, let us see. So, this is f inverse r inverse take something interesting, whenever you have two elements x and y in a group.

So if you multiply these two group these two elements in a group, take the inverse, it is actually y inverse x inverse and this property has a very curious name, which is called socks shoe property quite curiously. So, using socks shoe property I conclude that this is f inverse r inverse f and this is f r square f.

So, what would it be? So, I said some time ago that r is square same is f r. So, I have f and this is f r and f square is 1. So, this is just r which actually belongs to S and what about this, again this is f r f and as I computed here r f is same as r r f is same as f r square. So, this is f f r square. So, this is r square which does not belong to us

So, here direction should be from r f to f there, actually should be this and similarly if I do it for others direction, would be this and here the direction would be this and here r f and r. So, I have r inverse r f which is f, which belongs to S right. So, from r to r f, I will put direction like this what about r f inverse r are you socks are you socks shoe property. So, this is f inverse r inverse r which is f, which I mean belongs to S. So, I have both the directions.

Similarly, here also I have both the directions. So, as far as directed graph of D 3 is concerned, it is this. So, this is colored as well as directed. So in fact, using directed colored graph I can, I can very it is very easy for me to read relations in the group how is that. So, in order to reach this point, for example, what you do, you take r and then you take f, this green was f, take r f that is all or you do f r r f r square. So, r f is same as f r square other relations also you can see within this.

So, here you see the direction of the outer triangle is like this, which is anti clockwise, but for the inner triangle the direction is like this, which is clockwise what about the direction of, what would be directed color the graph for Z 6. You have to think about it, maybe that is an exercise for you and that will make you realize that the directed color graphs for Z 6 and directed telegraph for D 3.

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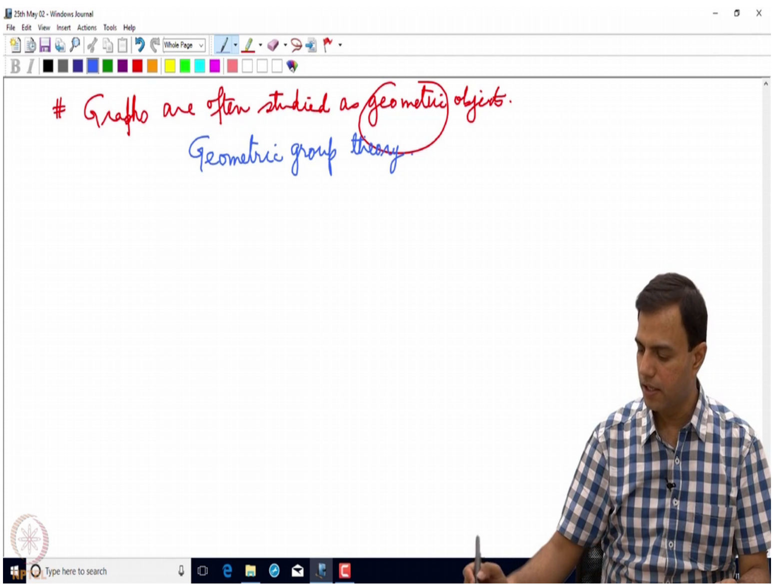

They are actually different, quite often all these graphs are often studied as geometric objects and we are not going to be bothered with that in this course, but to understand groups people use geometry and the branch of mathematics, which is called geometric group theory, but we are not going to talk about geometric group theory in these lectures.

