Groups: Motion, Symmetry & Puzzles
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Structure of groups
Lecture – 05
Parity and Puzzles

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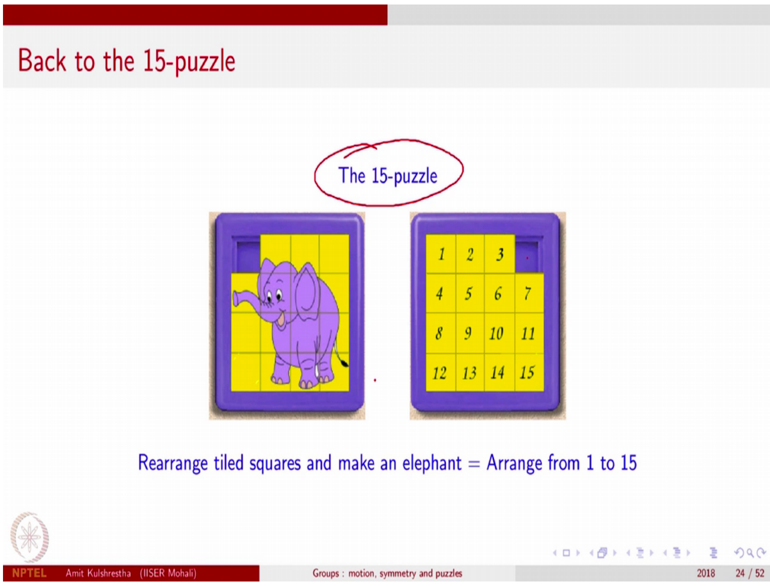

So, we are back and we are back with this interesting puzzle, 15 puzzle and you know what we are supposed to do here? There are 15 small tiles and then there is empty space you can move your tiles in this case 3 or 7 to the empty space and like that empty space will be moving all across. And eventually you can shuffle, reshuffle and bring it back to the same position so, 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15.

So, what is the, this is solved puzzle what is unsolved puzzle for this? What is the scrambled one? Some random sequence of some permutation of numbers 1 to 15 and this place may be somewhere else; so, that is the thing.

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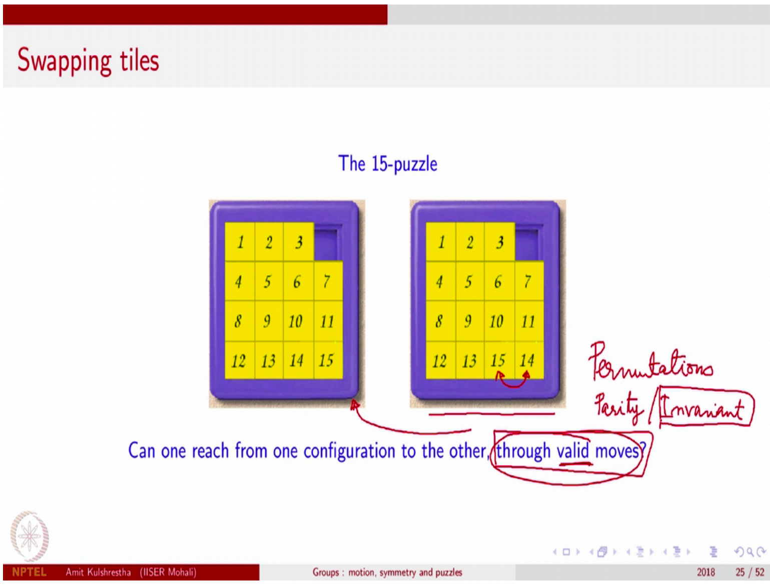

Here is the question here is 1 configuration, where you have all the numbers in order and here you see these two have been swapped. Now, the question is suppose this unjumpled versions is given to you, the scrambled version is given to you and then you are expected to make it into the identity make it into the original position, can you do that is a question. And supposed to do it through when it moves you can just take the tiles out and later on arrange it.

So, to when it moves you are supposed to do this. So, what do you see in this puzzle? You see of course, permutations and as you recall we are talking of parity, we are talking of invariant. Some quantity which is associated to a configuration that remains unchanged as you go through valid moves. As you play the game as you go through this puzzle through valid moves second quantity which you associate in certain abstract fashion is not going to change. So, now there is a whole game; whole game is about understanding what is the right kind of invariant in the situation.

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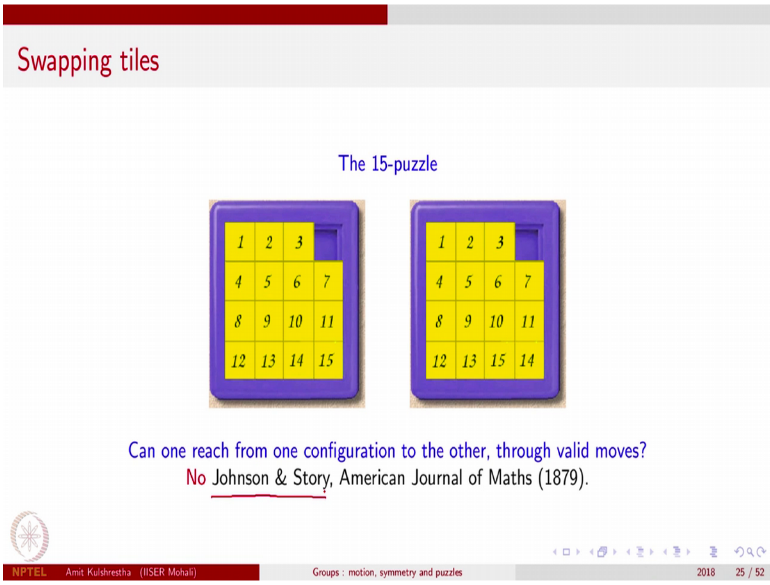

