Groups: Motion, Symmetry & Puzzles
Prof. Amit Kulshrestha
Department of Mathematical Sciences
Indian Institute of Science Education and Research, Mohali

Groups, as they occur naturally
Lecture – 03
More on group actions

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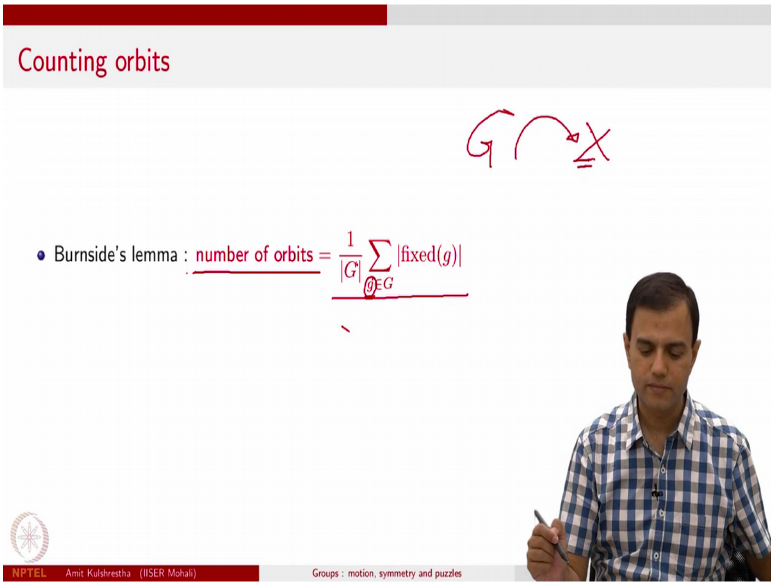

So, welcome back. so last lecture we had seen some motivations for definition of group some examples and group action. So, when you have a group you have a set one can make this group act ZX that is group action. And we saw formula for number of orbits, and it turned out that number of orbits is precisely number of average number of fixed points average number of fixed point meaning, we take elements in G. Consider number of elements in X which are fixed by G, and then sum it over divided by size of the group that is the average number of fixed points.

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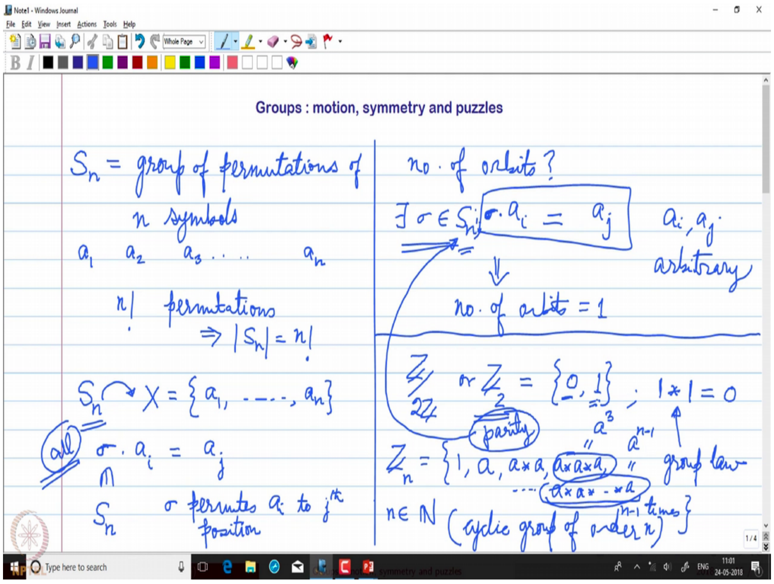

So, before we move let me mention to you certain examples of groups and actions as well and these we lead for further discussion. So, I will start with Sn; yesterday, I mentioned Sn is the group of permutations and symbols. So, I have symbols say a 1 a 2 a 3 an and permuting them and then there are n factorial ways. So, there are total of n factorial permutations.

