Groups: Motion, Symmetry & Puzzles
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More applications of Groups
Lecture – 17
Finite subgroups of SO (3)

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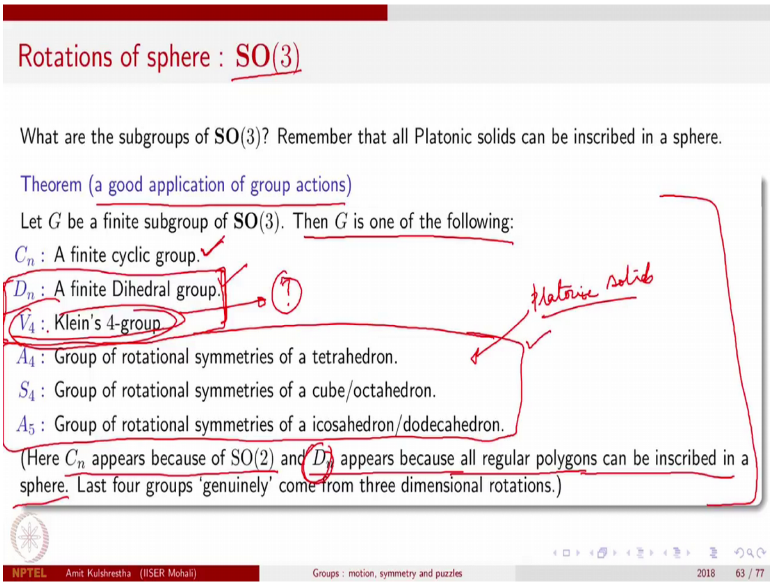

Now, I am back; and I had promised you last time that we shall discuss subgroups of SO 3.

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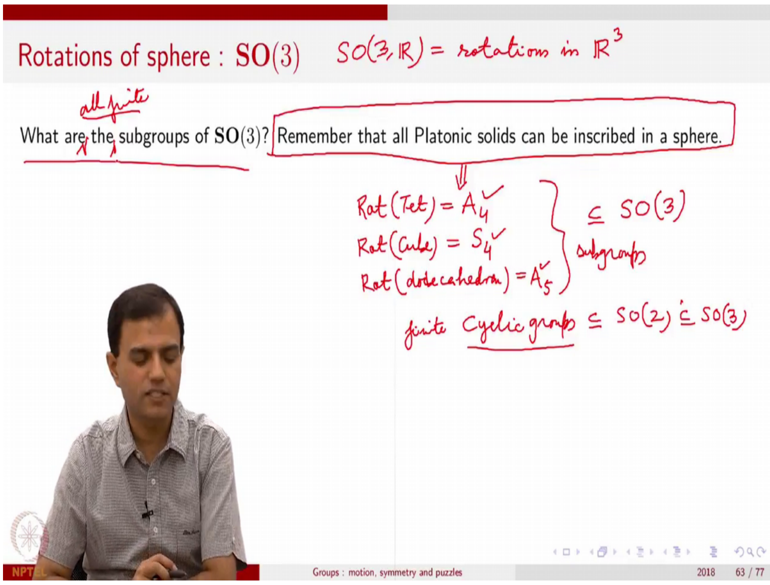

So, what are all subgroups of SO 3, when I say SO 3, I mean SO 3 R which is also group of rotations in R 3. And I had emphasized it towards end of last lecture that all platonic solids can be inscribed in this sphere, everything is perfectly symmetric. So, I can perfectly put them inside sphere; all these one of them I can put them inside this sphere, yeah.

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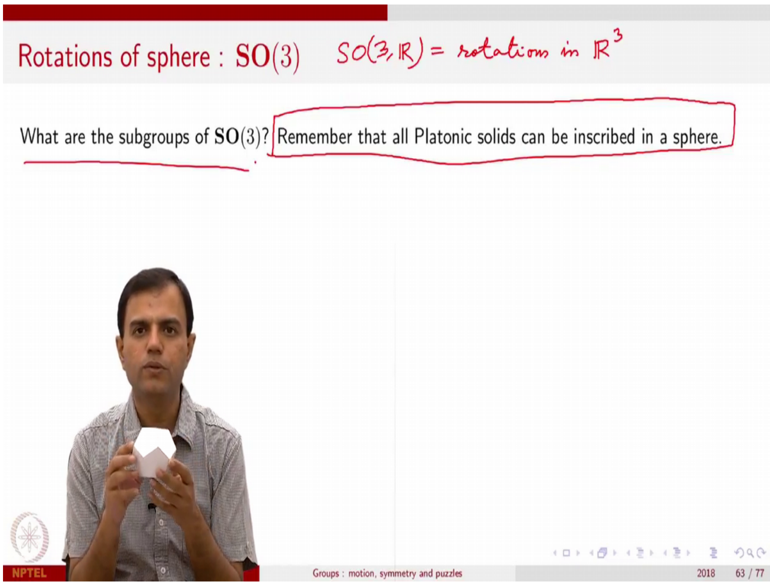

So, if I put them inside this sphere, and I do all the operations which are symmetric for these objects, I am actually rotating the sphere that is I am considering, therefore these symmetric operations of these objects as elements of SO 3, so that is easy. Therefore, conclusion is that all those elements which were groups of rotations of tetrahedron you remember what it was, it was A 4 rotations of cube that was A 5, rotations of dodecahedron or icosahedron that was sorry you rotations of cube is rotations of cube is S 4 and rotations of dodecahedron A 5.

So, from the fact from the observation that all platonic solids can be inscribed in a sphere, what we conclude is that all these are sub groups of they are sub groups of SO 3. And we had also observed last time that cyclic groups which are any ways subgroups of SO 2 are also subgroups of SO 3, say finite cyclic groups if you wish finite cyclic groups.

So, secondly, if you want to list finite subgroups of SO 3, you will have to have A 4, A 5, S 4 and all cyclic groups. But the question is what are all finite subgroups of SO 3. So, these are secondly few but what about others and that is where the fine part is going to be and we are ready with group action. So, it is a good application of group actions. Let me read out the theorem for you. If you have a finite subgroups of SO 3, then following are the options for it nothing else can be finite subgroups of SO 3. It could be finite cyclic group, finite dihedral group, Klein's 4-group or these three things which are coming from platonic solids. And this we have already observed; this also we have observed.

So, this we have to observe today, but more than that we are suppose to show that if you pick any finite group finite subgroup of SO 3, then that subgroup is one of them, there is no other, so nothing which is not there in this list is going to be a finite subgroup of SO 3, so that is quite amazing. So, as I mentioned C n appears because of SO 2, and D n should appear. And D n appears because regular polygons the two-dimensional ones all regular polygons can be inscribed in this sphere that is correct. You have this sphere, you consider the equator of this sphere that is circle and on that circle you can have all regular polygons.

So, if you have regular n 1, what you have is the dihedral group right. Because you can have just imagine as if this is circle, when I move, I get cyclic group; and when I rotate it along some axis on this sphere, what I am doing this actually I am flipping the plane of the equator. So, rotating in three dimension like this that is swiping northern south pole via rotations is actually same as flipping of the of the regular r-gon, which is there on the equator, so that is how you get the group D n. So, D n is fine. What about Klein's 4-group, how do we imagine Klein's 4-group, we are going to see after few slides. But remember the purpose is we shall start with arbitrary finite group which is subgroup of SO 3, and then we shall show that it has to be one of these which is there in the list ok.

