Groups: Motion, Symmetry & Puzzles

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Groups, as they occur naturally
Lecture – 04
Groups and parity

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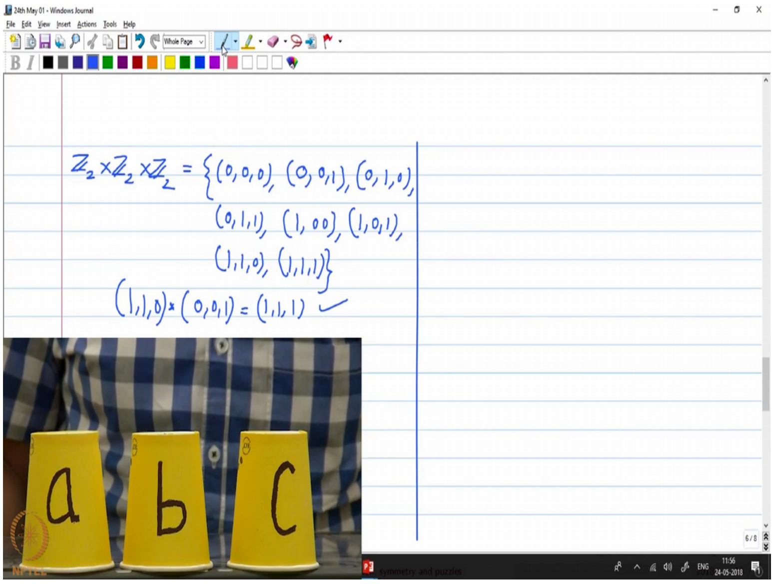

So we are back and as I promised, we are going to have some fun with these 3 glasses and I am going to use group action to solve one problem. First, let me ask you the problem we have 3 inverted cups and at one time I can pick 2 of them and change their orientations. So, if they are inverted, I can make them upright and then keep doing like that I take any of them maybe this I am back.

So, my purpose is to make all 3 of them upright by doings, and I am just trying it and I am trying hard it is not happening I cannot do that. I have tried to be couple of times 10 times something, but even if you tried thousand times it is not going to happen. And why does it? Why is it the case? Why I cannot make all 3 of them up right? So, it is a question for that let me first convince you that there is some group action which is going on here.

So, whenever there is group action there is a group and then there is a set. So, this is one configuration how many such configurations are possible? These are 3 and there are 2 positions for each of these glasses 2 to the power 3 8 configurations are possible and as I had indicated. In the last lecture, look at it this sequence of 0's and 1's there is some similarity right. So, we can say 0 is the inverted position and say 1 is the upright position

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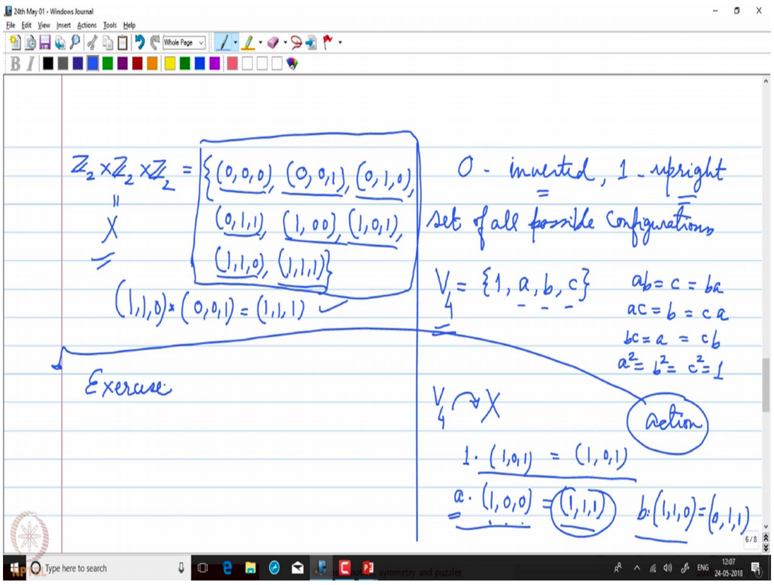

So this set which we constructed as a group, let us take it as a set. This is set of all configurations which are possible for these 3 classes.

So, this is set of all configurations all possible configurations and on the set of configurations. What is acting right? I am acting right it is a group there is a group which is acting which is changing these configurations. So, here is an interesting idea you can see they are the same, you remember clients 4 group as I have been mentioning and this time we denote it by 1, a, b, c there are 4 elements in these group, as you recall a square is identity b square is identity c square is identity and then abc ac is b and b c is a it is Abelian group so, abc is same as b a similarly this is same as ca and same as c b of course, a square, b square, c square is I think. So, what I am saying is that this group has something to do with this problem.