Therefore, the size of the group Sn is n factorial the way this symmetric group is defined the way Sn is defined it is very natural that, Sn x on x what is x? X is just the collection of those n symbols a1 into an. How does it act? It just. So, you take just element I will just called it sigma sigma is an element of Sn and I take an element of x say a i.

So, sigma was with changing the position of ai it may be moving into jth place or whatever place. So, then I take it to be aj. So, that is sigma has the property that sigma permutes ith element to jth position. So, that is how the permutation group is ok. So, we have an action what about orbit? What about number of orbits for this action ? So, that is not very difficult thing is you see Sn is collection of all permutations. Therefore, given any ai and any aj so, given any 2 symbols so, ai aj arbitrary.

So, given any 2 symbols there exists a permutation that moves ai to aj right. So, there exists with the property that when sigma x1 ai what you have is aj this property and therefore, as you can say as you can see number of orbits is just 1 because any point can move to any other point by action of Sn. Few more examples of groups I am going to mention we shall lead those groups, in the lecture very simple group it is 2 Z by Z mod 2Z or some times by Z2 what is Z2.

It is a grouped with 2 elements and there are 2 elements one of them as to be identity other has to be non trivial identity. So, I can say 0 1 or I can say identity and some other elements say a and what is the property the property of the symbol a is one together with the group operation I am it by a star 1 is 0 so star is this is group law.

In general, we have Zn and this is called cyclic group of order n. What is Zn? Zn consists of here, I will take about multiplicative notation 1 element which is identity the neutral element and then some element a and then cake operation of a with itself and once more. And you keep doing it until you do it this is n minus 1 times so the cyclic group they are n elements. So, for the simplicity invitation a star a star a we just denote it by a to the power 3 this has notation a to the power n minus 1.

So, cyclic groups are quite simple groups and there are some interesting actions of cyclic groups. In fact, a group which is as easy as this can be efficiently used in checking what I call yesterday? Parity it can be used to see parity please understand that may be in coming lectures. So, with the parity we come back to Sn there are 2 types of elements, there are 2 types of permutations in Sn they are. So, called even permutations and then they are so, called odd permutations.

The parity of even permutations is defined to be 0 parity of odd permutations is defined to be 1 so what is even and odd parity?

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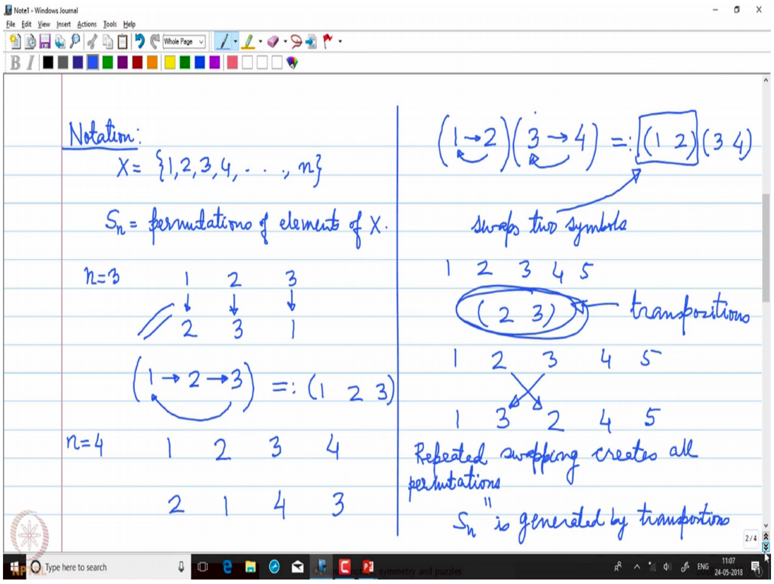

So, for that I will just revise some notation and here I would call those symbols which are being permuted by the group Sn I just call them simply 1 2 3 4 and so on. This is my X and my Sn is permutations of elements of X. So, how to express permutations? So for that we are going to have a notation for simplicity just consider the case of say ns by 2 3.

