Groups: Motion, Symmetry & Puzzles

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Groups as they occur naturally
Lecture – 02
Groups acting on a set / an object

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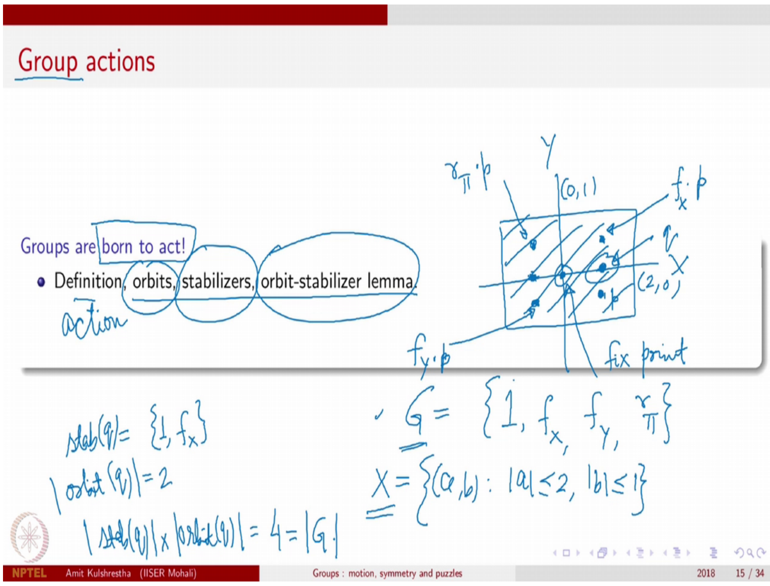

So, the previous lecture we have seen examples of groups, we have seen lots of situations where groups arise naturally and we also created we also synthesized the definition of groups after all the discussion. And I said last time that groups are born to act; so, group action is something which is very important concept all throughout all throughout the cases which are there in this course; I said symmetry, puzzles and motions. All these 3 things actions are something which are at the center, they are the most important that is the most important concept.

So, I am going to discuss what group action is and related object related concepts or which stabilizer lemma orbits and stabilizers. And you remember the example that I had last time, that is something that we are going to have the back of our mind that that was a rectangle and we could perform certain symmetry operations on it doing the thing, flipping about X axis flipping about y axis and rotating it by angle of pi right. And this we said forms a group; so, that is an example that we are going to keep in the back of our mind ok.

So, let me take certain point here let me call this point say p. So, group we understand, it is a collection of symmetries of this rectangle and I consider a set what is the set? The set consists of all the shaded region along with the boundary. So, suppose this is a 2 comma 0 and this is 0 comma 1; suppose that is a rectangle. Then I consider all those points a comma b for which mod of a is less than equal to 2 and mod of b is less than equal to 1 that is this rectangle right the shaded region along with the boundary let is X.

Now, let me take say f X what happens to p when I apply f x, that is when I when I flip about X axis what happens to p? p goes here ok. So, this is what p after the action of after the action of f X. So, through this example only I am going to define action and if I apply f X the other action the action of f y, then it will be somewhere here and that is what I call f y dot p and this point will be here somewhere r pi dot p.

So, under the influence of this group, that is through the group action p moves from this position to this position, this position and this pose these four positions it can moved ok. So, that is the basic thing; so formally I am going to define the action now in the action there is a group and then, there is a set. I have already given you a concrete example of action through the rectangle and this group case ok.