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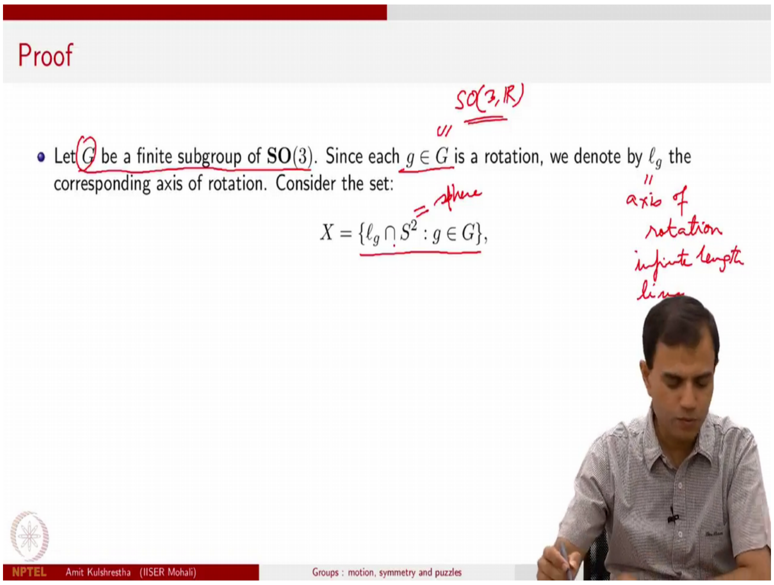

So, for this theorem, I am going to have proof. So, I have to start with arbitrary finite subgroup of SO 3. So, I start with a finite subgroup of SO 3. Now, what is there to start with when we can have what are called poles, we can have poles of rotations. So, what is that? I will just take one of these objects. So, I take this and I imagine that I have inscribed it inside a sphere. So, this these vertices will be touching this sphere right, phases will be slightly away from them only the vertices will be touching this sphere that is what is a meaning of inscribed. So, all those points where these vertices are touching, they are potential axis for symmetry right, I just hold these vertices and I just rotate.

So, points which are touching this sphere these vertices, so opposite vertices are forming potential axis for the for a symmetry operations. And these things are going to be called poles something more. I can have rotation through an axis which is the opposite sides, opposite phases right. So, if you just extend them, extend them to touch this sphere, similarly extend the bottom you touch this sphere, then the point where you touches this sphere is again going to be called a pole and similarly for your edges. So, all these things are going to be poles for this object.

So, I start with some considerations on poles. So, what I have what I have been given is just arbitrary finite subgroup of SO 3, which apriori has know geometry. So, I cannot really express the poles in terms of what I did, but I will do it in some other terms. One thing is very clear each elements of G is the rotations it is an elements of SO 3. So, this is every elements the rotation, there is an axis. Every rotation has an axis, axis of rotation. So, I denote by l g the axis of rotation and then I intersect axis of rotation with S 2, S 2 is this sphere. So, what do I get that is how I get poles. So, l g just take it to be infinite line, infinite length line. So, all the axis all the way is going to intersect with S 2. And those two points where intersecting with S 2 is those points are going to be called poles.

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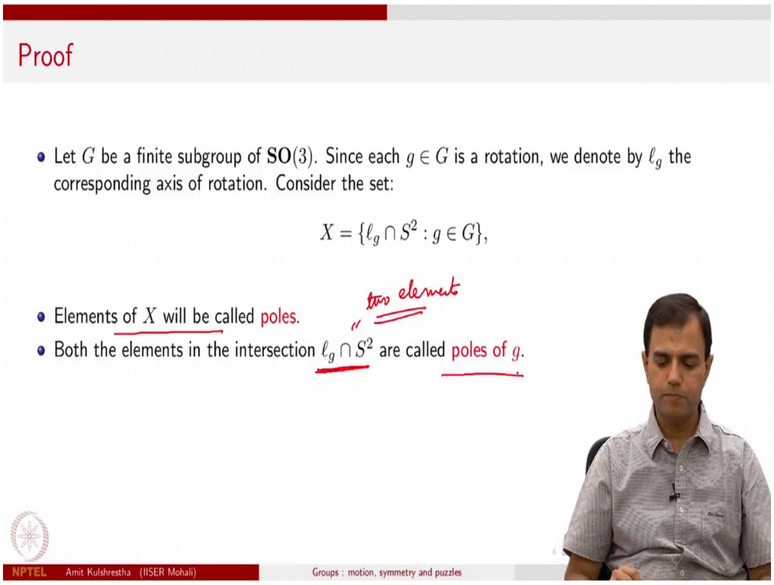

So, it is a definition elements of X you call them poles. And l g intersection S 2, these are two elements, the two elements. This intersection one just say North Pole, other you just say south poles. So, they are called poles of g, poles for that particular axis.

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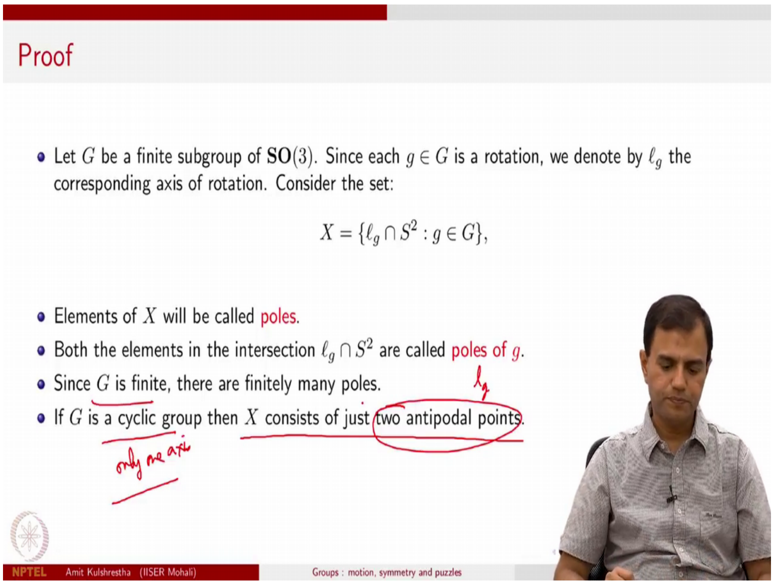

And it is clear that since G is finite for each G you have one l g. So, there are finitely many poles, number of poles is finite. And if G is cyclic group, then there are only two antipodal points in the pole right, because when you have some cyclic group again I am taking this cyclic just imagine as this is sphere. When I am having cyclic group, there is only one pole, there is only one axis. Cyclic group meaning is only one axis. Therefore, X will have only two elements just the antipodal points.

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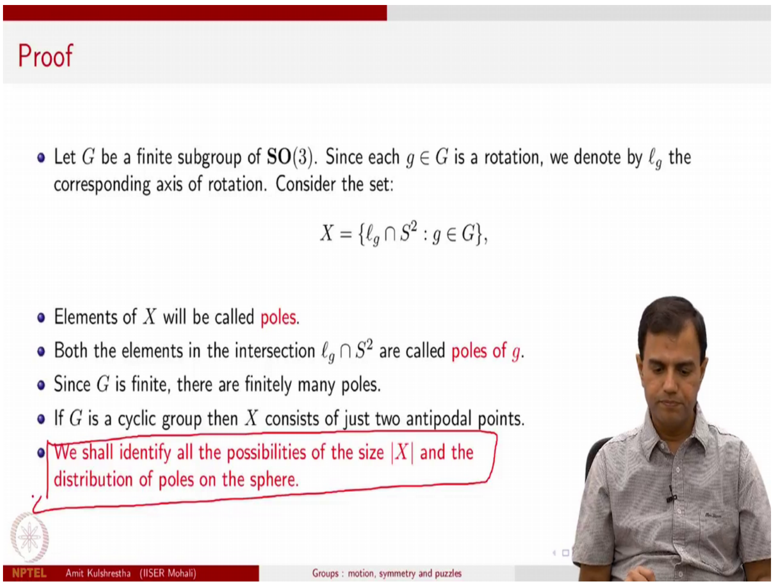

Now, what is the idea of the proof. Remember, we are proving there are finitely many possibilities for subgroup of SO 3. We have defined these poles; we shall do some juggler with those poles. We shall make some observations about how those poles are distributed all across this sphere. And depending on our observations of the distribution of poles all across the sphere, we are going to make predictions about what G could be; in fact, that distribution would completely be determining what our group is ok.