So, answer to this is no. So in fact, that is what you cannot reach from this configuration to this configuration of course, without breaking the puzzle. So, you are not supposed to break it we are supposed to do only valid moves and there was a research paper by these people Johnson and Story. So, what did Johnson and Story do?

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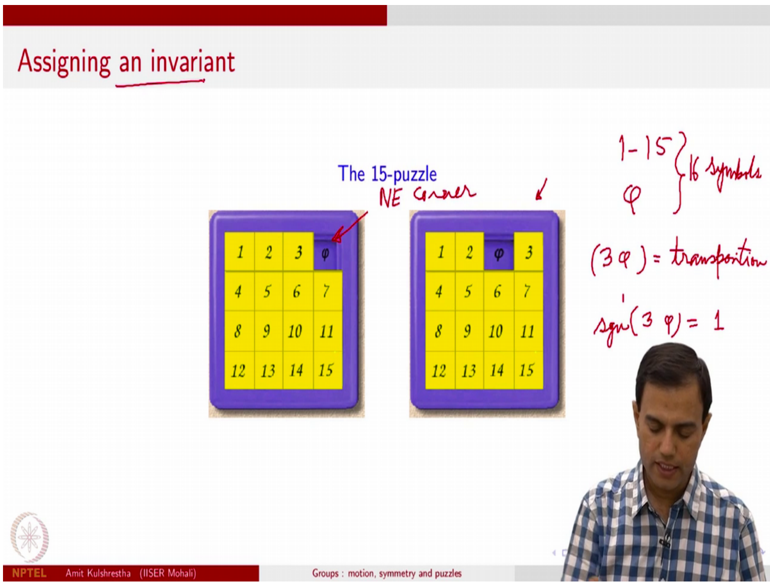

They assigned an invariant to all possible configurations some of them are possible configurations, some of them are not. What you consider all possible permutations in this. So, regard this North-East corner that is empty space give us a notation phi. So, there are 16 symbols; 1 to 15 and then empty set. So, these are 16 symbols where all this permutation is happening.

And as you can see this particular configuration, the permutation is what? 3 phi 3 goes to phi phi goes to 3 and everything else is intact. So, this is as you recall from the previous lectures, this is transposition and its parity or signature, signature of 3 phi is 1. Signature of 3 phi 1 because, there is only one is the odd number 3, there is only one transposition which is used to express this particular permutation ok. So, how do we assign the invariant in this case?

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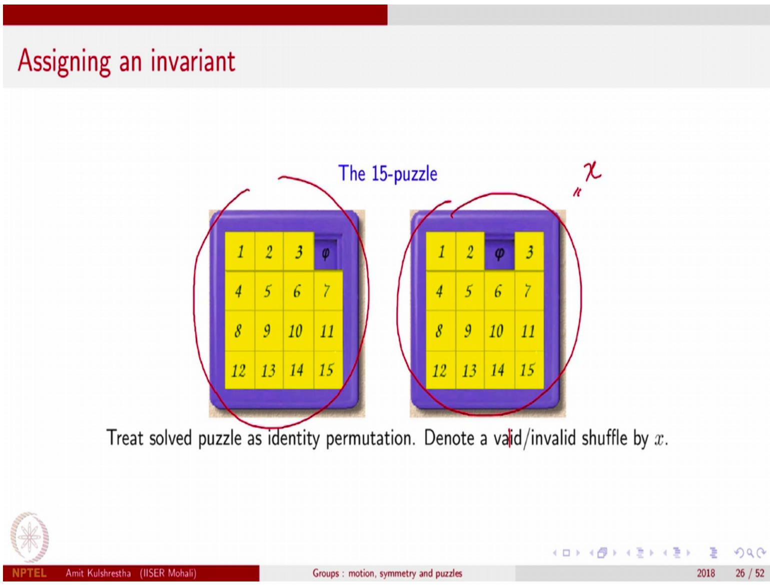

First you treat the solved puzzle as identity permutation and denote any valid or invalid shuffled by x. So, this is x, this is x into x I am supposed to associate some invariant.