So, this is the original position of these 3 symbols and suppose, after permutation you have something like 2 3 1 so, what it what is happening here? 1 is going to 2 2 is going to 3 3 is going to 1. So, I express it like this 1 is going to 2, 2 is going to 3 and 3 is going to back to 1 it is going to back 1 and that is all there is no other symbol and it is 3. So, there is no forth symbol. So, this is how we denote this permutation. So, this is a shorter mutation and yet another notation for this is simply 1 2 3 and we read it like 1 goes to 2 2 goes to 3 and 3 comes back to 1 let us take few other ones let us take n is equal to 4.

So, let us see suppose, these are original position and suppose 2 comes here, 1 goes there, 4 comes here, 3 goes there; that means, after permutation at the first seat 2 is sitting 2nd seat one is sitting 3rd seat, 4 is sitting and 4th seat 3 is sitting. So, how do we express this so, notation for this is going to be what happens to 1? If 1 goes to 2 what about 2? 2 comes back to 1 and when we have complete cycle, we just close this bracket and then because there are another symbol.

So, we can start with 4 we can start with 3. So, I have 3 3 goes to 4 and 4 comes back to 3. It is a complete cycle and for the case, for the sake of simplicity, I just write it as 1 2 3

4 this is notation. So, simplest types of permutations are these are permutations is simply say 1 2.

That means, 2 symbols are being swapped. So, I have 1 2 3 4 5 see for example, 5 symbols and there I can consider 2 3 what is 2 3? 2 3 is the 1 which takes 1 2 3 4 5 is there, which takes 3 here, 2 here, it just swaps right swaps like this and everything else is intact. So, these special symbols these special permutations which just swap 2 positions are called transpositions so. In fact, just by allowing 2 symbols to swap repeatedly.

So, interesting fact that we can construct all applications this keep swapping keep swapping and that is how you construct all possible permutations? So, so repeated swapping creates all permutations. And in other words, the way we sake in the language of group theory Sn is generated by transpositions. So, for one transposition for example, for this how may swapping have an appoint only one right only ones is swapped.

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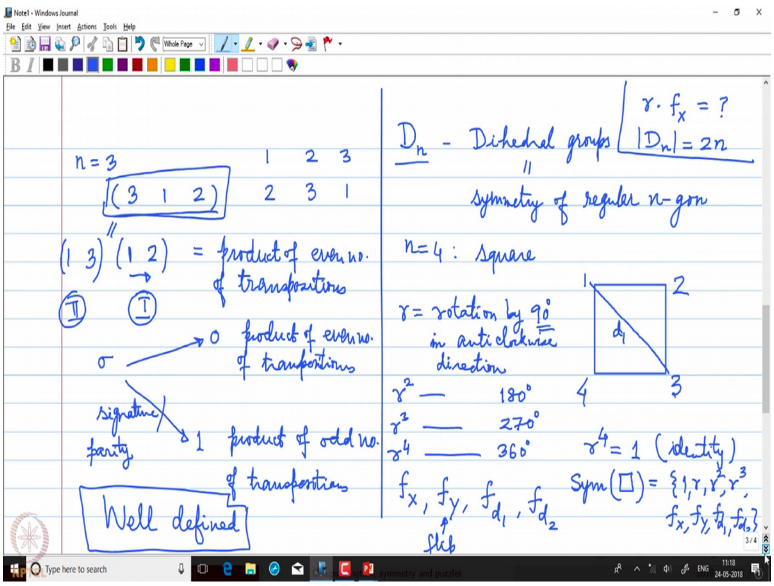

So, if take an element for example, I am simply taking n is equal to into say 3 and I am considering the element say 3 1 2 what is that? 3 is going to 1 3 is going to 1 1 is going to 2 and 2 is going to 3.

So, I am taking this permutation 3 1 2, I can express it as product of 2 transpositions.