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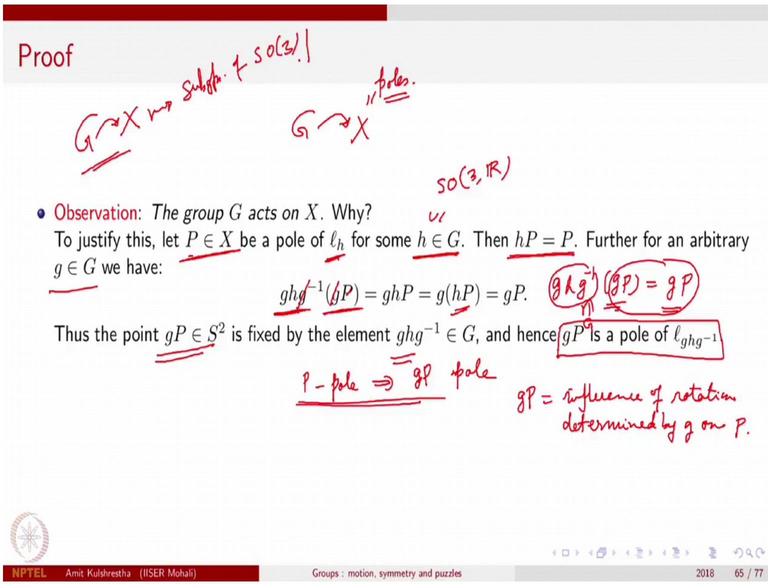

So, it is first observations. First observation is that one can give a very nice action of G on X. So, what your G, we started with it actually acts on set of poles of it, poles of G which in the set of poles is being denoted by X. It is easy to see let see. So, I take some elements which is a pole. And pole for axis for some h. So, h is eventually an element of SO 3; and for that SO 3 element and I have axis and then I have pole.

So, what is clear is that, h this rotation does not move this pole that is correct. If I am rotating like this, then north pole and south pole they do not move, so h P is P. And now if I take any arbitrary element in G in G, then I mean this calculation g h g inverse acting on g P, g P is what happens to this pole and influence of g. So, g h g inverse g P is simply g h P, which is g h P, h P is not moving P because h is precisely the rotation for is pole. So, I have this to be g P.

So, what is happening therefore g h g inverse g P is g P that means there is some element in G, which is not able to move g P, that means, this element g P is fixed by g h g inverse, and therefore it is fixed under the influence of some element some rotation because every element of G is a rotation eventually. So, therefore, g P is also a pole and for this pole, the axis is l g h g inverse, so that particular axis is having g P as a pole. So, what we have concluded if P is a pole, then g P is also a pole. What is the action, it is just simply that g as a rotation is trying to move P that is a rotation. So, g P is influence of rotation determined by P on sorry determined by g on P so that is an action G acts on poles.

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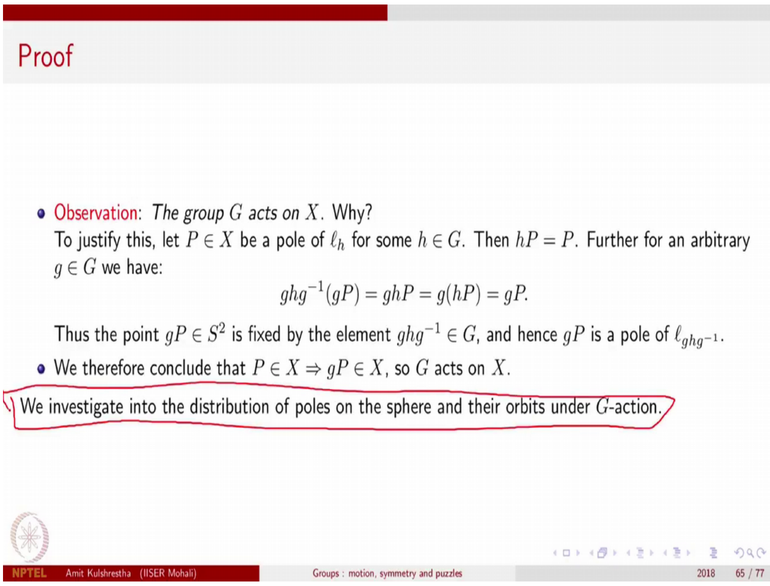

And as I said we are going to investigate distribution of poles, and what are the orbits of that action all that is what we are going to analyse.

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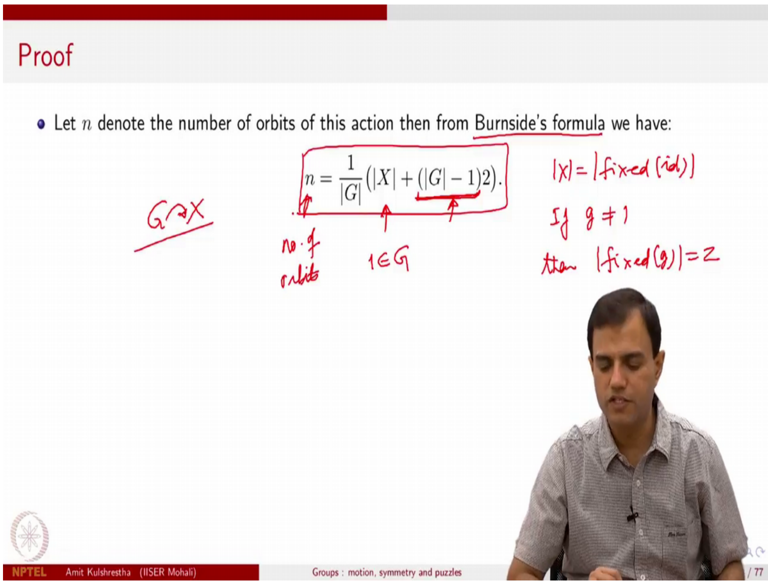

So, we start with you remember Burnside's formula in very beginning I had mentioned Burnside's formula, what it is number of orbits is equal to average number of fixed points. So, this is this number of orbit for this action, this number of orbits for the action in quotient is this. And why this is there, so for identity, identity is going to fix all the poles how many poles are there, size of the set X, X is precisely the set of poles.

And what is this rest of the mod g minus 1 non identity points they are fixing only two poles which are on the axis which are the two sides of two edges of two nodes of the axis. So, those are mod G minus 1. So, these mod x is precisely size of fixed points of identity and for non identity. So, if g is different from identity, then fixed points are going to be only two right, so that is how I get this. Again and again quite frequently I am going to use this formula.

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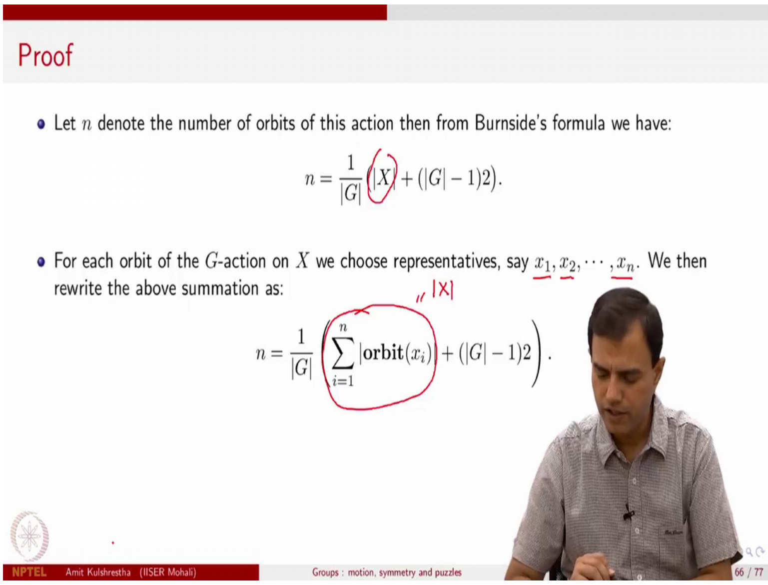

Now, I am just writing this in slightly different way. So, for each orbit, I choose representatives x 1, x 2, x n. So, from each orbit, I am picking one element, and then I rewrite the summation in this form. So, everything is fine here the only thing is in case of x I am just writing this. So, for each so I am having for each orbit, one element and that element is contributing how much equal to the size of its orbits. So, this is precisely the count of all orbits and union of all orbits is precisely your set on which this group is acting. So, you have this.