So, I define the group action exactly in the same way as I am playing this game V4 is acting on X. What is x axis? All possible configurations and then there should be an action. So, let us pick an element from V4 well if there element is identity then whatever I pick from X, I pick something like say 101 or then it is simply 101 nothing is happening interesting thing is for example, when I pick a from V4 and then what is it is effect on say 1 0 0 that is what I am going to explain.

So, effect of a on 1 0 0 is that, this position. So, let us call it position a, position b, position c. So, position a is unchanged while 2 other positions are swapped. So, 0 becomes 1, 1 becomes 0. So,1 0 0 becomes 1 1 1. So, if I have this to be 1, 1 is upright position 0 is inverted position. So, if a is 1and b is 0 and c is 1.

So, if I have this, then identity is not doing anything to it yeah and here say 1 0 0. So, it is 1 a 0 and c 0. So, when I apply on this configuration the element a from the client's 4 group what happens? All 3 become upright. So, I take I do not touch a, I touch only other 2 and I just change their position. So, all 3 become upright if I take something else say, I take say b and consider how b is going to act with say 110.

So, I am not supposed to touch sorry so I am not supposed to touch b. So, this one will remain intact while the 2 other will be swept so 1 becomes 0 and this 0 becomes 1. So, it becomes 011 so let us see so the 110 meaning a is upright, b is upright, c is inverted and when b is acting when b is acting on 110. So, I am just fixing this, I am not touching it and other 2 I will just swap. So, what I have is therefore, 011 yeah so this is how we define the action, but be carefully.

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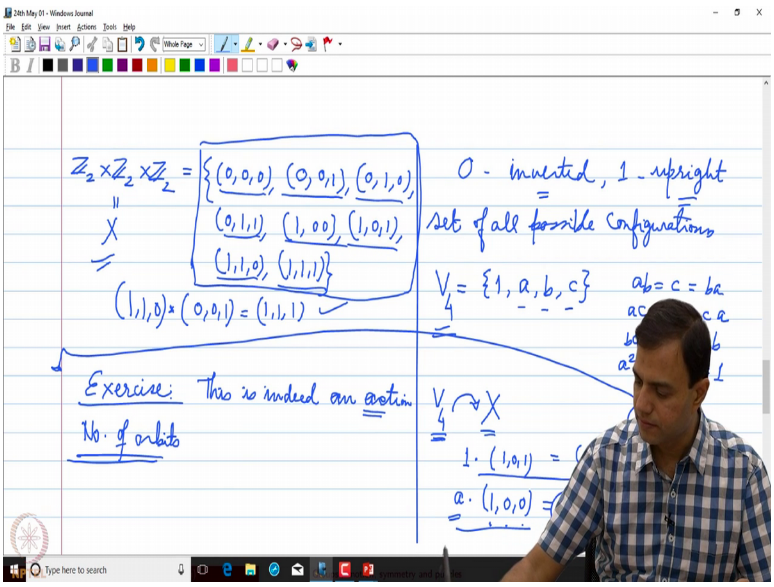

So, exercise for you is that this is actually in action and I guess you remember the definition of action.

So, if I define the action of clients 4 group action of V4 on this set of 8 elements then, it is actually an action ok. Now, the question is well let us see if number of orbits how many orbits are there that, would be something interesting. So, we estimate the number of orbits for this action.

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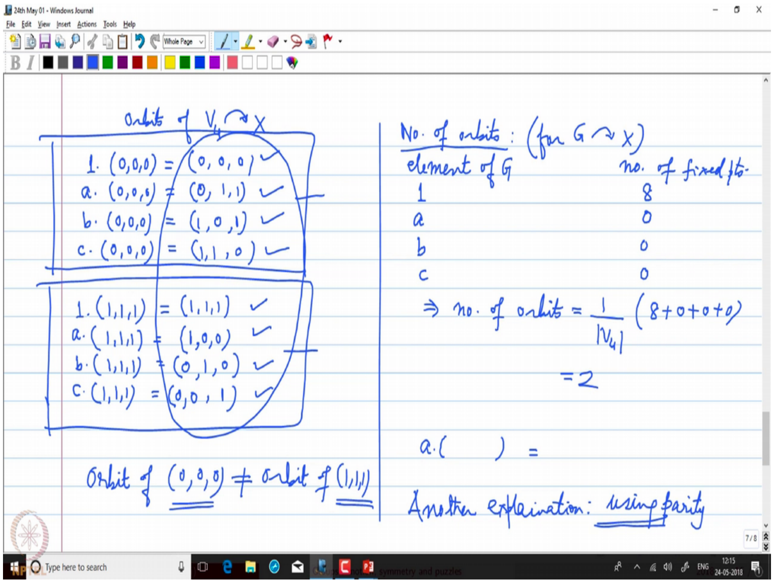

So, I consider say 000 that is all the elements are all 3 glasses are inverted.