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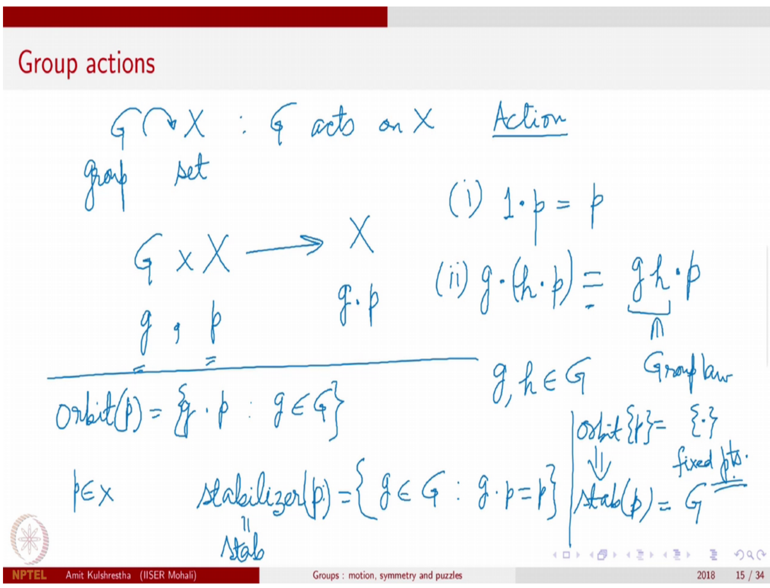

So, what is a group action? I have a group we call that group was collection of objects along with second operation, which followed certain axioms that I discussed in the last class.

So, here is a group and here is a set. So, I have group, I have a set and I say that G acts on X when do I say this? When there is an action and what is an action. So, as was happening earlier; I pick a point in the set and I pick an element in the group. And under the influence of this element of the group the point p moves somewhere within X and this is what I call g dot p that just a notation I call g dot p.

So, I have I pick a pick some element from the group pick an element from set and it goes to some other element of the set. There should be certain axioms which have to be satisfied before we call it an action, identity element of the group is not allowed to move any point that is first axiom action of identity on p is just the axiom. So, these are action axioms for the action 1 dot p is p; second is trusting one I take the point p and I take 2 elements g comma h g and h in group.

So, first with the help of h I try to move p that is I am considering the image of h comma p under this map. So, h dot p and then I am trying to move whatever is the resultant I am trying to move it by action of g. What I could have done is that I could have tried to move p, by the action of g h; whether I g h g h is an element of g this is the group law as a group law right. So, for this map g cross X to X to be called an action, I want this to be equal I want that action of h on g followed by action of g on h p. So, that that action is just same as the action of g h on t. So, that is action and as you can see this is precisely what happens in the example of rectangle that ahead, when on the rectangle the group of 4 element acts and back here now there are other concepts orbits and stabilizers.

So, let us pick this particular point orbit meaning, under the influence of the group where all can this point p go. As we can see here if point is somewhere like this it can go here, can go here, it can go here after the action of this is f x, after the action of f y, and this is after the action of our pi. So, it can move across these four points what if the point p were here?

On the X axis it says if that were the case, then under the action of f y it would go to this point, but under the action of f X it will remain here and yet the action of r pi it will again go here. So, a point here is going to have 2 elements in its orbit ok. So, what is orbit? I will just write here, orbit of a point p; p is in X what is orbit? Orbit is start with p and where all you can p go under the influence of the group, g dot p such that g is in g that is orbit.

So, as you have seen this point has four elements in orbit well this point has only 2 elements. Now orbit and if I take this origin then orbit is the origin because whatever thing I do 1 f x, f pi r, r pi this point is not going to move there is only one element in the orbit. So, that is orbit, this is another concept which is that of stabilizer.

So, stabilizer is stabilizer for a point p is the connection of all those points of the group, that is all those points of the group g in g; such that g does not move g does not move p that is g dot p is simply; all those points which all those points of the group all those elements of the group, which fail to act on p in the non-trivial fashion that is called stabilizer. So, what is orbit? Larger the orbit meaning the point is moving much the point is having much more space in the set X, but if orbit is small then point is moving slowly right you see this point, this is having a smaller orbit its moving only here and here, but if point is this then it is having four elements now orbit and step.

So, if orbit is small as the intuition says stabilizer will be bigger. So, smaller the orbit bigger the stabilizer. For example, you consider points which have singled an orbit that is orbit is just one; no point in group is moving that that. So, no point of the group is moving a given point of the set that means, stabilizer is everything right. So, if orbit of p is singleton then stabilizer of p. So, I will also call it just stab; stabilizer of p is everything right and such points are called fixed points they are just fixed throughout the action no element of group is able to move those points as you can see in this example, this is the fixed point this is the fixed point in action ok.