So, I start with 1 what is happening to 1? 1 is going to 2 I just close it and then I write it like 1 3. So, 1 goes to 2 and here nothing happens to 2 so eventually 1 goes to 2. What about 2? 2 goes to 1 and 1 goes to 3. So, 2 is going to 3 what is happening to 3? Nothing is happening here and here 3 is going to 1. So, this and this the same thing first to perform this permutation and then you perform this permutation. So, if you perform this permutation followed by this permutation you will have the same effect.

So, 3 1 2 can be obtained by this product 1 3 and 1 2 how many transpositions are there 2. So, these are this is product of even number of transpositions. So, permutation sigma is sign the parity 0 if it is product of even number of transpositions and it is a sign parity 1 if it is product of odd number of transpositions. One important thing is, it needs proof and I am not proving it that this assignment is well defined. What is the means of that meaning of well defined? Is that, if you have a permutation and suppose, there are 2 different ways of expressing it as product of transpositions.

And if one of the ways the number of permutations which are require the number of transpositions which will be required is again even. So, is the same thing, but that could differ ah. So, one this even number could be 4 that even number could be 6 that is quite possible. So, this is well defined and this map is called signature.

Signature map for this lecture series I will be more often using the word parity. So, parity is the word to express how many transpositions are required to express a permutation as product. Let see few more examples of groups and we will lead those groups in coming lectures. So, this lecture is full of examples.

Let me tell you what Dn these are dihedral groups what dihedral groups are? Yesterday, we had seen symmetry of rectangle. So, today we are going to see symmetry of regular n gone. So, for n is equal to 4 the regular angle n gone is square. So, I simply take square so, you take this square.

So, I just have some notation say 1 2 3 4, I can have varies symmetries of these square I can rotate it 90 degree, I can rotate it by 180 degree by 270 degree. I can also flip it right so let me denote r to be rotation by 90-degree in. Let us say, anti-clockwise direction and then with that I keep the notation r square to perform this thing rotation 90 degree in anti-clockwise direction twice.

So, that is rotation by 180-degree r cube therefore, it rotation by 270 degrees what about r to the power 4? That is rotation by 3 6 degrees back to the same position. And for that reason, we see that r to the power 4 is identity. So, I this one represents identity doing nothing the original position is back. What else can you do with this? You can actually flip it about x axis yeah, it will again remain a symmetry, you can flip it about y axis that will remain the symmetry, you can flip it about diagonal say d 1 say. So, this diagonal is d 1 and other diagonal d 2.

So, you can do all these operations so certainly the symmetry group of square is going to have all these elements 1 r r square r cube fx f by f d 1 f d 2 f is for flip. In fact, as I said yesterday composition of 2 symmetries is again a symmetry. So, it would be a going to exercise for you to think what is say r composed with say fx you first a flip it about x axis and then you rotate it by 90 degree the anti-clockwise direction what is it? Is it a new element which has not been explored here? No.

So, interesting thing is if you try to compose all these things you realize this, this set is self-contained with respect to that composition. And therefore, this is the group of symmetries of regular for gun square which has 8 elements. In general, for n gun what we have? Is dihedral group, this name is dihedral group and number of elements in the dihedral group of order n of the, of group of symmetries of regular n gone. It is dihedral group is if the order is the number of elements is 2 1, this has some contrast with the symmetries of rectangle and I will just tell you what that is.

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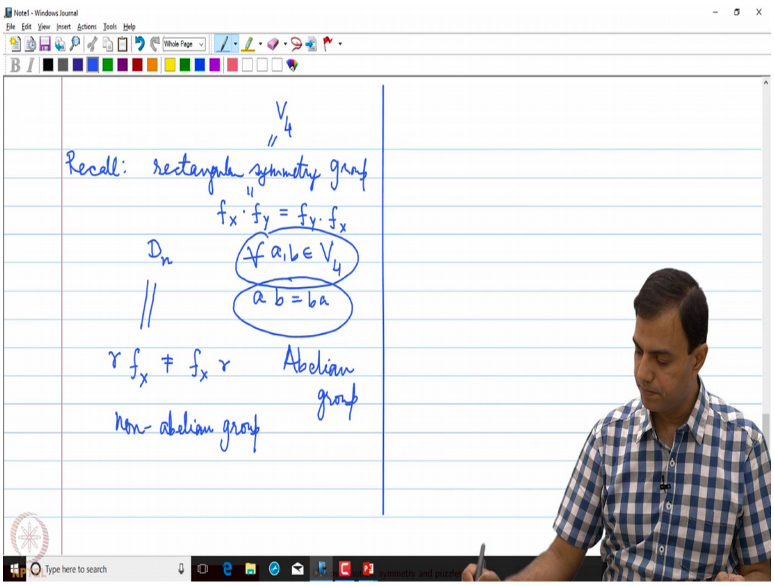