So, I am looking at orbits of this action. So, in all 3 glasses are inverted action of 1 is again all 3 classes are inverted action of a on 000 is 1st position becomes 0 sorry. So, action of a is 1st position remains unchanged and 2 are changed, what about b? When b acts on 000? What happens you have the middle position does not change? And this changes to 1 this also changes to 1 and. Hence, it happens with c what you have last position does not change and first 2 positions change.

So, 0 0 0 can move to either of these positions what about 111 and you have been influence of 111 goes to 111, no change what about a acting on 111 that is simply position of a the first position does not change other 2 positions become 0. What about be acting on 111 is simply this does not change and these 2 things change and c you have this position c does not change while other 2 positions change. So, orbit of 111 has these 4 elements fine.

Let us see how many orbits are there this action so average number of fixed points. So, I pick 1, so I pick element of G and number of fixed points. So, I take identity identity fixes everything right so I am using the fact that number of orbits is equal to average number of fixed points as I have explained. In previous lectures, what if I pick a how many elements are fixed?

So, let us see if a acts on anything it is not going to be fixed right because other 2 positions will change. So, number of fixed points is 0 same is too with b and same is too with c. Therefore, number of orbits is 1 divided by size of V4 which is 4 and number of fixed points you just add them which is just 2.

So, this action has 2 orbits and we have already identified this orbit and nearby identified this orbit and that is all this covers all the elements of X. So, there are 2 orbits these 4 elements are in 1 orbit these 4 elements are in 1 orbit and therefore, orbit of 0 0 0 is not same as orbit of 111.

Therefore, by means of this action therefore, by the rules of this game we cannot change position then all 3 cups are inverted your position where all 3 cups are upright. So, that is very interesting application of the group actions the same thing the same problem I could have ah. We would have understood in much simpler terms not even in terms of group actions, but using the concept of parity.

So, another explanation to this would have been using parity so the notion of parity something that we are going to learn. So, let us see in this case what would parity mean?

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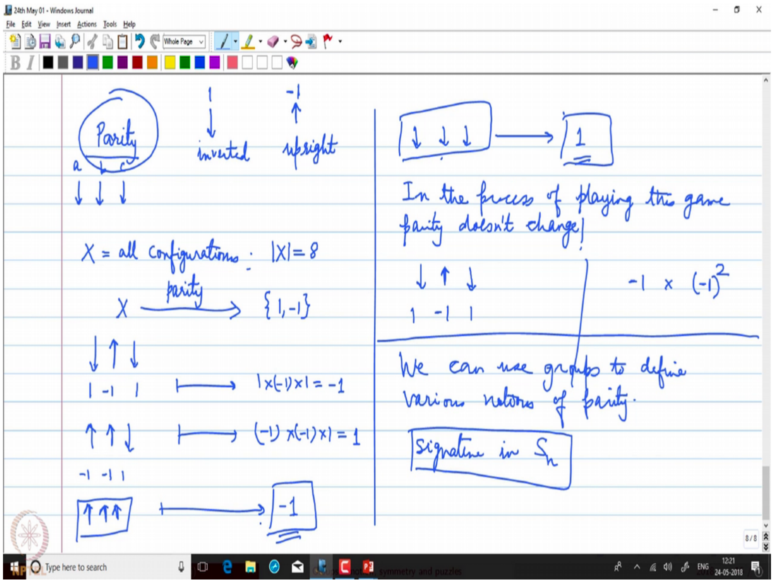

So, here to each state I can assign parity in the following sense so suppose these are 3 positions for the 3 cups abcs this is inverted. So, this is the notation inverted this is notation for upright.

So, whenever I have this down I say one when I have upright I say minus 1. So, X is a set of all configurations so size of X is 8 and this map goes parity maps goes from X to say 1 minus 1. And what it is suppose? I take this particular thing. So, this is a inverting I am doing 1 1 and this is minus 1 this is the product of these 3 1 into minus 1 to 1 which is minus 1.

So, to this state I I assign the parity of minus 1 some other state I have minus 1 minus 1 1. So, the product is 1 so here if you check. Here, if you observe this has parity minus 1 while this has parity 1 1 into 1 into 1 and this is minus 1 into minus 1 minus 1 parity is minus 1.

So, parities are different and interesting thing is that the process of playing this game parity does not change that is the thing. Why? Because suppose you have this state. So, you have 1 minus 1 1 so we have minus 1 after that, whatever you do you are actually doing minus. You are changing the position of 2 you are changing the configuration of 2. So, we are actually multiplying with minus 1 square minus 1 squared is just 1.