So, here in this example there is only one fixed point and for this fixed point all four elements constitute stabilizer. So, this some kind of conservation law right orbit and stabilizer. Orbit is bigger stabilizer is smaller, orbit is smaller stabilizer is bigger. So, what we have therefore, is orbit stabilizer lemma. So, its true for finite groups.

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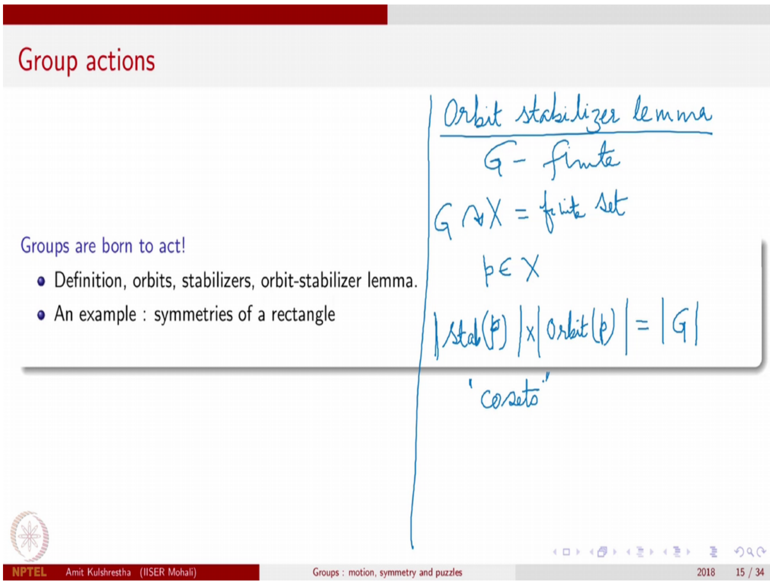

So, I will explain what it is. So, I will just mention it here orbits stabilizer lemma what does it say? It says that take a group which is finite and you take a set X such that g acts on that set and see X is also finite set, and then you pick any element in X. You can consider with the stabilizer of this point this is set from considered the size of this; this is a this modules size of the stabilizer, how many points are there in the stabilizer and then you consider orbit of p and then you consider size of the orbit of p.

These are 2 numbers multiply these 2 numbers, multiply the size of the stabilizer with the size of the orbit and what you have is actually size of the group. So, that is called orbit stabilizer lemma and lemma, this is some kind of conservation law. Larger the orbit smaller the stabilizer and why should it be true that involves some quotient groups or rather taking cossets, and I am not going to say much in detail about it. So, this has something to do with cossets.

For example in this case if your point is this, then the orbit size is 2. So, let me call this point q; for this point we take stabilizer of q what is it? Stabilizer of q is of course, identity is always stabilizer and apart from that if you take flipping of about X axis that is also stabilizer for this point. So, stabilizer is size 2.

And what is orbit size? Orbit of q is again there are 2 points in the orbits this point and this point. So, the product; so, stabilizer of q times orbit of q is 2 into 2 4, which is precisely the size of the group just through the example we have tried to understand this thing.

So, I hope now it is clear to you what an action is, what orbit is, what stabilizers are and what is orbit stabilizer lemma ok.

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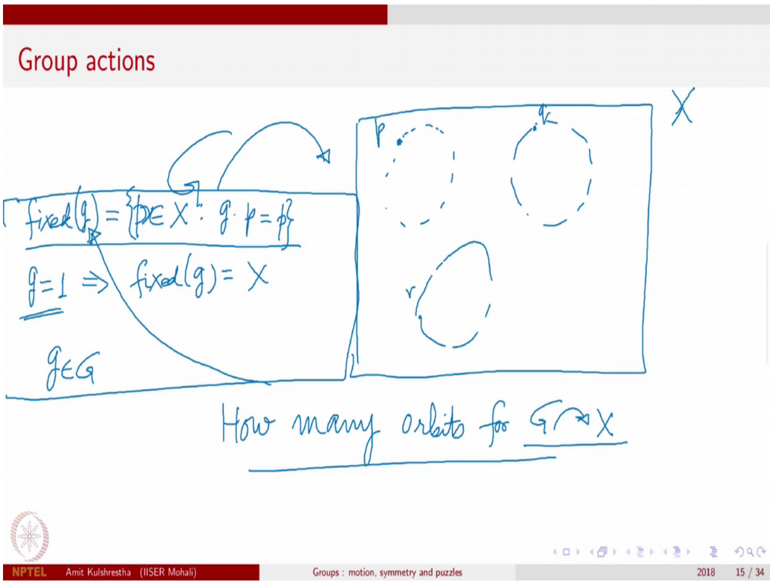