So, you recall rectangular symmetry group and here you have dihedral group this group had the property remember fx fy was same as fy fx. So, if you have a rectangular, you flip it by x axis and then flip it by y axis. It same as first to do flipping about y axis and then you will do flipping by flipping about x axis is the same thing. So in fact, rectangular symmetry group let me denote it by is actually notation. So, I just skip to that notation that notation is called V4, it is called clients 4 group V4 notation. So, for every element in the group of symmetries of a rectangle we have let a b same as b a.

But if you take D n and try experiment in it we take r and then you take say fx is actually not same as fx r. So, if you first rotate by 90 degree and then flip about x axis it is not same as first flipping about x axis and then rotating about 90 degree. So, when you have this property for a group for every element for every pair the order in which you are having the operation is in material. Then, this is called Abelianess, this is called Abelian group. While, Dn is an example of what is called non Abelian group.

So, Abelianess is an important properties of groups let me construct new groups out of even groups.

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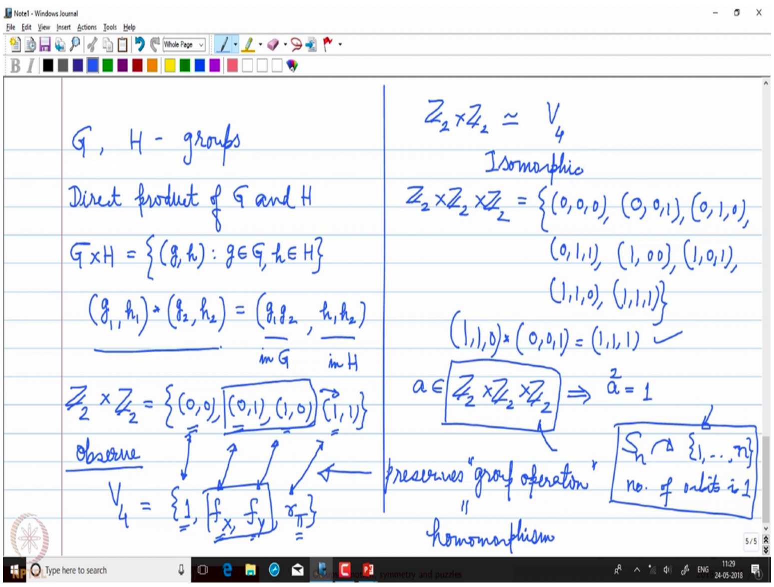

Suppose G is a group, x is the groups so 2 groups are given, you can take what is called direct product of G and H. What is that notation is G cross H? As I said, it is just a collection G coma H just like what we have in cartesian product just like that G comma S. So, G is in G and H is in H cartesian product and how do we take the group operation?

How do we multiply 2 elements from this? This is one element G1 H1 and other element is say G2 H2. You simply take the first coordinate to be G1 G2 and where is it happening? Where having this operation in G and then you have 2nd coordinate as H1 H2 and where is it happening? It is happening in H?

So, you can simply construct a new group out of 2 given groups. So, for example, if you have Z2, you remember Z2 it has just 2 elements. You take the cartesian product with Z2, what are going? What are you going to have? You are going to have, we are going to have 0 0 0 11 0 and 1 1 and you. Of course, know how to add them so here, if you add 0 1 with 0 1 answer is 0 0 and 1 0 with 1 0 answer is again 0 0. So, here you should observe that what I had in clients 4 group, you remember there was a neutral element there was 1 symbol fx. There was another symbol fy they have meanings to this symbols and then r pi.