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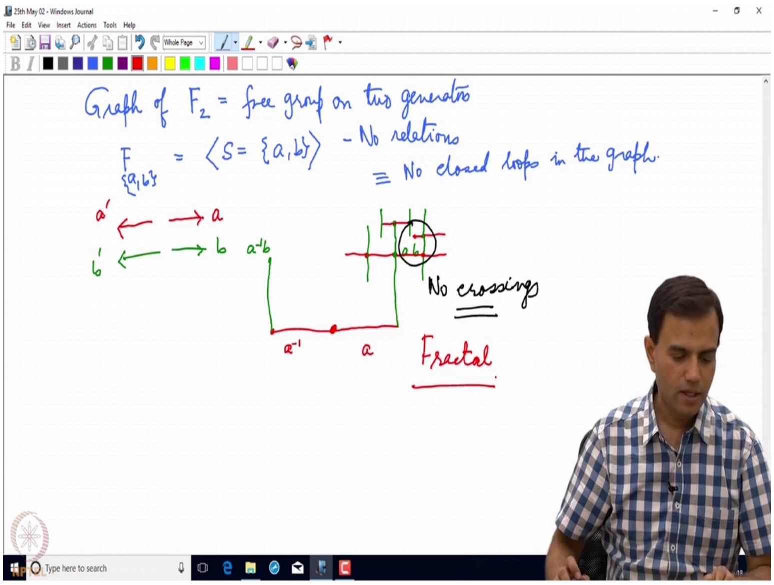

Let us see one more interesting graph in this discussion which is a graph of f 2. What is f 2 free group on two generators. Do you remember free groups on two generators or n generators, there are two symbols a and b and we make words of all possible lines out of these two alphabets and we also put two more alphabets a dash and b dash and we use all four alphabets to make words we make dictionary.

And then there are some equivalent words, collection of all those words under juxtaposition, it is just back to back we just put those two words and form a new word that is called; that is a group operation for free groups. So, how to draw the graph Cayley for f 2? So, I take f a be free group on two generators a b, you just naming them as a b, what are the relations? There are no relations and since there are no relations, there are no circuits, there are no closed cubes in the graph. So, no relations means no closed cubes in the graph, how does it work.

So, let me take. Now I am taking color coded thing, let me take this as a and let me take this as b so that. So, I am taking like this and then the reverse direction I have a dash and I have b dash, rather I would use b vertically, I need horizontally just a, just a way to avoid relations. So, then there as far as possible, so I have so I have a, then I have b. So, this is a b, this point is a b or I have here inverse and then I have here b. So, this is a, a inverse b inverse a inverse b and here again if I take a. So, I take, if I can do a, I can do a inverse, I can also do b or I can come back here.

So, like that I have and here again I have options of going up, coming down, going right, going left and similarly at this point also I have the option of doing this, option of doing this, an option of doing this. And once we reach these points again I have option to do this, to do this. Sorry the horizontal, the vertical will be green. So, to do this and to do this and to do this, and similarly here I have option 2 at this point, I have option to do this or I have option to do this, I have option to do this. Be careful, be careful here no crossings.

So, eventually what you get it is an image which you may like to call your, heard of this word fractal, it is a huge graph, it is a graph with infinitely many vertices. So, many vertices that is quite, quite big and you cannot see relations in this, because there are no close the group. The group is free group, what is the application, what is the use of having all these graphs Cayley graphs.

We are going to talk about all this in coming lectures when the interesting application would come in understanding puzzles, in understanding Rubik's cube. In fact, whenever there is a group action you can construct a graph out of it, and some more graphs we are going to join coming lectures, and lots of those graphs are going to come from symmetry. We are going to understand symmetry groups, groups of symmetries of various geometric objects, various things like this you recognize what it is.

Every side of this is triangle and then there are 1 2 3 4 vertices, it is quite familiar I guess you saw it in your chemistry course, its tetrahedron tetra stands for 4, hedron stands for number of faces, it has 4 faces. So, tetrahedron we are going to understand symmetries of this. As I told symmetry is going to be one of the important aspects of this course.

So, symmetry is what we are going to understand and if you try to write Cayley graph of this, you can try to write the colorfully directed graph of this and all the action, all the movements of this we can understand through the movements on the graph. So, to know more about these things, this is just one example of what are called platonic solids. So, to understand all this, keep watching I will see you next time