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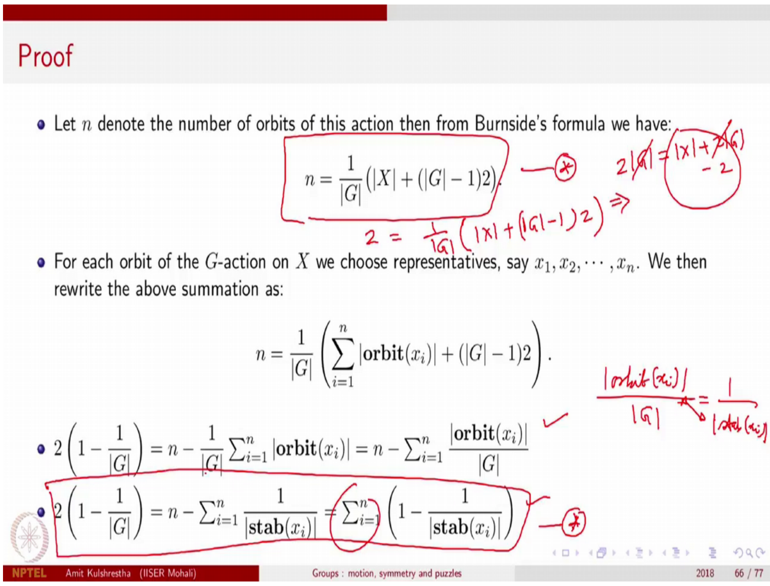

And in fact some manipulations I would do with this. So, I take this to the other side and divide by G. So, what I get are these quantities. So, this is n minus 1 by summation G summation of all the orbits. And here I I write orbit x i and take mod G inside. And you remember orbits stabilizer formula. So, orbit of an element divided by orbit of G is 1 divided by stabilizer of x i because orbit into stabilizer size, size of orbit times size of stabilizer equals size of the group. So, so what we get after all these manipulations is the 2 times 1 minus 1 divided by order of group is equal to 1 minus 1 divided by size of stabilizers and take the summation. So, this formula again I am going to use multiple times, this also I am going to use multiple times.

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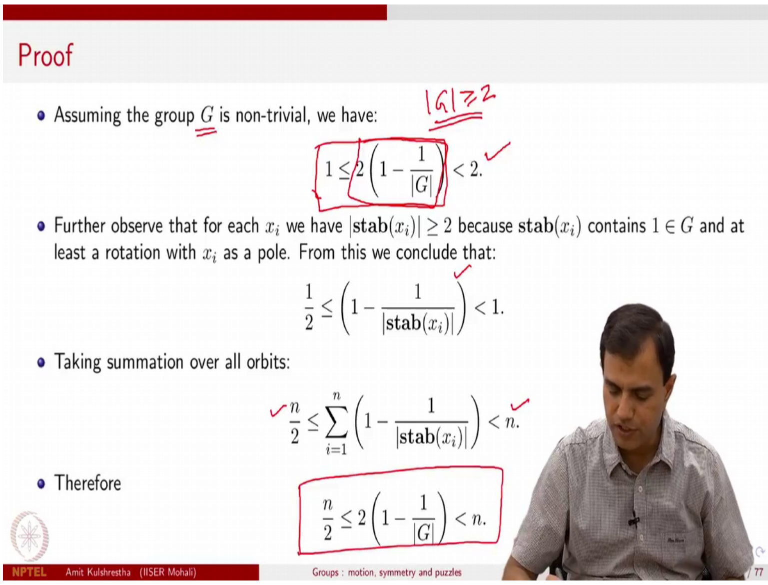

If group is non-trivial that is group size is more than better than equal to 2. So, I have this; this because group is non-trivial. So, here I have 1 by n, where n is greater than or equal to 2. So, this number is certainly more than 1. And since this is less than 1, twice of something less than 1 is less than 2. So, this is what I have. And this expression 2 times 1 minus 1 by G from this is summation of 1 minus 1 divided by size of stabilizers. So, just that much I replaced here, so 1 by 2 here, 2 here.

So, I get this expression. And now I am taking summation overall orbits, this is a constant half number of orbits is n, constant 1 number of orbits is n, and here the summation is happening right. So, what I have is n by 2 is less than equal to twice of this because this is precisely the summation is precisely twice of 1 minus 1 divided by size of G, so I get this expression.

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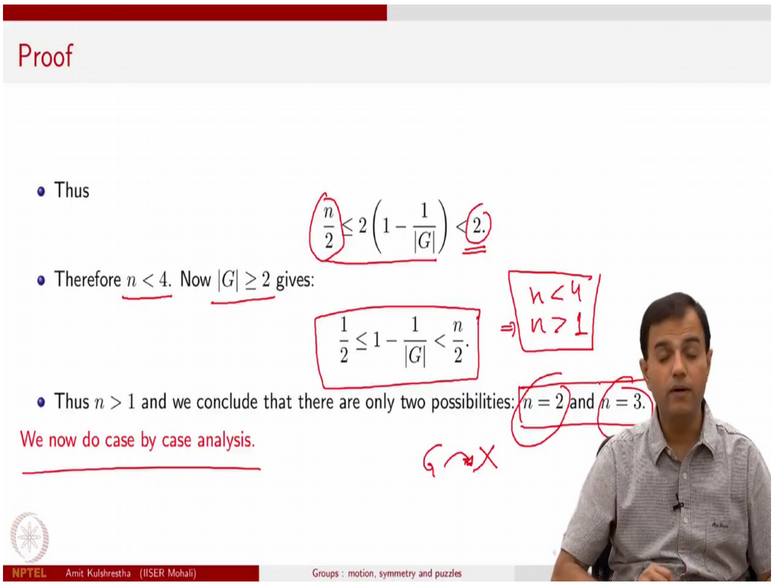

So, n by 2 is less than equal to this is what I get. And this expression is any way less than 2. So, when I combine this, this is strictly less than 2; and therefore, m which is number of orbits is strictly less than 4. And also by group as size greater than equal to 2, so when I take that into considerations. And look at this when I look at this what I get is half is less than 1 minus 1 divided by mod G less than n by 2. So, again this is going to be quite useful for me.

So, from this what I conclude is that n is greater than 1. So, n is greater than 1 and n is less than 4, that means there are only two possibilities that whenever group sub finite sub group of SO 3 acts on its poles, number of orbits is either 2 or 3 that is quite substantial information. And from this information, we are going to get all possible subgroups of SO 3. And in fact, we are going to show that they essentially come from catering solids or polygons, which is in case of SO 2. So, case by case analysis first we do for n is equal to 2 case; and then we do analysis for n is equal to 3 case.