So, let me just try to give you a picture, just imagine just imagine this the some set X on which a group G is acting.

Now, I take a point p and the I can consider all those points where p can go under the influence of G. So, p can go to various points. So, I consider all those points which are there in the orbit of p. So, whatever is outside it, some other point say q I consider orbit of that. Some other point r I consider orbit of it. The question is how to count how many orbits are there and that is quite interesting question has very interesting answer and has very interesting applications. The question is how many orbits are there for a group action for a given group actions? G is a group axis, a group action which is happening is finite group how many orbits are there? And its quite interesting answer and before I could give that answer to you, I would define one more concept which is that of fixed points.

So, here I pick an element in the group, and I find out all those elements in the set which are fixed by g; that means, those points are not moved by this particular g.

So, I considered all those points p in X such that g fails to move those points. For example, if g is identity, then by definition of action itself, everything is fixed one dot p is equal to p is it that is a first axiom in the definition of growth right. So, when gs identity everything is fixed. So, fixed points are quite interesting those points which are not moved by a given element in a group.

And what I am going to say is that, question at how many orbits are there for a group action that question has connection with number of fixed points.

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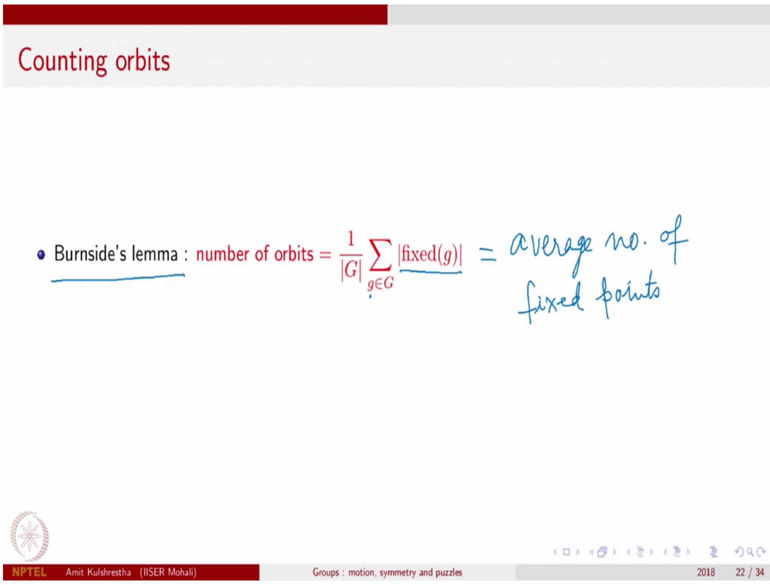

Now, what is the connection? And that is burnsides lemma what does it say it says that for each g you consider how many fixed points are there? Take the summation and divide it by the size of the group and that is actually number of fixed points in an orbit its popularly called burnsides lemma. So, what it is? This is average number of fixed points.

So, average number of fixed points equals number of orbits is what burnsides lemma says. Now how to prove such a statement? Its not very difficult and its quite useful, and I am going to tell you 2 interesting applications of burnsides lemma ok.