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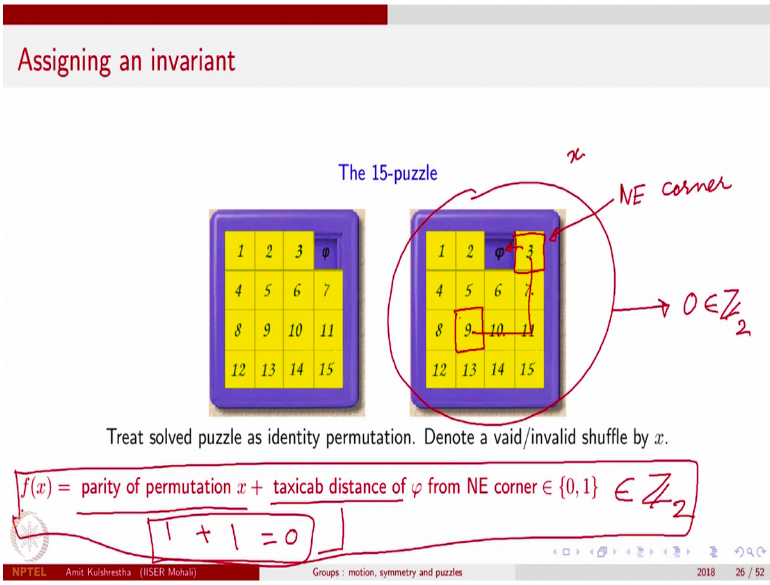

So, here is how I am doing it first parity of the permutation x. So, this is as I said this is x and you as a parity of this transposition 1 plus taxicab distance, what is taxicab distance? Here is this North-East corner. So, here this is just 1 step. Suppose every set were empty symbol were somewhere here then we just make L shape. So, taxicab distance is some L shape thing, go from here to this. So, this is how many steps; 1 step 2 step 3 step 4 steps right, but in this case for this particular x its just 1 step.

So, we do 1 class 1 and where do you do this you do this? In the group Z mod 2, so here you get 0. So, to this particular configuration the number that is going to be assigned or the element of the group Z mod to Z that is going to be assigned is 0, the identity element. So, this is how you assigned to a given configuration and element of Z 2. This is the idea of those two people Johnson and Story who published this paper in American journal of mathematics 1879.

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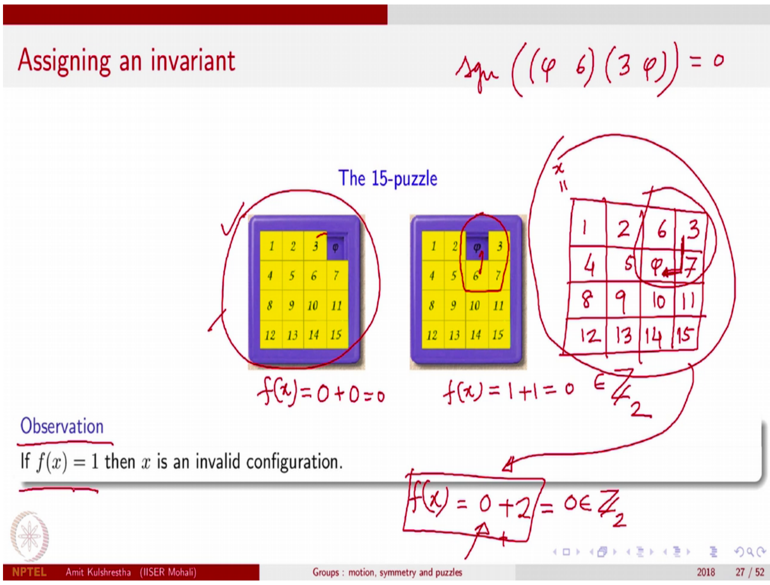

That is a crucial observation that is a crucial observation. So, when we play this game what is happening? Let us observe here and here; here identity so, if you if you write f x for this particular x, how much is it? Identity permutation whose signature or whose parity is 0 plus distance of phi from north east corner which is 0. In the next step here for this particular x how much is parity how much is the invariant it is 1 plus 1 which is again 0, this calculation is happening in Z mod 2 Z mod 2 1 plus 1 equals 0.

So, you can observe here. So, now let us take one more situation. So, suppose 6 goes up that would be interesting 6 goes up. So, I am just trying to so, since 6 goes up then you have 1 here 2 here 6 here 3 here empty set here right, I am just swapping these two in the next step and then 4 5 7 and then everything else is intact 8 9 10 11 12 13 14 15. Suppose this is a configuration this is x then what would f x be? So, to this x I am going to assign f x.

So, first of all we have to understand what permutation it is. So, in this permutation 1 2 4 5 8 9 10 11 12 13 14 15 all of them are intact. So, the only change that is happening is here. So, let us write down the permutation. So, let us see. So, what is happening to phi, phi goes to 6. What happens to 6? 6 goes to phi. So, over and that is on right, but here are I have seen 3 and phi they have been exchanged.

So, from the identity position first we did the change of 3 and phi and then 6 and phi right. So, the signature of this, this product of even number of permutations therefore, the signature will be 0. So, f x is 0 plus the distance of empty set from the distance of empty symbol from northeast corner which is 1 step and 2 step; 0 plus 2 and 0 plus 2 is actually 0 mod 2 So, in z mod 2 this is 0.

So, f x is 0. So, f x is 0 here f x is 0 here of x also 0 here. So, as you can easily observe as you play this game as you move according to the valid moves the quantity is 0 remains same. Meaning if you assign the quantity in this fashion which is parity of the permutation plus the distance of empty symbol from the northeast corner, that sum remains unchanged. And one can actually conclude that if f x is 1 then x is an invalid configuration. You cannot actually move from thus from the solved puzzle to that particular configuration or from that particular configuration cannot solve it in this fashion if f x is 1.