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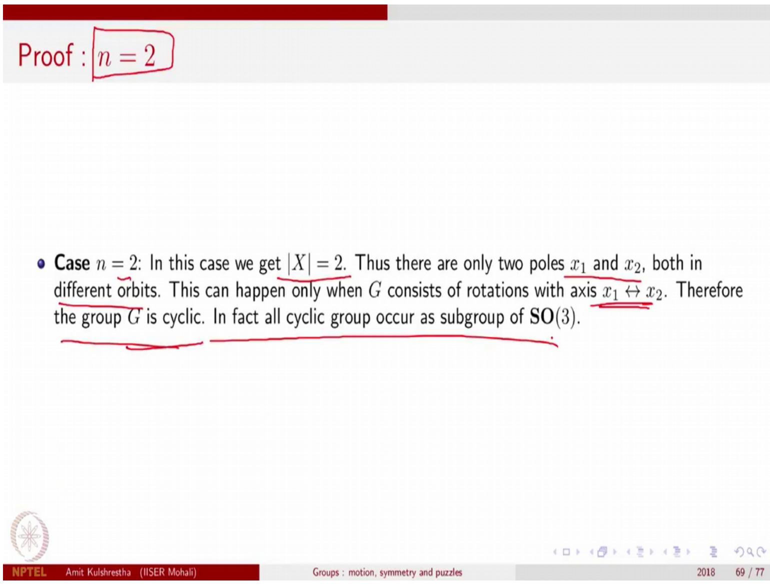

So, n is equal to 2 case. There are only two orbits. So, if n is equals to 2, then size of x is 2. Why is that? Let us go back and look at this. So, if n is 2, then 1 divided by size of G times mod X plus size of G minus 1 times 2. So, what happens here in order to get 2 here, so let me just get mod G that side. So, twice of size of G is therefore, size of X plus twice of size of G minus 2. So, this cancels out and you get size of X is 2. So, size of X is 2, if n is 2.

So, number of poles is only 2. And if there are only 2 poles they have to be antipodal. So, they have to be antipodal. And both of them are in different orbits, because the number of orbits is 2. So, I cannot move this pole to that pole right. So, what is the picture now that these are actually antipodal, I cannot move this from that and this is happening when x 1, x 2 the line joining those two poles is actually the axis. And now your group is moving and this is rigid the only possibility is your group is cyclic. So, all cyclic groups occurs as subgroup of SO 3, and n equals to 2 possibility is that one that was quite straightforward, n (Refer Time: 24:27) to be precisely cyclic groups.

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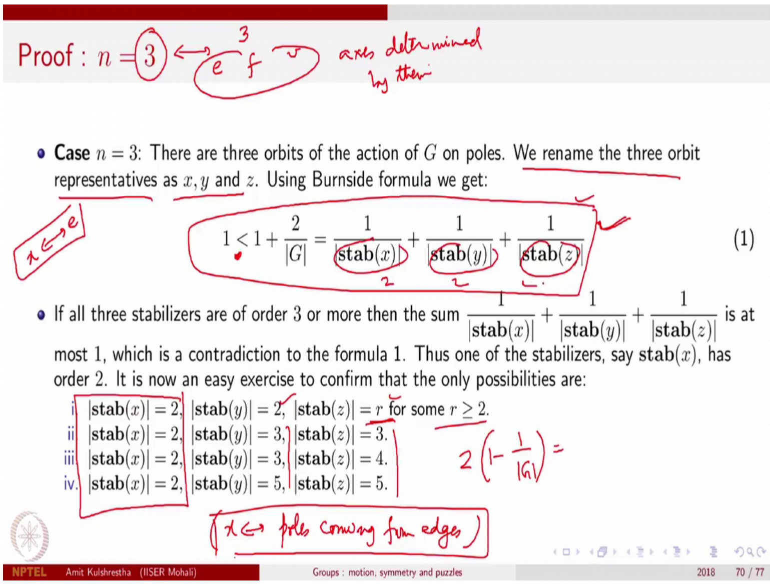

Now, n is going to 3 cases something quite something which is interesting. So, can you imagine what n is going to 3 would be a corresponding to, when you think of platonic solids like this say like this. There are three types of poles, poles which are coming from vertices, poles which are coming from edges and poles which are coming from mid points of phases. And pole which is coming from midpoint of phases cannot go to for that it cannot go to the pole which is on the midpoint of edge or a vertex that is the interpretation. So, this 3 stands for, 3 stands for edges, phases and vertices, axis determined by them, axis determined by them. And for that matter every vertex is as good as any other vertex.

As good as in what sense they are in the same orbit. Similarly, any midpoint of edge is as good as any other, because they are in same orbit by certain rotation, by certain action of the group of symmetry of this, you can obtain one from the other. So, n is 3. So, what we do we take representatives from these three orbits, and we are calling them are your x 1, x 2, x 3, x n let us simply call them x, y, z.

And one of the formulas that we had earlier, remember what formula it was, summation of stabilizers. This formula, this formula, this formula which says that twice of 1 minus 1 by this is summation of stabilizers. So, using that you conclude this thing that 1 is less than 1 plus 2 divided by G is precisely some of reciprocal of sizes of stabilizers. Now, here comes an observation. How large could these be, how large these could these be. If all of them are greater than 1 by 3, then this quantity will be less than 1, but here it is actually more than 1. So, what we are therefore force to this that one of the elements say x, one of the elements representatives of orbits has to have its stabilizer to be of order 2; otherwise we cannot have this inequality. So, in order to respect this inequality, one has to have size of stabilize of one of the orbits representatives to be equal to 2.

What does it mean, it means that edges are there. So, 2 corresponds to edges. So, x here therefore is corresponding to poles which are coming from edges. So, x corresponds to poles coming from edges, but we do not need all this these proceed I am just making size remark. Now, what are other possibilities, other possibilities are precisely these. So, I have written all possibilities. So, stabilizer 2, then next element other orbit has stabilizer size 2, and other one has says stabilize size r, for some r and similarly 3, 3, 5, 3, 4, 5 like that ok.

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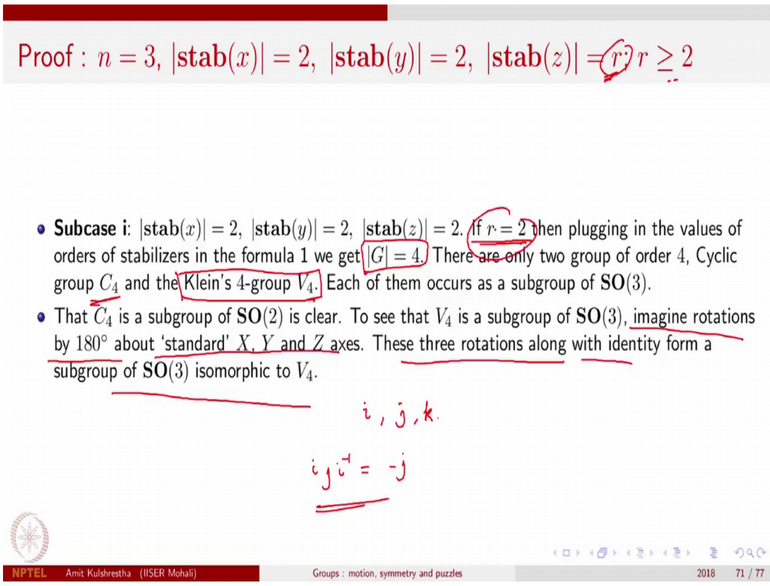

Consider this situation. When n is 3 stabilizer of axis of size 2, stabilizer of y is also size 2, and stabilizer of z is of size say r, then what happens? So, if r is equals to 2, r is this. So, stabilizer of z is of is also of size 2, then in one of the earlier formulas as if we put all these values which formula, this one. So, here everything is 2, 2, 2. So, 1 by 2 plus 1 by 2 plus 1 by 2 is 1 plus 2 by mod G. And that gives us that order of G is 4.

Now, as you know there are only 2 groups of order 4, one is cyclic group, and another one is Klein's 4-group. Cyclic group is certainly there in SO 3. What about Klein's 4-

group? Klein's 4-group is also there in SO 3. How to see that and that is quite interesting description. We think of it is sphere and you think of three axis x-axis, y-axis and z-axis, so that forms a system, that forms a orthonormal system and consider with x axis, the minus direction as well with z axis, minus direction as well and with y axis also minus directions. So, the six the you just consider imagine the three vertex three axis intersecting at the centre intersecting at the origin.