So, we are not changing the parity in the process and since the parity of this is different from parity of this parity is minus 1 here parity is 1. So, we can easily conclude that you cannot obtain from this position all inverted positions to all upright position so same question.

Therefore, had 2 different answers 1 in terms of parity and 1 in terms of group actions. And, interesting thing is that we can actually use groups to define various notions of parity. As, I was saying in the mock in the previous lecture the signature the notion of signature in Sn that is 1 some kind of parity.

So, I am going to use that parity and this kind of parity Z more 2 parity, this kind of parity to solve for you, some other puzzle and somewhere permutations are there groups are there.

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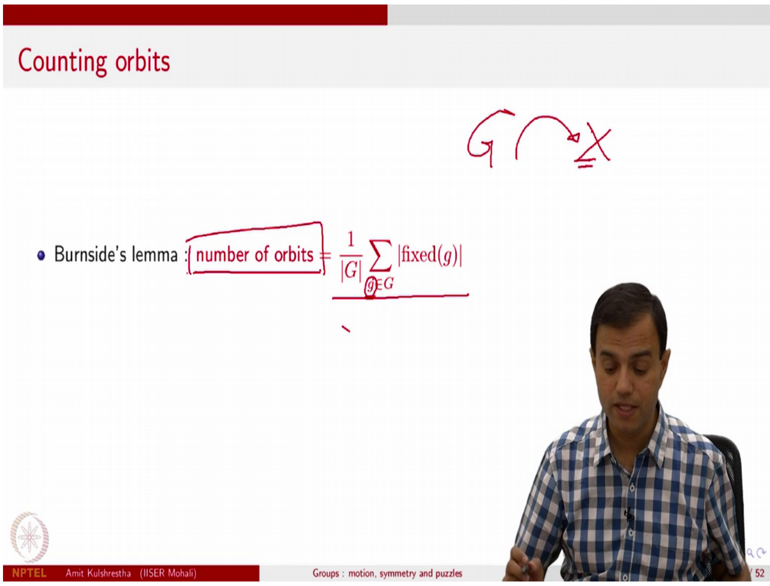

So, today we saw how number of orbits how calculation of number of orbits could be useful for some puzzle.

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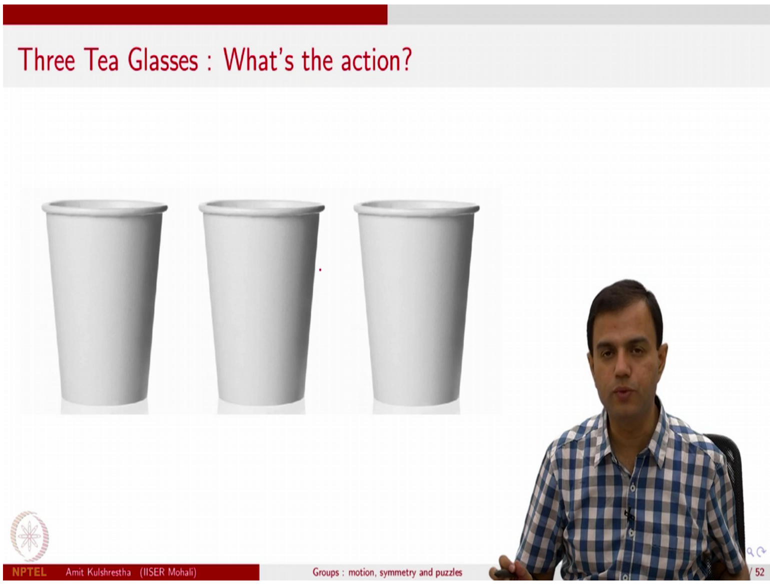

Inverted glasses puzzle we understood we understood it through group actions. We calculated orbits the 2 orbits we explicitly could tell that they are 2 distinct orbits. Therefore, you cannot move from 1 from 1 configuration in this orbit to other configuration other orbit.