So, as you can see 1 corresponds to 0 0 fx corresponds to say 0 1 fy corresponds to 1 0 and r pi corresponds to 1 1 [noise.] In what sense do they correspond this is identity element? Here, this has the property that, it compose to the itself is identity. This also compose with itself is identity same is to here same it to here like this moreover when you compose f x and f y you get r pi you remember.

Similarly, when you compose 0 1 and 1 0 in this fashion, in the sense of direct product, you actually get this. In fact, all the relations which are there all the group operation relation which are there are preserved when you make this kind of a assignment. So, this correspondence preserves group operations any map that preserves group operation is called homomorphism. And then when homomorphism, which is also invertible is called isomorphism.

So, therefore, Z2 cross Z2 and V4 they are isomorphia a very interesting example of a group that we are going to use to resolve one problem.

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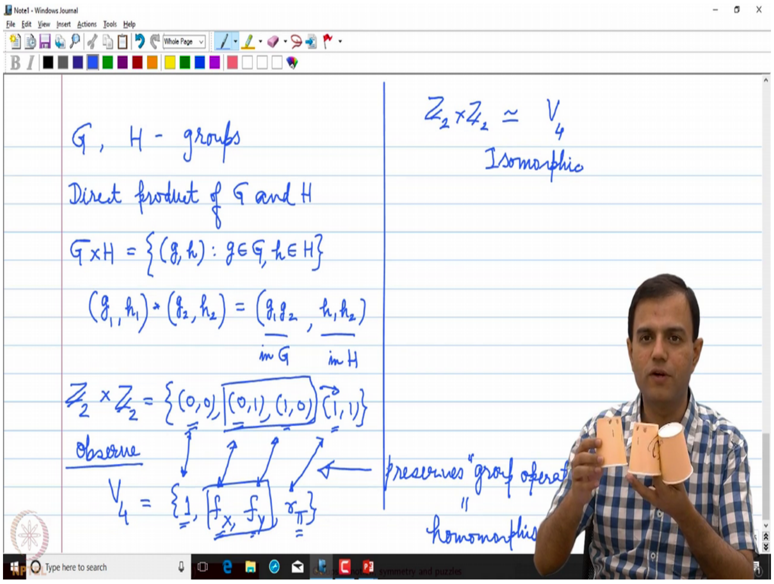

If you call this I had shown you last time, the puzzle with the problem with the glasses, inverted glasses. How to change their orientation from all inverted to all a pride? That problem so that problem actually, we are going to use a group and then introduce you to that group now that group is actually Z2 cross Z2 cross Z2. What is that Z2 is 0 comma 1. So, here I have 0 0 0 so I am just taking once more, direct product and settling the same fashion as we do it for Cartesian product 0 0 0; so, 0 0 1 0 1 0 0 1 1 and then 100 101 1 1 0 and 1 1 1 right.

So, these are 8 elements and you have the group operation exactly in the same fashion as here. So, for example, if I add 1 1 0 add meaning to the group operation. So, 1 1 0 with 0 0 1; what I get is so 1 1 plus 0 is 1 1 the 0 is kth 1 in 0 plus is 1. So, this is a group operation exactly what I had defined here in this same fashion. So, this is group of 8 elements all these elements have the property. So, if let me just say if a belongs to Z2 cross Z2 cross Z2 then order of a the a square is 1.

So, order of a is 2 except for the identity element where s square is 1, but the order is just 1. So, this group is what we are going to be use in the permutation puzzle. So, I hope you could understand this group and I hope you could also understand that, in Sn when action is natural action, action is from 1 to 3 n the natural action of Sn then number of orbits is. And this is in some sense answering the question that I had raised in previous lecture and this is going to be used in the next lecture when I try to answer problem of making all 3 cups up right. So, keep watching see you.