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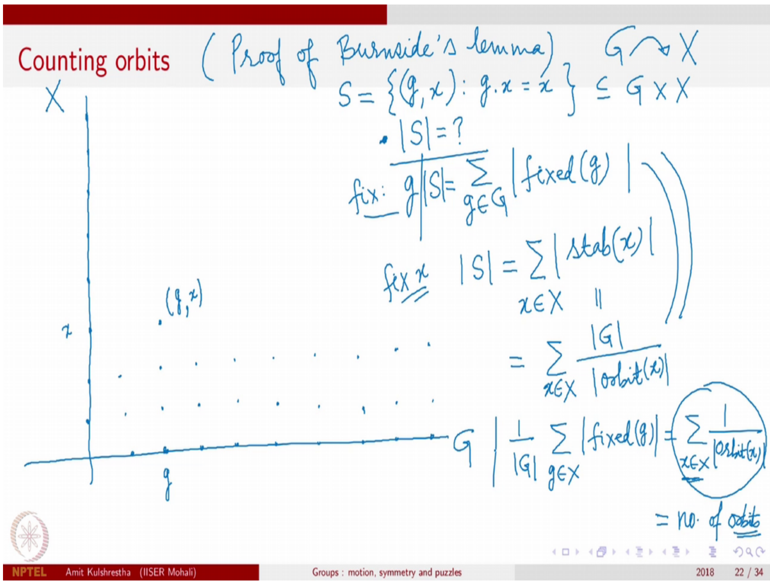

So, I am going to prove burnsides lemma. So, we have to have some (Refer Time: 24:01) let us do this I am asking all the elements of the group and all the elements of the set to be expressed here in a Cartesian format.

So, let me take this as G axis this is X axis, I have action of group on a set X g G is acting on X. So, here I am marking elements of G just a pictorial representation and here I am marking say elements of X ok.

So, here just take some point here, suppose this element is g this is X then this is say g comma X ok. So, all these things are presented in a Cartesian format and now I consider a subset of this. So, you can think of his lattice which is there. So, many points are there on this lattice right it is like lattice it is g cross X as a set is just g cross X that is some lattice.

So, in this inside this lattice i define a set s what is the set? The set is g comma X. So, this is a subset of G cross X. So, all those elements G comma X for which g dot axis x; that is X refuses to be moved by g or g is not able to move X is the same thing. So, g dot axis X and every the proof is just about size of S.

How many such ordered pairs g comma X are there on this lattice for which g dot axis x? No there are 2 ways count it and that is precisely what gives proof of both sides gamma. So, what are 2 ways? One way is I fix a g and then look for all the elements in this column for which g dot axis X and then add it with the same with the similar number for another g another g another g like this or I fix one X fix one X and then in the same row, I look for all those gs which are not able to move this.

So, I can do this calculation on size of S in 2 ways column wise and row wise 2 ways and that is what is going to give me the proof of one sides lemma. So, let us do that. So, let me first fix g and fixing g. So, a fixed g and then I ask for how many X are there which are not moved by this g that is fixed g right size of the fixed g. And then simply I am summing it over the whole group right and this is what is size of S.

So, here is another way of estimating size of s which is you fix X fixing the element X in the set and then ask like how many elements of the group are fixing that x, that is the size of the stabilizer of X and then you just take summation and the 2 numbers are equal. This summation of fixed points when g is moving all across group is same as summation of all the stabilizer, where X is moving all across set X and I just equate these 2 numbers. So, before I do that, let me just make an estimate for the summation of sizes of stabilizers.

So, summation of size of stabilizers actually equal to we have seen orbit stabilizer lemma, the same as mod of size of g divided by size of orbit of X orbit into stabilizer size; size of order, size of orbit and sizes of stabilizer is equal to size of g. So, here it is X and X. So, when I equate these 2 what do I get? I just take g on the other side and divide I get that 1 divided by size of G summation g in g and then size of fixed points is equal to summation X in X 1 divided by size of orbit of X.

Now let us pick one element here now our job is to estimate this. In fact, we are supposed to say that this is precisely the number of orbits in in an action and g action as this is the number of orbits. So, let us see how much is the contribution from one element? Contribution from the one element X in axis, 1 divided size of the orbit of X and therefore, how much is the contribution in this summation from one orbit? It is 1 divided by size of orbit of X times size of orbit of X which is 1. So, one orbit is contributing exactly once in this summation and therefore, this is precisely number of orbits, this is precisely the mode of orbits in action.