Now, what is the symmetry, what are the symmetries of this? So, you consider rotation by 180 degree for each of them. So, you imagine rotations by 180 degree about this standard x, y and z axis. So, then these rotations along with the identity, they form a subgroup of SO 3, which is isomorphic to V 4. In fact, here you can think in terms of quaternions because rotations will be in terms of i, j, k. And you can think of i, j, i inverse these kind of things you can think of and i, j, i inverse is what is minus j, since the rotation by 180 degree. So, these kind of things you can think of. So, we have obtained that V 4 is also a subgroup of SO 3. So, here r was greater than equal to 2, so this is r equals to 2 case.

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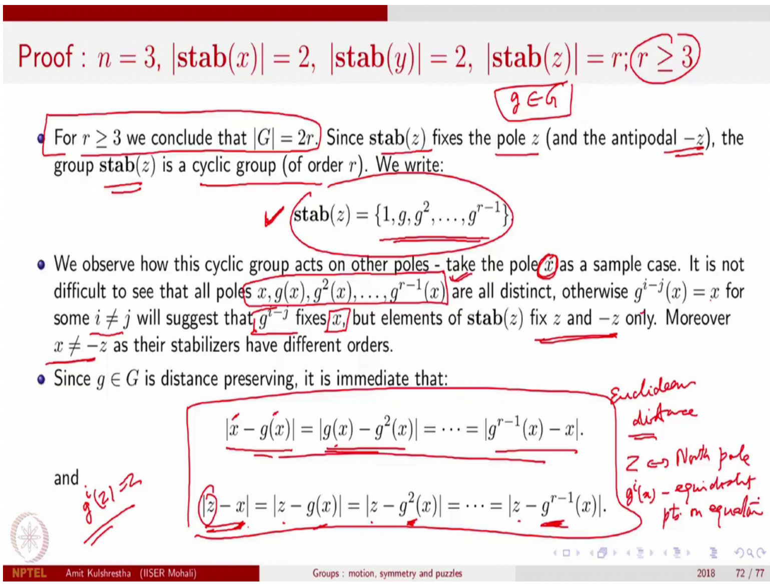

So, if r is greater than equal to 3, then what happens, so that is where we make use of the fact that G is an isometry. G meaning the elements if the element of my group, then it is an isometry that is it keeps the distance is preserved. So, first thing to observe is that if r is greater than equal to 3, then size of G is twice of r. How do we get it? Again from one of the formulas this one, I obtain that. Just put 1 by 2, 1 by 2 and 1 by r. So, you will get that G is of order 2 r.

And I pick this stabilizer. Now, this stabilizer; obviously, fixes the pole z stabilizer of the pole z of course as to fix the pole z. And of course, the antipodal pole is minus z also has to be fix by this. Therefore, this is actually a cyclic group of order r; and stabilizer can be written as 1, g, g square and so on. And now this stabilizer this cyclic group acts on other poles. So, I just take one other pole say x. So, I consider x, g x, g square x and g to the power r minus 1 x. I came that that all these are distinct; if that were not the case some smaller power of g would be fixing x in some smaller power which is different from 0. So, for some i which is different from j, I would be getting that g i to the power i i minus j fixes x, but elements of this all these powers are they are allowed to fix only z and minus z, and certainly axis not z and axis not minus z.

And therefore, these are all distinct. And now we see the distance preserving observation, so x and g x, the difference of x and g x distance on this sphere the Euclidean distance, these are all Euclidean distances. So, Euclidean distance is same as g x and g square x. So, I have put f transform this and this both by g and so on. So, like this I have. And I do the same thing with z. The point is g to the power i z is z. So, everywhere I am having z, z, z. So, this kind of picture is what I am getting.

So, what is that that means, x, g x, g square x g to the power r minus 1 x, they are equidistant from z. And among themselves also they are equidistant. So, picture is something like there is north pole something like this z and each. So, you can imagine z like north pole, and g to the power i x these are equidistant points on equator. You can think of like this.

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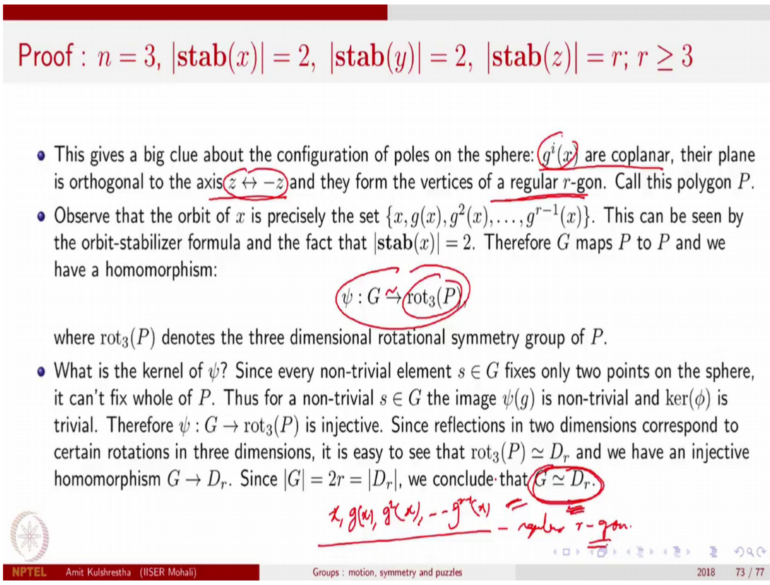

So, like this important clue about how the poles are configured g, these are coplanar as I said they are on the equator, and their plane is orthogonal to what joins north pole with south pole z with minus z. And therefore, these are forming vertices of regular r-gon. And when I observe rotations of this regular r-gon, I realize that these rotations are actually isomorphic to g. This some small calculation which is involved there, the small argument which is involved there. And using that essentially, what is happening is that G is nothing but the regular r-gon with regular r-gon which is made by which is constitute which is x, g x, g square x and so on, g to the power r minus 1 x these distinct points they make regular r-gon and that is how G becomes isomorphic to D r.

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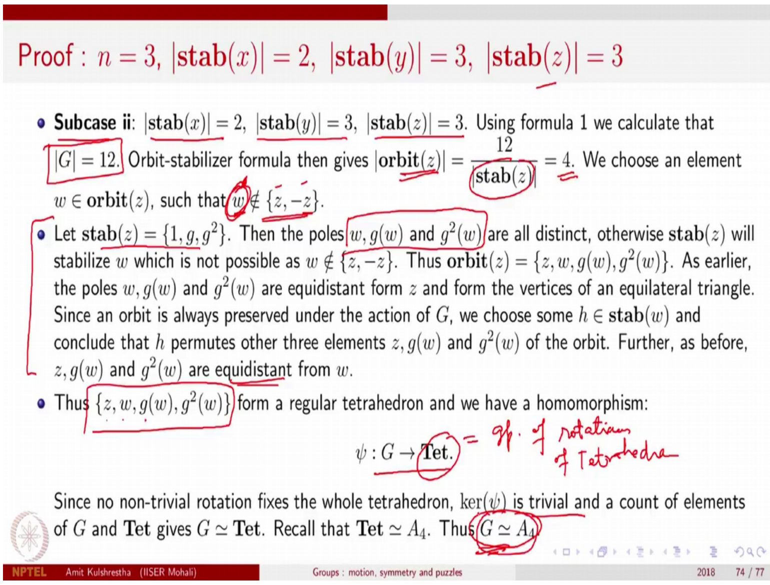

In other cases as well we have similar kind of analysis. Let us take this analysis, where I had sub case 2, in which case stabilizer was 2, this stabilizer was 3 size, this was size 3. So, again I can use one of the earlier formulas which one this formula. So, I had 2, 2, 3.