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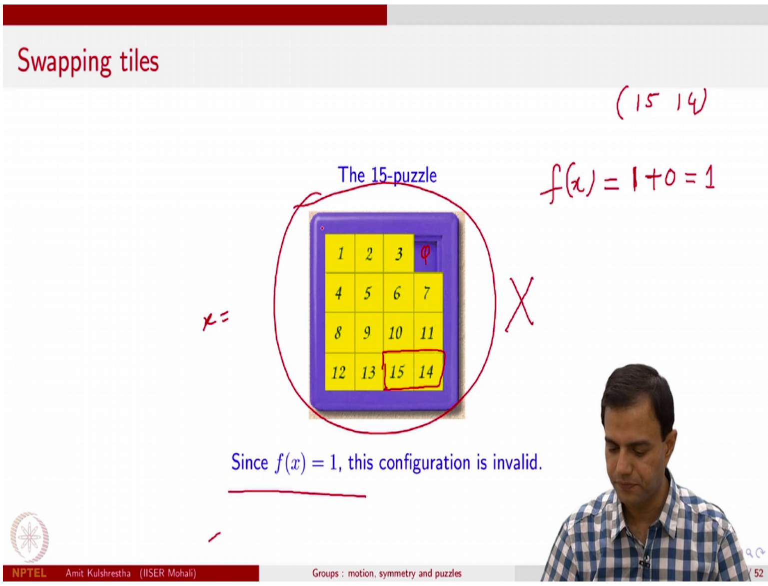

Now, let us see what is there in this particular situation. In this particular situation empty is here empty symbol is here and if I call this configuration as x then what is f x, f x is the parity of this permutations. So, here you would see from the origin the identity position there is a swapped; 14 and 15 have been swapped. So, this is exactly 1. So, the 15 and 14 they have been swapped.

So, the parity is 1, signature is 1 for the permutation and how much is the distance of empty symbol from the northeast corner; it is 0. So, f x is 1 here and since f x is 1 this configuration is not possible. So, just with a very simple idea of parity and using the group Z mod 2 the group Z 2 which has 2 elements you could conclude quite easily that this particular configuration is not possible; till that half the configurations are possible half the configurations are not possible, that is why in another lecture as mentioning about even permutations.

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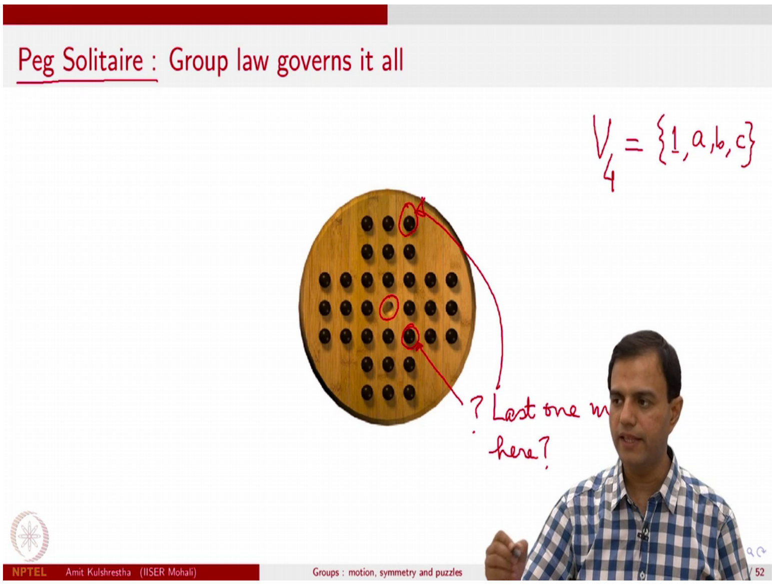

And then we have this quite interesting puzzle as I mentioned to you earlier Peg Solitaire. So, before I explain what it is let me just play this puzzle we are going to play this and then I am going to come back to this.

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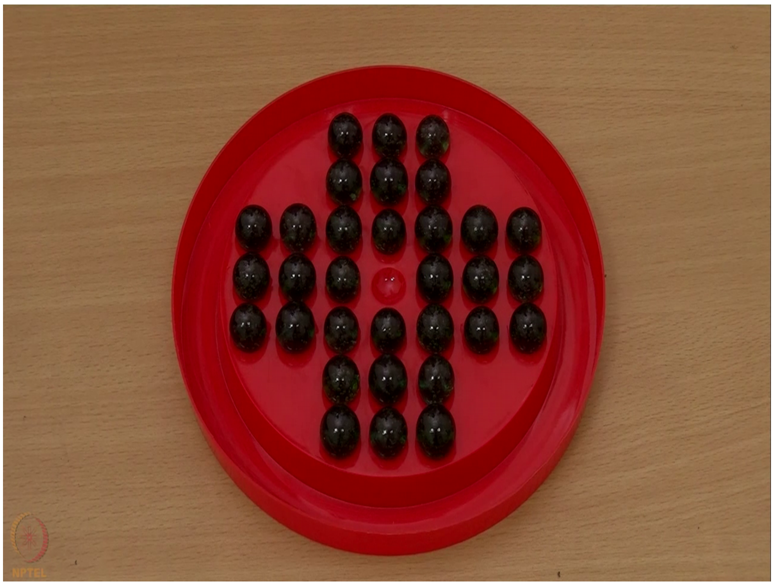

So, this is Peg Solitaire and rules of the game as I was just mentioning during this lecture. Here is empty space which is available I can pick say this 1 jump over this and knock it out. Now there is some more space created I have option of doing something like this and knock it out say something like this and knocking out. So, here is the space I can do a knock out and eventually I am seeing what happens in the end can I say as minimum as possible the number of marbles that are safe denied they should be as less as possible. So, here I would request my TA Kanika to play this game and show you if you can say won.