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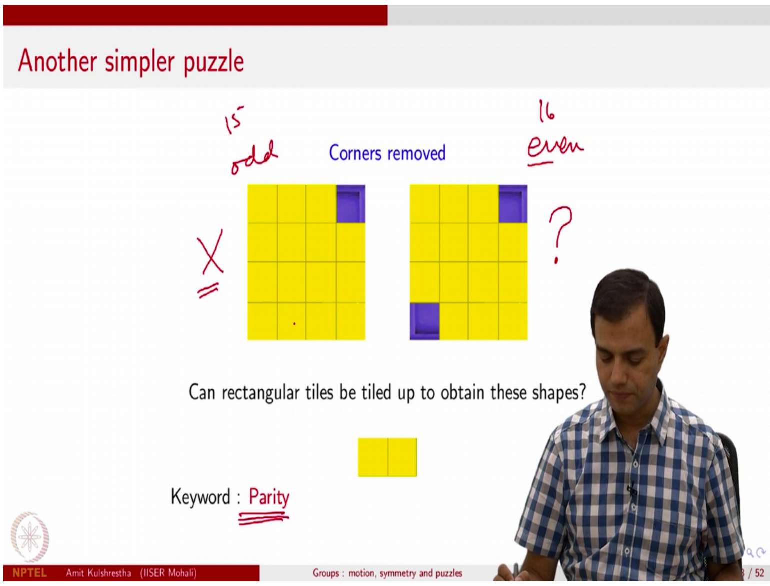

There is a some some there is few more puzzles that we are going to discuss.

So, here is some here is a puzzle which has to do with parity and then we shall use groups to assign parity. So, consider this so I have this square the tiles 16 tiles 1 corner is removed, here is a similar set similar set of tiles the only thing is that, the 2 corners which have been removed from this and as a very simple question. Extremely, simple question is I am I am given these kind of tiles rectangular tiles and the point is can I tile up these things.

So, here there are 15 here there are 14 can I tell these things using these kind of rectangular tiles. So, rectangular tile tiles be used to obtain this shape and this shape, what would be your answer it is difficult as you can easily observe this has odd number of tiles 15 is odd.

So, suddenly you cannot tiled up this using this these are 2. So, you are using some odd what about this does it always mean that if the number of tiles which are here they are even. And this is an even number you can always tiled in this fashion when there is not here application, but using parity we could at least cross this possibility we could cross right.

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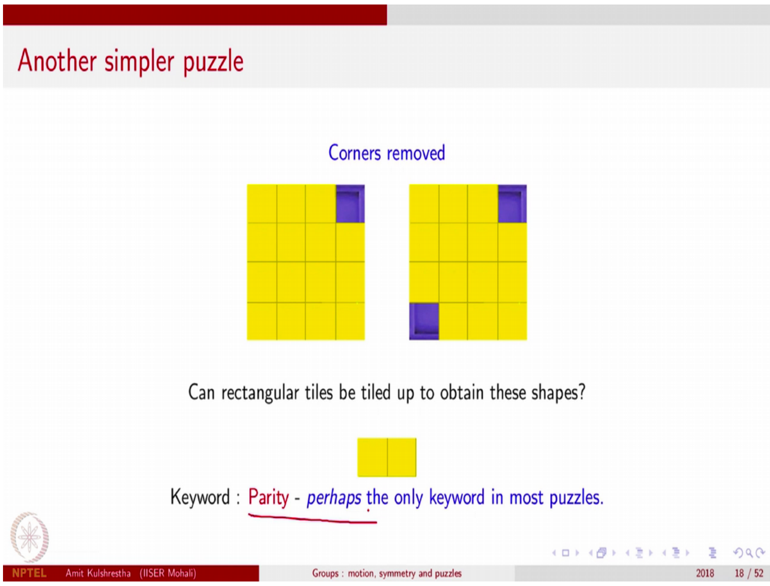

So, the various puzzles actually key word is parity, where it is quite important a keyword in puzzles.