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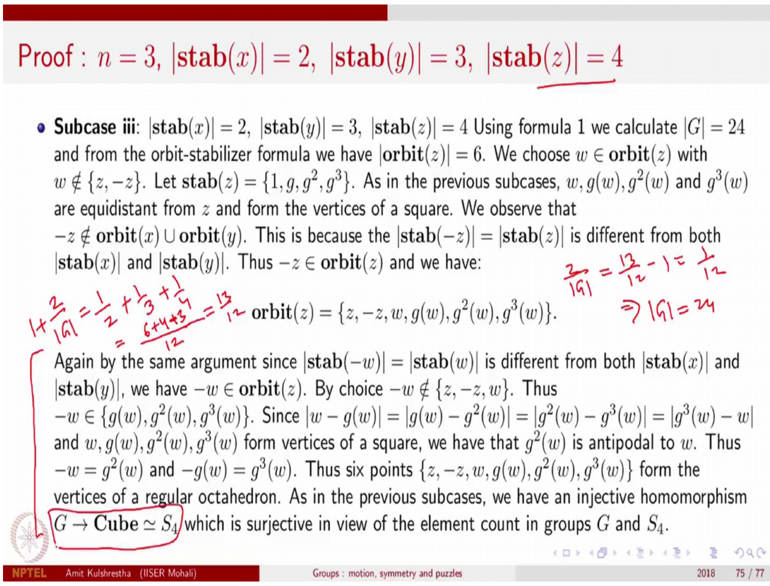

So, let see 1 plus twice of order of G is 1 divided by 2 plus 1 divided by 3 plus 1 divided by 4. So, if I do this calculation, what do I get here I get 12, and then 6 plus 4 plus 3 which is 13 by 12, and then I subtract it. So, 2 divided by this is 13 by 12 minus 1 which is so which gives me the so this is 1 divided by 12. So, size of G is 24. And similarly in the previous case size of G is 12. So, I have actually done the calculation for next case. So, in the previous case also the similar fashion size of group is 12. And because of orbits stabilizer formula, we get that stabilizer of z is having this relation orbit of z size is same as 12 divided by size of stabilizer which is so and this ratio is 4, because stabilizer z is 3. So, 12 divided by 3 is 4.

Then what do we do we pick an elements which is there in the orbit of z, and elements which is different from z and minus z. We can do that because orbit of z has four elements. So, one is z, other is minus z and two more elements. So, I can pick say w and now I do some jugglery with all this. And if this stabilizer of z is say stabilizer of z is cyclic it has three elements. So, let us call it 1, g, g square. And then I look at the fact of this size of stabilizer of z on this element w on the orbit, and conclude that the three distinct elements w, g w and g square w.

And using previous kind of analysis is not very difficult to forming to conclude similar kind of argument using the fact that g is distance preserving that z and w, g w, g square w they actually form a regular tetrahedron. So, since they form a regular tetrahedron, I can have a homomorphism from the finite group g that is given to me to the group of this is group of rotations of tetrahedron. I can have that. And again it is not very difficult for me to conclude that size of g is A 4 because you can make count of elements and all that. So, the group in this case is A 4. Again when stabilizer z is size 4, I can similar kind of analysis exactly same kind of analysis, and I can conclude that G is S 4.

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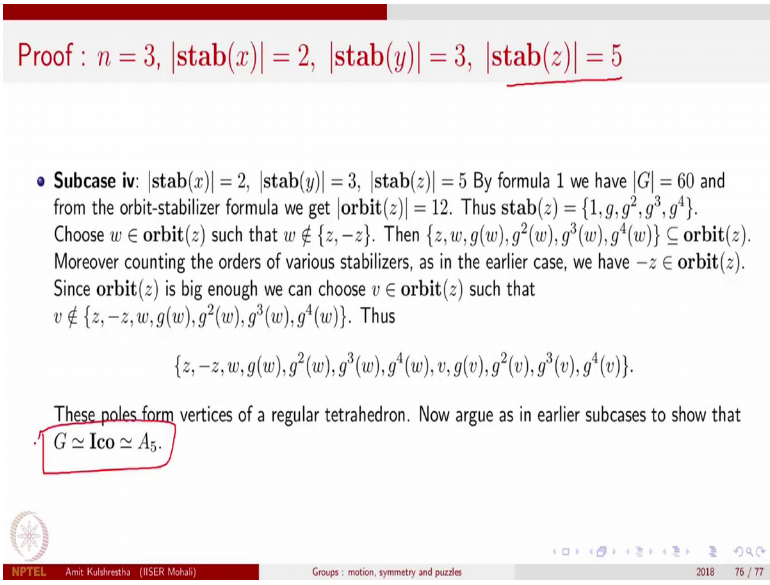

And similarly when stabilizer is of size 5, I can conclude that G is actually group of icosahdedral which is A 5, and then all the cases are finished. So, I had this case of A 5, which is finished.

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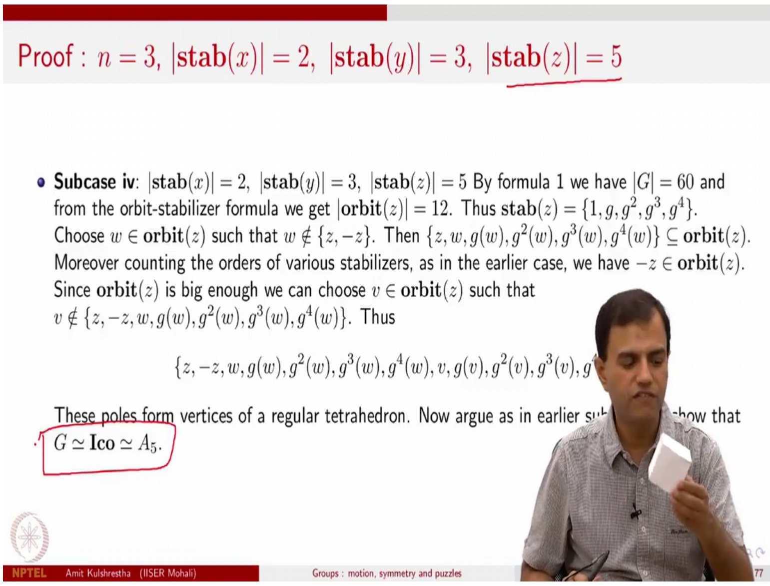

Before that I had case of cube which is finished and before that I had case of tetrahedron, which is also finished.

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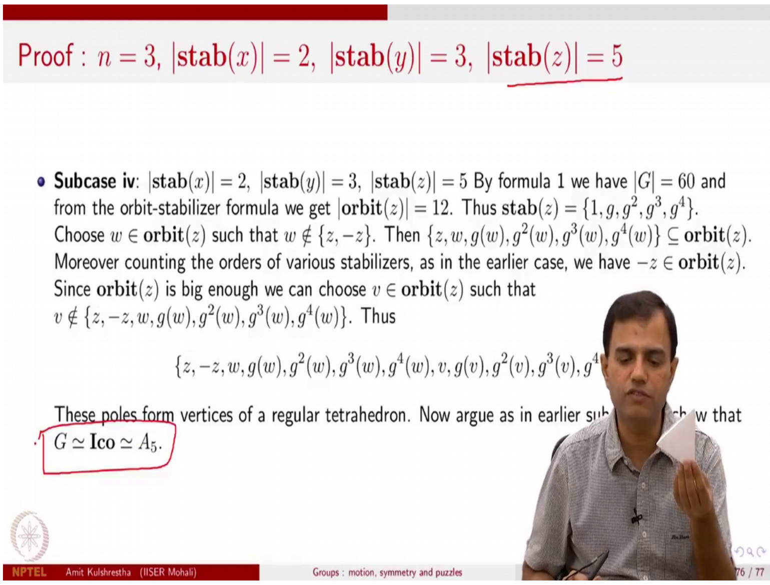

So, we are therefore, back to the original statement which is that these are all subgroups of SO 3. What goes there you carefully examine what are poles, how stabilizers of other poles are acting on those poles and that would eventually lead either to platonic solids or to a polygons, which is right there on the equator or a situation is a Klein's 4-group situation where you have the three axis x, y, z axis there are which are inscribed inside sphere, so that is how we prove that SO 3 has finitely many groups. And those groups are what I have already mentioned to you in this lecture. So, this was all about our group action and how group action is useful for understanding subgroups of SO 3.