Wow that is amazing great Kanika wonderful great job. So, here you see the last one is here at the middle and just before that the positions were like this and she jumped over this knocked it out and therefore, you have last one here. Let me also try to play and see what happens ok, Kanika you win. So, here you see the last two which have come they are at this position the last ones which are saved are in these two positions and the last single one was at this position at the middle position.

I can ask Kanika to do something difficult thing which is I can challenge her if she can have last marble here single last marble, but at this position and I am sure if he tries for couple of times she will declare that it is not possible. But how to logically arrive at that conclusion, we are going to learn through group theory and the group that we have already seen claims for group the group of symmetries of rectangle; very surprisingly that plays a role in the rules of this game. So, I hope you enjoyed all that and now, let us come to the explanation of it.

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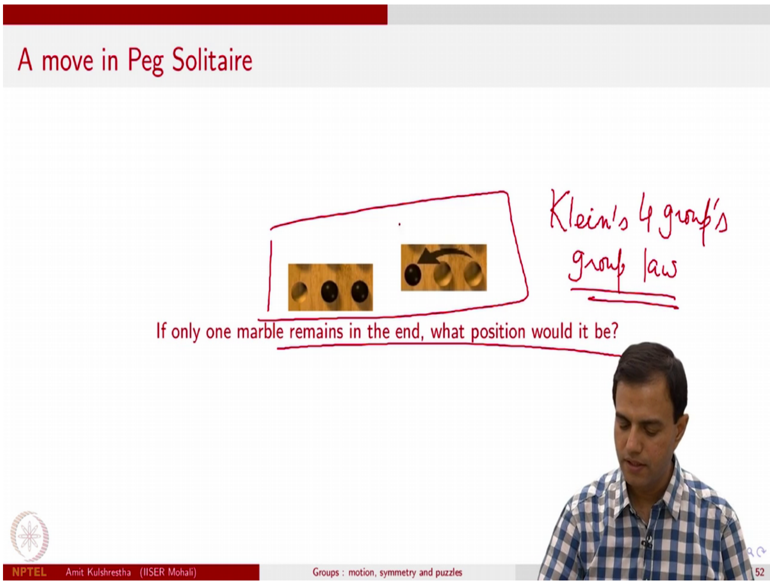

So, this is our peg solitaire and as you have seen what is one move? What constitutes one move here? So, there were 2 marbles I jumped I from this marble to the empty space and in the process 1 marble was knocked out right; that is how we played all this and we just keep doing like this. Now, my question is in this game if you play like this; in the end we saw we could actually save 1 marble. And you remember what was the location of that marble? When, as you could see it is possible to save 1 marble here.

I change the question. I say ok, can you have last 1 marble saved at this position or if I give you some configuration and ask you that it is it possible to reach that configuration; how will you answer that question? The answer is quite surprising and shocking and in fact, the whole game has something to do with our favourite group. We have seen this group couple of times Klein's 4 group, 4 element square is 1, b square is 1, c square is also 1 into b c and so, on.

So, I hope you have understood the question, I just find some position I just some position I just mark some position and they want last 1 marble to be saved there at that position. And I hope you have also understood what the rules of the game are, whenever you find empty space you are allowed to jump like this and then knock out this marble. This marble goes out this jumps and comes here and this is what you have.

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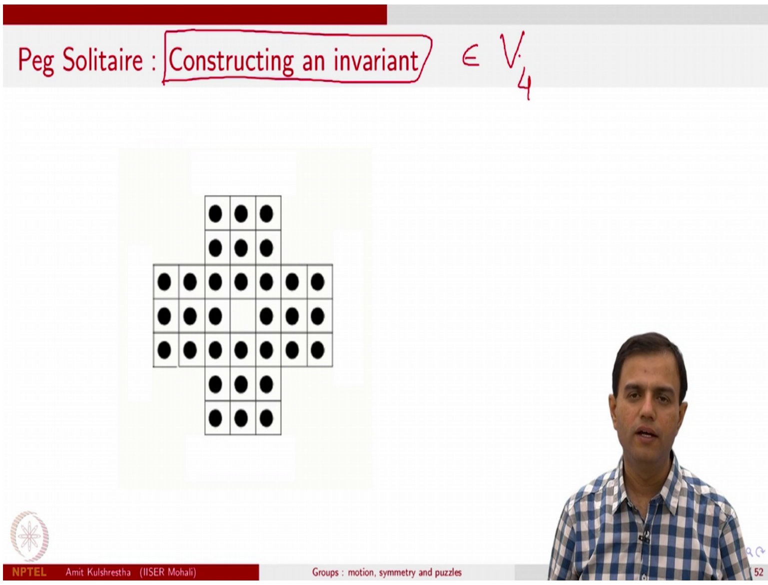

So, that is a question the 1 marble is there what position is we will that have ok. So, again here we are going to construct an invariant and interestingly this time the invariant is going to take place in Klein's 4 group. So, to a given configuration to given arrangement of marbles on the on the on this game we are going to have we are going to associate an element in the Klein's 4 group. How do we do that? It is quite smart quite interesting and everything has to be realized here that somehow in this one move, Klein's 4 group, the Klein's 4 groups; group law is there ok. Let us do one very smart labeling.