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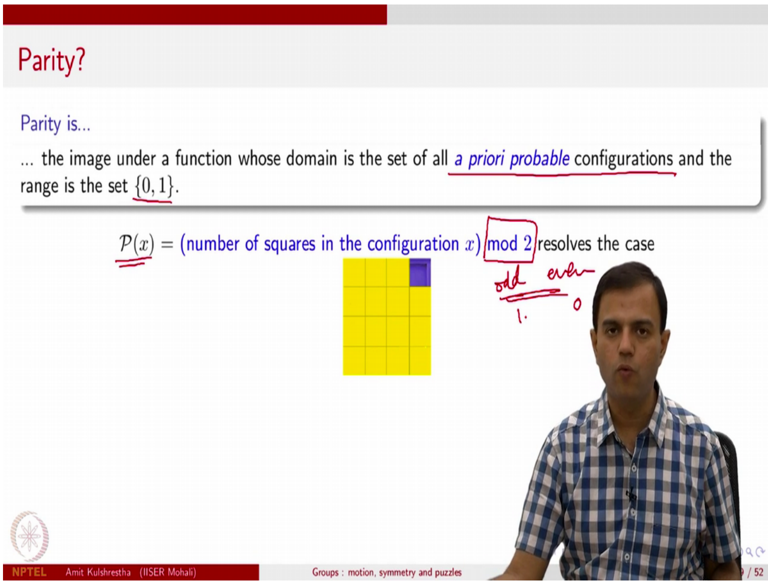
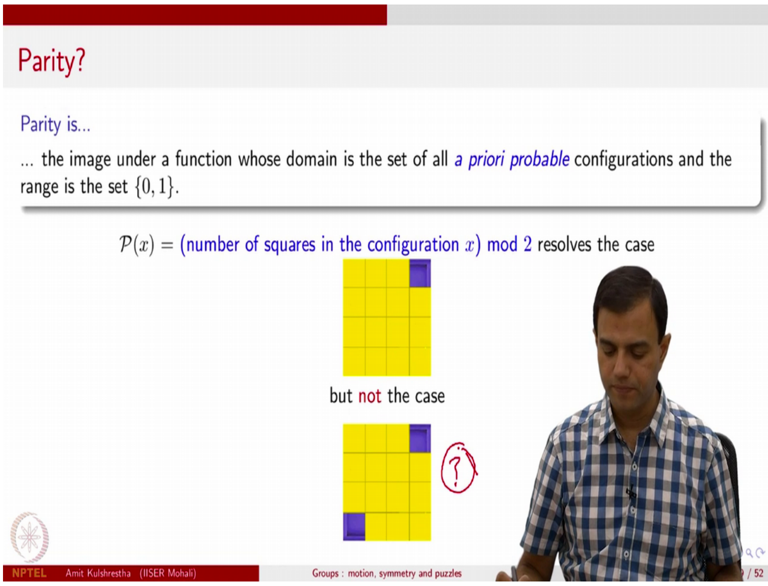


So, what is parity? So you take actually all a priori possible configurations of a puzzle or all possible situations. You take and then you assign each of those configurations a number it could be 0 comma 1 one of these numbers either 0 or 1, it could be minus 1 1 could be color say red and blue it could be.

So, what resolved in that previous situation, the if the case it was this notion of parity, which was just take number of squares in the configuration. So, with this configuration number of squares is 15 mod 2 so 15 mod 2 is what resolved even and odd right that was essentially parity. (Refer Slide Time: 23:59)



So, even may be represent by 0 odd maybe represent by 1,But this care case could not be resolved using the notion of parity.

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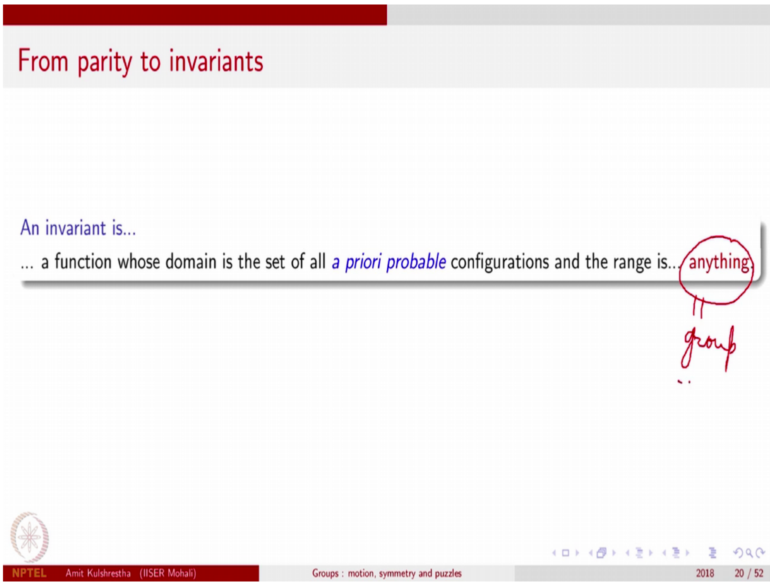

What is interesting? Is that when you have some configuration when you have some puzzles it is interesting to see if while that puzzle is being played. While that game is being played something remains unchanged. For example, in the case of 3 glasses the parity remained unchanged.

So, parity was an invariant so an invariant can take values in any set quite often the interesting situations are when this set is actually a group.