So, we arrive at one side lemma which says that number of orbits is precisely the average number of fixed points in an action what is the application of thi? So, I highlight I present 2 applications of burnsides lemma.

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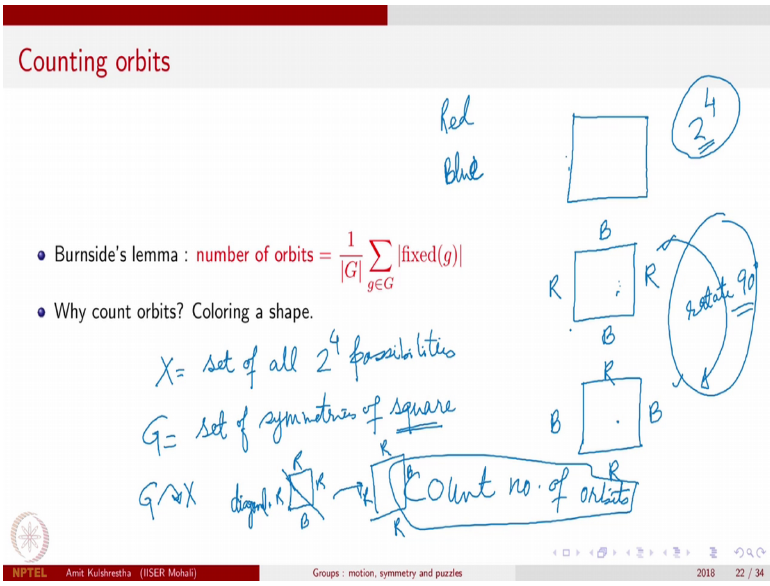

So, here is quite interesting example coloring a shape. So, let me put a shape very simple one. I will take a square and suppose there are 2 choices of colors available to me say red and blue I have 2 options. So, each edge each side of this can be colored either red or blue there is possibility where all four are red, there is a possibility where all four are blue ok.

So, question is how many different shapes like this how many different squares like this can you make. Well there is a possibility where I have this red, this red, this blue, this there is a possibility where I have this blue, this blue, this red and this red right. Probably these 2 things look at 2 these 2 things look like 2 different possibilities, but not quite if you treat these colored things as toys you know, one can actually rotate this one can rotate it by 90 degree to obtain from this shape other one right.

So, as toys they are same actually right. So, if you are supposed to count number of different toys that you can create by this kind of coloring scheme, its not its not correct to say that this is 2 to the power 4 choices right. For each edge I have 2 choices there are four edges they 2 to the power 4 toys are possible that is not quite right to say because its easy you see there are certain situations where although in this count that 2 distinct possibilities, but as toys they are same because you can obtain this from this just by rotation. So, as toys they are same.

So, what exactly should we do in order to count right number of toys? And idea is simple count number of orbits and then you should ask here is the action, because we called orbits only when there is an action. So, then we have to see what is happening, what is x, what is g ok.

So, here I can take X to be all 2 to the power four possibilities in the set of all 2 to the power 4 possibilities and where is the action? Action is this these are the things which keep the toy look same in the shape, but probably different in the coloring configuration. So, what is G? G is the set of symmetries of this object in this case this object is square. So, G is a set of symmetries of square and what is the action? Action is quite that strange one, action is you take a particular configuration take a possibility it could be say R R R B or something and take symmetries of rectangle.

So, in this case is a square, take symmetries of square, there are eight symmetries of square and that could be an exercise for you the way we proved the way we got convinced that there are four symmetries of a rectangle, in the same way we can also estimate that for a square there are 8 symmetries ok. So, suppose this symmetry is something like diagonal flipping across one of the diagonals, says say this this this diagonal then this possibility goes to what. So, when you flip about this diagonally B goes here R comes here R comes here and R comes here so.

When in there this action you count number of orbits and that is when you actually get the right number of different toys that you can create, and in the exercise said we are going to have some interesting problems and some problems which involve cube and tetrahedron. So, in the exercise said we are going to have that that is one thing why we should count orbits.