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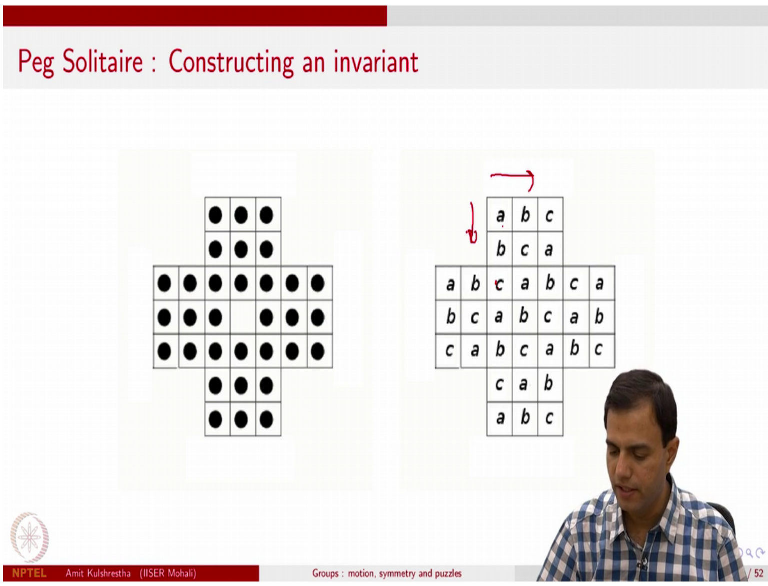

What I do? I just start labeling these locations in cyclic fashion a b c and in b c a, then a b c. So, that here also have a b c a b c a b c a then b c a b c a b c a b c a b c c a b a b c. So, you read it like this or like this it is quite cyclic ok.

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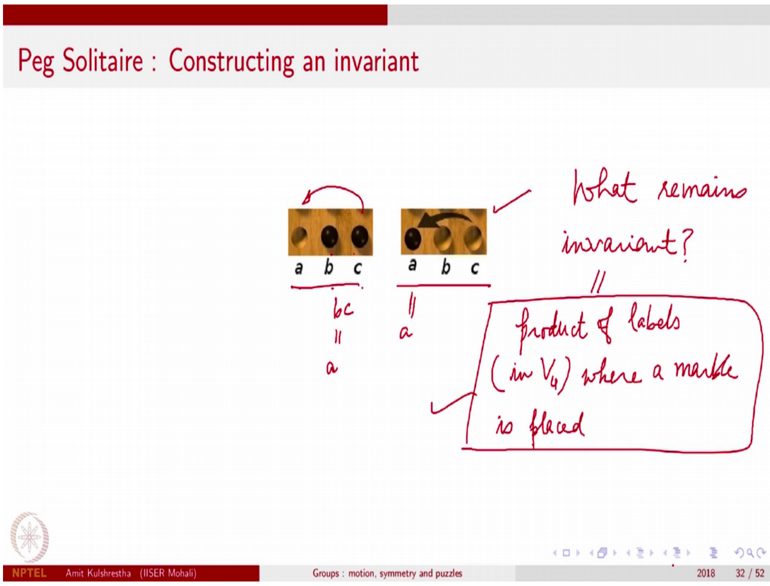

So, if you pick three consecutive locations, one of them is a, one is b, one is c does not matter I could have said b c a. So, a b c a b c this is there and then I am jumping like this here and I am having this situation. Now, what remains invariant here and that is the main observation that is the main observation. Since, I already mentioned Klein's 4 group and those a b c; anyways of Klein's 4 group associated I am labeling using those a 3 labels.

So, here if you observe we get the answer of this namely the product of labels. So, product in Klein's 4 group where marble is placed and this is what remains invariant is not it? You see marble is placed here and here. So, b into c I have b into c which is a in a Klein's 4 group and here the marble is placed only here, so product here is just a right and this is just one step. So, this is a quantity which is being preserved here and other locations are anyway untouched in one move and in one move this product is unchanged. So, that is easy how to decide how to associate, how to assign an element in Klein's 4 group to a given configuration to a given distribution of marbles on the board.