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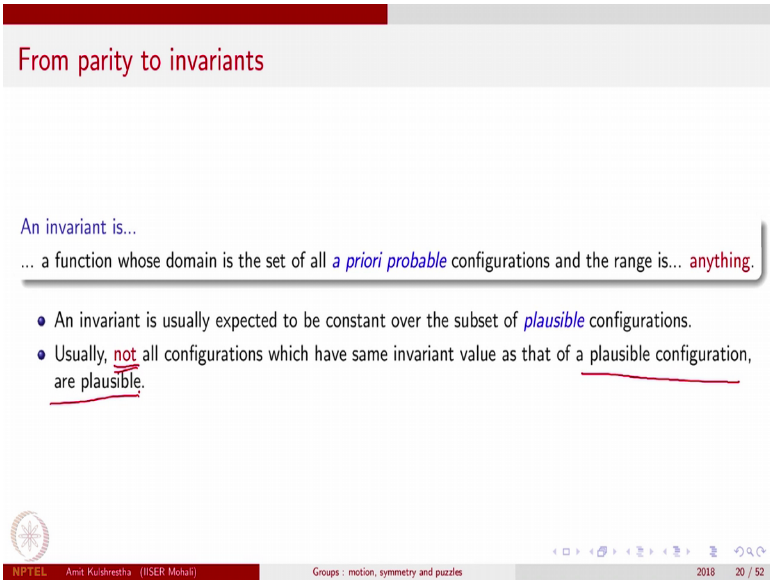

So, as I said invariant is expected to be constant over the subset of possible configurations.

So, if it is if a game is being played so all the possible all the plausible configurations are arriving and it is invariant. So, certain thing is constrained some parity is constrained. So, over all those plausible configurations then variance is constant and just being constant does not mean that your plausible configuration is actually there is actually a reality.

So, we are going to understand all these things through an example.

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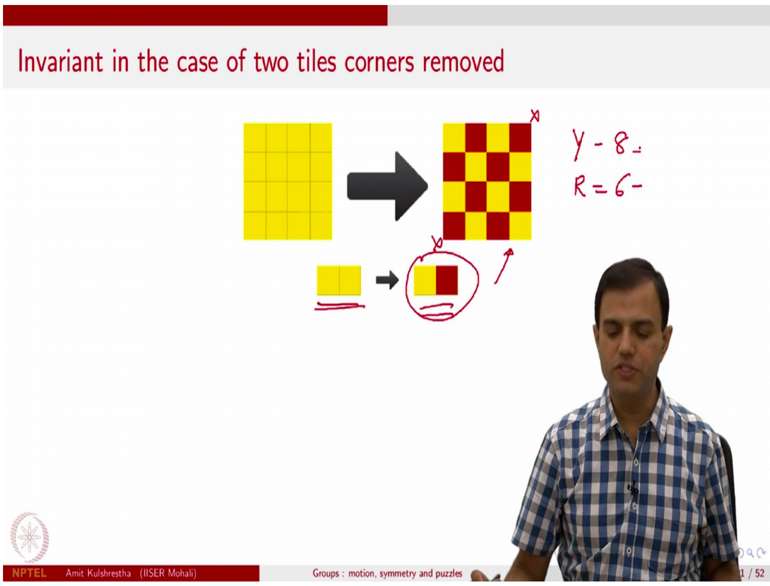

So, in the case where 2 corners were removed, what can we do? We can use colors and I am going to mathematically express that. So, rather than seeing it like this, I could have seen it like this. And when corners are removed this corner is gone and when this corner is gone.

If I want to tile it with tiles like this then the number of yellow tiles should be same as number of red tiles, but here I have removed both tiles which are red. So, number of yellow tiles is 8 while number of red tiles is 6 so it is very clear that if I want to use these tiles to completely cover this knocked off thing with 14 squares. I cannot because number of red tiles is not same as number of yellow tiles all this I would say in terms of in terms of parity, in terms of some invariant.

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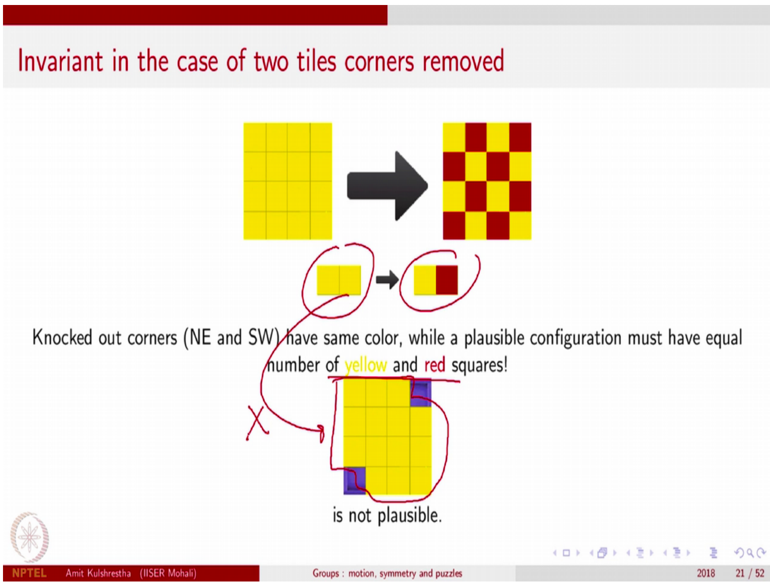

.So, using the concept of color yellow and red this using the concept of parity I have concluded that this is not plausible. So, I cannot cover this, I cannot do this using these kind of tiles it is impossible.