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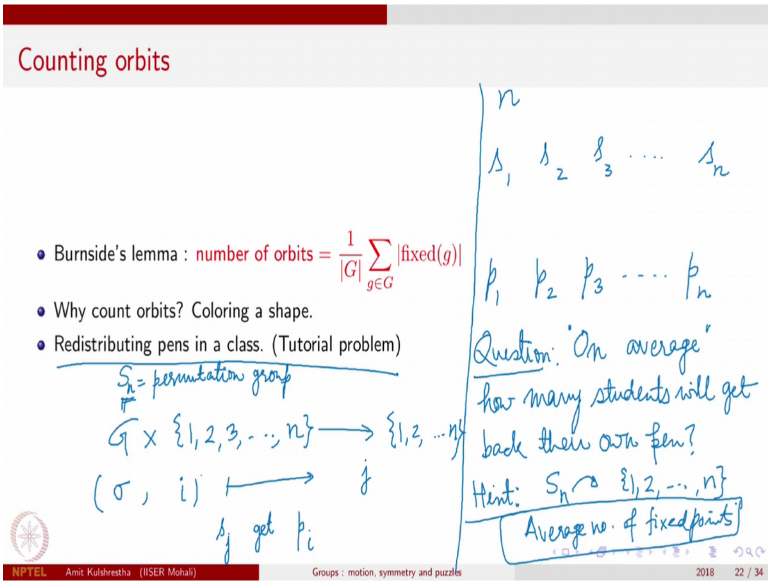

Here is another interesting problem. So, maybe during the exercise a part the assignment part I will give you this problem, but let me explain you what the problem is. If the class suppose there are n students s 1 s 2, s 3 s n there are any students and each of them is having a pen p 1, p 2, p 3, p n.

So, as students as a n students are there pens each of them is having their own pen, teacher comes and randomly distributes these pens to these students. So, teacher comes collects all this point all these pens and distributes it to students randomly. So, possibly s 1 gets the pen of p 3 maybe, s 3 gets the pen of p n maybe s n gets the pen of p 2 all this (Refer Time: 40:04) happens.

Now, question is on average and one has and one has to device the right meaning of on average. So, on average how many students will get back their own pen and for this one has to understand what is the action this is going on, once has to understand what is the group involved and what is the set which is there.

So, I will just give you some hint. So, what is the group what is the action. So, group is actually permutation group, the permutation is happening here right. So, permutation group which is the group. So, and the action is what. So, this G is S n which is permutation group what is permutation group? I have said the last lecture that you have n distinct points you permute them and once again you permute them, what you get is, again permutation of the original configuration permutation of the original objects.

So, if you collect all possibilities all n factorial possibilities of all possible permutations, it forms a group. When compose to permutations and it is again the permutation and the original configuration itself is one of the permutations, identity permutation and for each permutation there is a reverse permutation, which brings it back to the original configuration. So, set of permutations is a group and this is a group of size n factorial right there n factorial possibilities of a permuting n objects of course, we are assuming that those objects are not tied to each other they are not bound to each other ok. So, there is no restriction the permutation we are permitting them as discrete objects as independent objects ok.

So, that is permutation group and the notation for the permutation group is S n and the action which is happening in this problem is this. So, what is happening? I take an I take a permutation let me call sigma and then you pick one of the pens suppose 10 of the ith student and then you take it to j, where s j gets ten of the ith student that is a that is the action. So, hint is that of course, permitted the action is this action is S n the to the permutation group acting on the set of and the elements naturally action, and what we are supposed to count is average number of students who get back their own pen.

So, maybe we are talking about average number of fixed points at has a hint. So, probably it has something to do with average number of fixed points, I would not give you hint beyond this because this is going to be one of the exercise in the assignment. So, next time we are going to learn about applications of group actions in understanding some puzzles, applications of group law in understanding some puzzles, and the key word is going to be parity, groups can be used for parity checking.

So, I will see you again in the next lecture.

Thank you.