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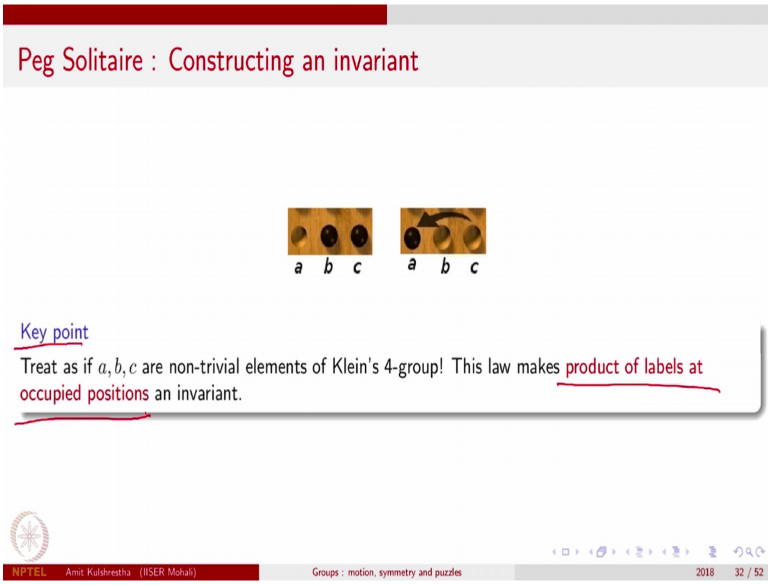

So, as I said the key point treat as if a b c are non-trivial elements of Klein's 4 group and the product of labels at occupied positions; those positions that marble is placed is not going to change throughout the game that is crucial ok.

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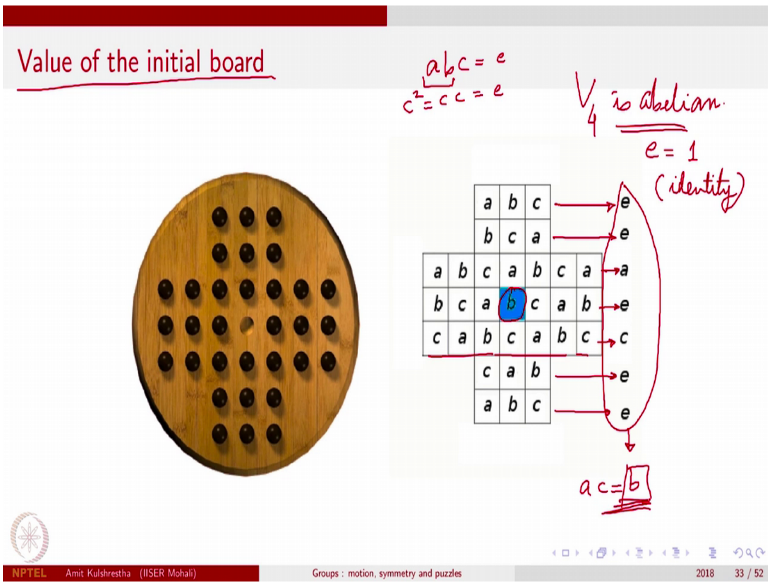

So, the beginning the value of the initial board if you call it or the invariant which is associated to the initial board is this, except this location you have marble everywhere. So, I just take the product of all these things. I am very fortunate here that Klein's 4 group is obedient. Therefore, I need not worry whether it is a product a into b b into a the same thing. So, I am just multiplying them. So, let me multiply them row by row. So, a into b into c is e; e is identity, identity. So, a into b into c is e; this is also e and how much is this, a into b into c is identity and a into b into c again identity and then a. This is simply a, b into c into a I would not take this is unoccupied position I am considering only occupied position and now c into a into b which is e.

So, e is identity aims identity. I hope you are with me when I say a b c is identity. How am I saying because a into b is c this c into c is actually e, c square is identity right; that is what happens in Klein's 4 group. And then c a b is identity c a b identity c what remains is c and c a b is again identity, many a b c is again identity and then I take the product of all these and what I have is a into c which is b right.

So, position of the initial the value of the initial board as per this assignment is b and we known as we move along as per the valid moves this quantity is not going to change. Whatever, is the configuration what was the distribution of marbles over this board if we assign to that the product of labels at occupied positions it is going to remain be throughout and that is a key point.

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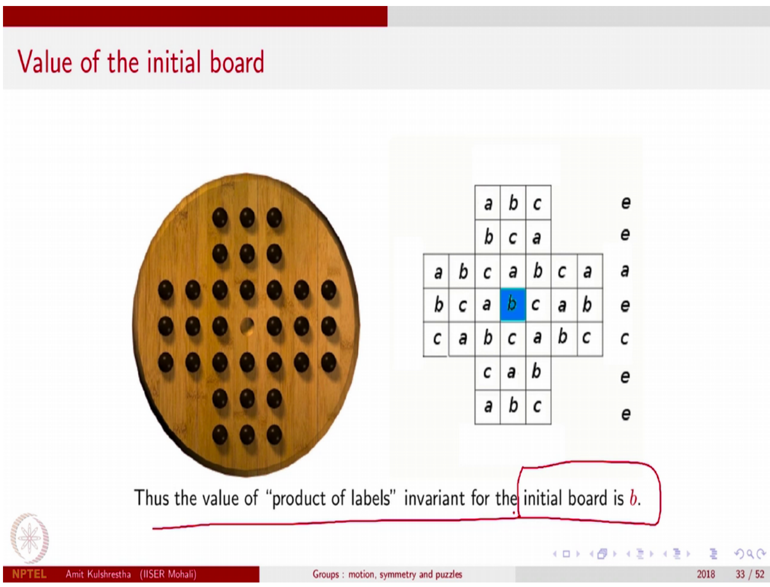

So, the value of product of label invariant for the initial board is initial board it is b ok.