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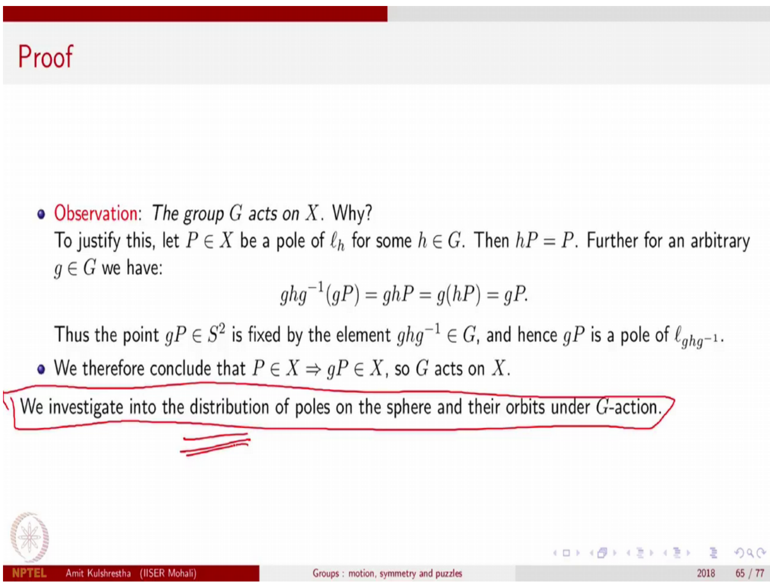

And crucial things has been is the distribution analysis of distribution of poles one this sphere and their orbits under G-action. So, as you have seen throughout this course throughout all these lectures, group action is quite an important thing. You can use group action to understand certain symmetries; you can use it to understand certain puzzles, certain games. And also while doing more mathematical while exploring other aspects of mathematics or the branches of mathematics such as representation theory, matrices and all that, also our groups actions are imported they are quite inevitable.

To end this course, I would just mention to you certain open problems in group theory. So, those open problems are most of the open problems are difficult to state there in much more mathematics. Then what has been covered in these course in these lecture series, but nevertheless I manage to find certain problems which are which can be understood in terms of the parlance in terms of the terminology that we have used in this lecture series, and I would mention it to you.

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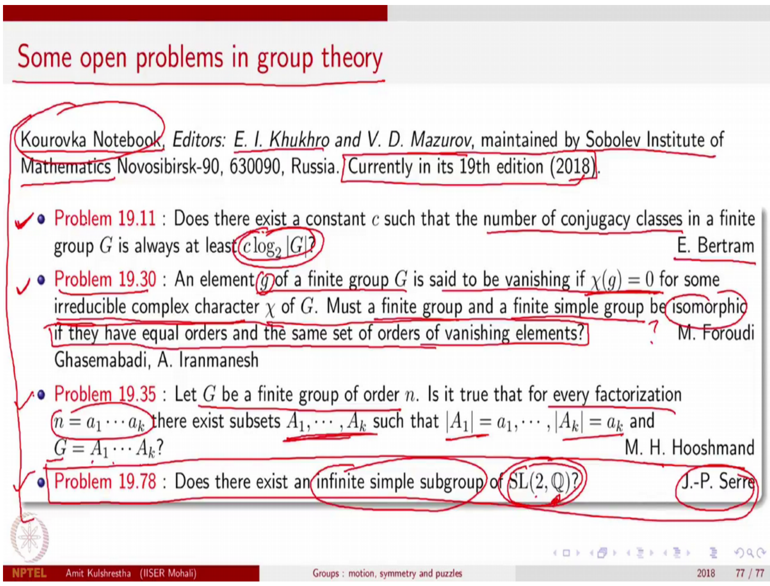

And all these problems I have taken from what is called Kourovka's notebook. So, Kourovka's notebook is maintained by Khukhro and Mazurov, these are two (Refer Time: 43:52) group theorists. They work at this place in Novosibirsk, Russia, Sobolev Institute of Mathematics. And this Kourovka Notebook is currently in its 19th edition 2018, so latest one just came out couple of days ago. So, I am just mentioning few problems which are open problems.

So, even basic group theory that we just discussed during these courses has makes us capable to understand at least some statements if not proofs. There are many problems in this book for which you need much more background; and those problems we need lot of probably in some cases geometry, lot of algebra, but these are very straight to understand problems. So, I am mentioning these.

So, this is problem 19.11 from Kourovka Notebook which says that can you estimate conjugacy class size for can you estimate number of conjugacy classes for finite groups. So, it says that can we actually express number of conjugacy classes as some constant times log of 2. So, it is quite interesting the count to conjugacy classes how many conjugacy classes are there in terms of log of group size; it is interesting. So, this problem is attributed to E-Bertram another problem, problem number 19.30.

And it is in terms of complex characters. Remember in one of the earlier lectures, we have seen what complex characters are. So, an element a finite group is set to be vanishing if chi g is 0. What is chi g, so it is chi g is 0 for some irreducible complex character. So, if on g, if on g, some irreducible character vanishes, then you call this element to be vanishing.

Now, the question is if you have a finite group, and then you have another finite simple group. And suppose if they have equal orders and same set of so if the finite symbol group and finite group they are same order and the same set of orders of vanishing elements, so you take vanishing elements and the order of vanishing elements. Suppose, the order of vanishing elements these two sets are same, can you actually conclude that these two groups are isomorphic, so it is quite interesting.

Another one which is due to Hooshmand which says that we take a finite group of order n and you factorize this order, so A 1, A 2, A k, these are integer factorization. So, it is true that for every factorization. You can break, you can find partitions, you can find subsets A 1, A 2, A k of the group such that A 1 has size A 1, A 2 has A 2, A k has size A k, and every element of G can be obtained as some element of A 1 times some element of A 2 times some element of A k, so again it sounds quite an interesting problem.

Last problem which is due to neither than but the famous fields medallist the famous mathematician of our times J.P. Serre and the problem is very simple. It is asks about infinite simple subgroups of SL 2 Q. So, SL is collection of SL 2 is collection of 2 by 2 matrices, and here over rationals. So, 2 by 2 matrices over rationals, which have determinant 1; in that group, can you find infinite simple subgroup; and it is quite surprising that such problems are still unknown. So, if the the straight with the whether this such statements are true or not is still unknown.

So, today's I just gave glimpse and each year this Kourovka's book Kourovka's Notebook, they keep adding problems two thousand, so 19th edition these are actually the latest editions, the latest edition they have added some 111 problems; and over all there are thousands of problems in Kourovka's book which are still open. They also keep updating what problems have been solved or if some progress is there what is the status current status of that. So, it would be quite interesting if someone can even comment even small comment about these or some examples that would be quite worth it.

So, with all this with I am stopping my lecture series, purpose of all this was to make you appreciate what groups are. Although, in usual courses in many regular group theory courses you get to know of various aspects of group theory, you get to learn what Cauchy theorem is, what Sylow subgroups are. And then are many other aspects how many groups how many elements of which order and all that.

But we fail to understand connection of group theory with various real life problems. And through this lecture series, I just wanted to bring that out, so that with the small definitions with not much baggage in group theory we could enjoy, we could appreciate why group should be there. And the right from those basic definitions, we could also come we could also see all these hard problems, all these open problems which are there in Kourovka's notebook ah.

There is not much that I would say in this course now; I hope you enjoy it all these lecture series. If you had any query, you can write to me, you can ask me through online inter mode, inter phase, or whenever I get live on YouTube, you are always welcome to ask all the questions.

Thank you very much, thank you for being for all these lectures throughout.

Thank you.