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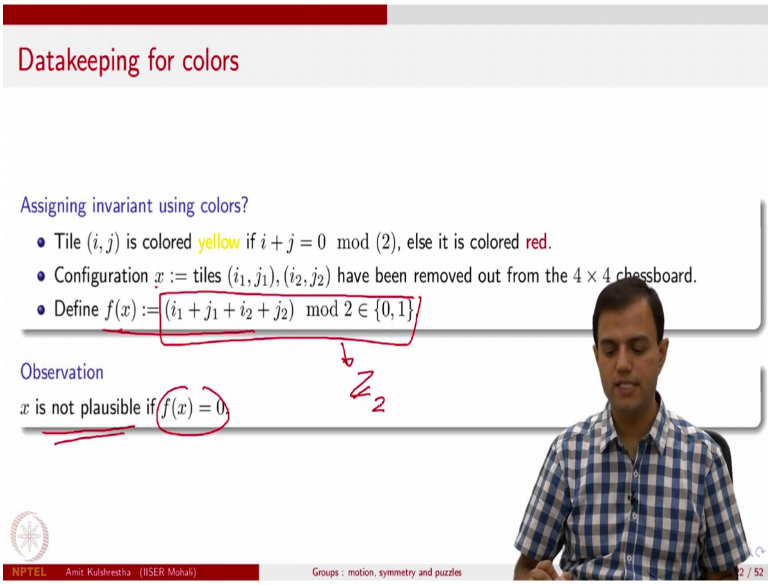

What was the invariant in the situation? What was the parity in the situation? So, all this colors all the all these colors which came to resolve this problem I am going to express that through this expression.

So, tile ij is colored if it is it is colored yellow if i plus j is congruent to 0 mod 2. So, i plus j is having value 0 in the group Z 2 otherwise it is colored red. And what is configuration x configuration x is tiles with these coordinates i1 j1 it is 1 1 tile other tile with cognates i 2 j 2 they have been removed from the 4 cross 4 that chess board. And then if you define your number the invariant associated to be i 1 plus j 1 plus I 2 plus j 2 mod 2. And that resolves the problem and the observation would be that if f x is 0 then that particular configuration x is not plausible.

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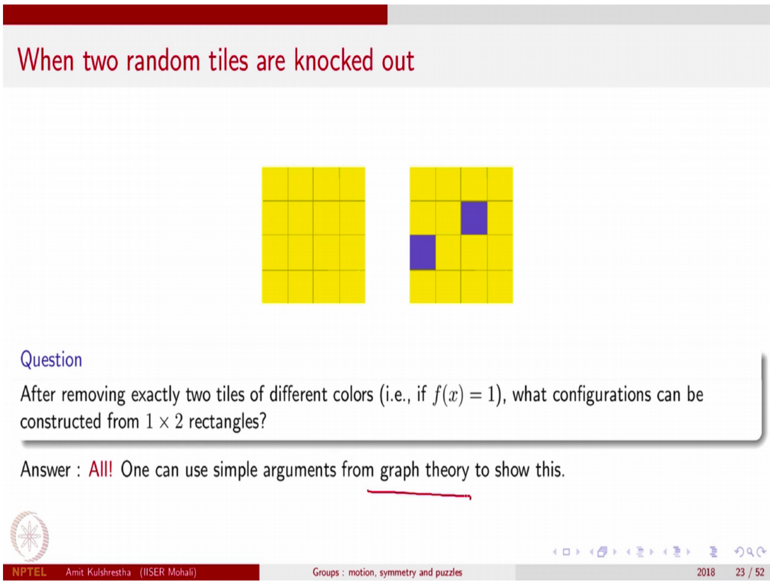

So, for example, if I knock it out knock it out this particular one then they are of the different color. So, after removing exactly 2 tiles of different color what you would have? Is that the number the invariant with that I have defined like this is actually one. So, then question is what configurations can be constructed from those rectangle then that rectangle that I have 1 1 by 2 tiles. Those, smaller rectangles and answer is all it is not very difficult are some graphs which are required to understand this part I am going to do it here.

So, basically what I want to say is that the concept of invariants and some mod 2 invariants can be used to resolve certain puzzles and here your function is lying in the group Z 2. So, we shall have some more interesting some quite interesting puzzles later on so I hope you enjoyed all this. So, in coming lectures we will try to see more applications of groups we will see 15 puzzle we will see Rubik's cube. And you will see how we can understand? How we can use our understanding of group theory to resolve the issue the resolve the problems which arise in those puzzles.

Thank you.