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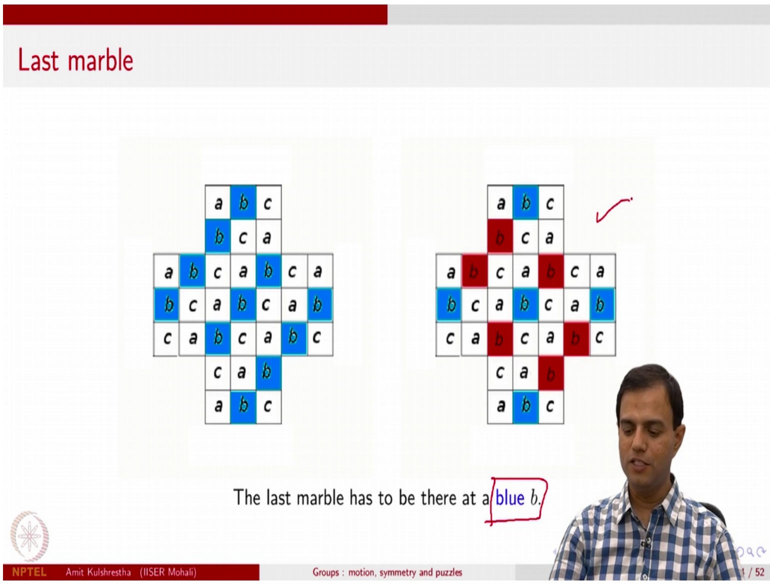

That means what? That means, when there is only 1 marble after playing this game. There only very few positions where 1 marble could be and that 1 marble could be at all these positions, wherever b is there right; So, these are the only possibilities and does not mean that marble will be there in the end, it is not there it is not really required. But if at all 1 marble can be saved then those positions could be some of these the other ones where a and c is there no single marble can be saved there in a.

And now, I make use of symmetry. I say that this is not a possibility and I mark it red. How do I argue? I am arguing it because if this is the possibility then just imagine as if you are playing in a mirror this would also be possibility right. But here I have a, here the level is a and I know that I cannot have 1 marble here. And therefore, just by playing the game in the mirror we are forced to conclude that you cannot have marble here in the end. And then you are applied this argument and then you would conclude that all the b is that I have marked with red, they are actually not possible. Because, if they are possible if suppose this is possible again by playing in the mirror I should be able to save 1 marble here.

So, what are the uncrossed b's? Uncrossed b's are these, only these and in the end when we saved 1 marble in the end and situation was essentially like this. So, here you have two options either take marble from here, put it in the center knock out c or you take this marble from c to b knock out a and keep your last marble here. So, these two are secondly positions and therefore, by symmetry all these are actually possible positions where you can have marble 1 single marble in the end.

So, this very simple idea otherwise if we think it in some other fashion it could be extremely complicated. But thanks to our observation, what observation our observation that this 1 step is following the group law group law which is of Klein's 4 group that observation we could stretch all across. And we would conclude using the ideas of invariant that the only possibility is there 1 marble could be same that these.

So, therefore, the last marble has to be there at a blue b there is no other position possible; So, very simple ideas just very basic use of group theory, but when one of your connections. You remember we started when we discussed Klein's 4 group, you started with the rectangle and Klein's 4 group turned out to be symmetries of rectangle. But interestingly Klein's 4 group is also having beautiful applications in this puzzle.

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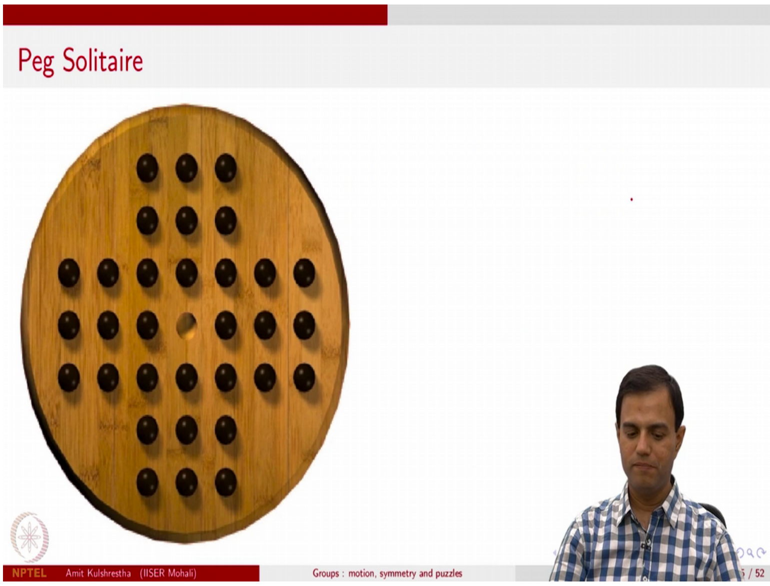

So, I hope this was enjoyable for you and you would also devise some such games. You could change the rules of such games and try to discover, if there is some other group which is playing a role in that and you find such a thing, please do write to us and we will enjoy.

Thank you.