Groups: Motion, Symmetry & Puzzles
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More applications of Groups
Lecture – 16
Rotations symmetries of platonic solids

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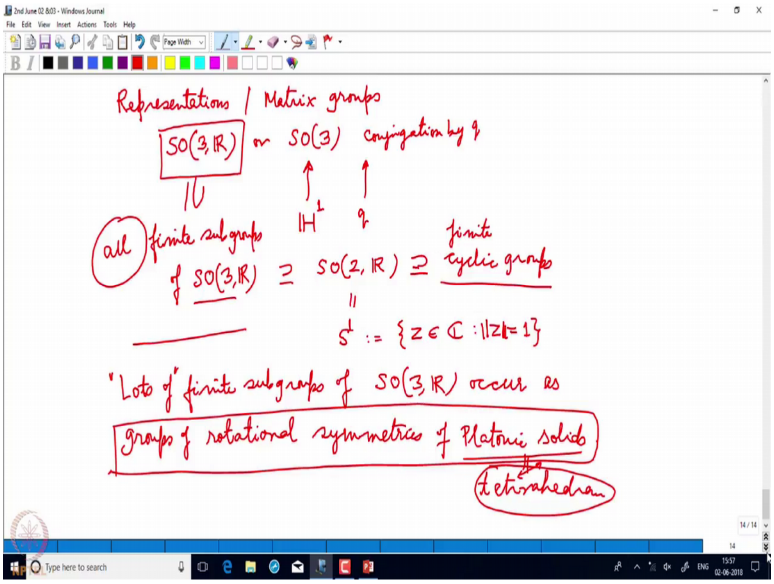

So, last time we discussed about the representations and I said that it is a good thing its quite handy thing for calculations to think of abstract elements of a group as matrices or some other thing may permutation so that calculations are easy. And we also saw certain matrix groups last time, and one of the matrix groups that I promised I will discuss today is SO 3 R or simply SO 3.

And a couple of lectures ago we had seen that SO 3 can be understood through its double cover which is unit quaternions, quaternions which have norm one you remember that, right. Whenever I have a unit quaternion to that I can associate a map which is conjugation by q inverse conjugation by q. So, that is q x q inverse that kind of map and in the understanding of rotations this map is quite useful.

So, what is the purpose for today's lecture today we want to understand finite sub groups of SO 3 R. So, you have group of rotations group of rotations there are inverse so many axis infinity many axis and there are infinitely many angles about which you can rotate. So, thing is this in finite group we want to find out number of finite sub groups.

So, before we do that it is very clear to us that SO 2 is a subgroup of SO 3. So, these are rotations in 2 dimensions. So, I can always fix back see I am fixing the z axis and I am rotating. So, that rotation is just a rotation in 2 dimensions and therefore, SO 2 can be thought of as a subgroup of SO 3. Rotations in 2 dimensions can be thought of as rotations in 3 dimensions as well yeah. And what are subgroups of this? What are finite sub groups of SO 3 R? So, what is SO 3 R? It is not very hard to see that SO 3 R is actually same as s one which is unique circular.

So, you want those complex numbers whose now whose mod whose norm is 1, no 1 complex numbers. So, finite subgroups of S 1 finite subgroup of circle they are what they are cyclic groups finite cyclic groups. So, therefore, it is clear that finite cyclic groups are also subgroups of SO 3 R. But our these all we want to know all finite sub groups of SO 3 R and that is quite interesting problem which involves group action and heavenly uses one side formula for counting orbits that we had discussed a couple of lectures ago.

Before we do that let me mention that lots of finite sub groups of SO 3, well I should probably not be seen lots, so what essentially all. So, they occur as groups of rotational symmetries of platonic solids I hope you remember what platonic solids were I had shown you certain 3 dimensional objects. I will show you again and that is going to your first job. First job is going to be understanding of groups of rotational symmetries of platonic solids. In fact, in one case in case of tetrahedron we had seen this group. In fact, we had written this group and we had also made any graph of this group in case of tetrahedron. Let us recall platonic solids.

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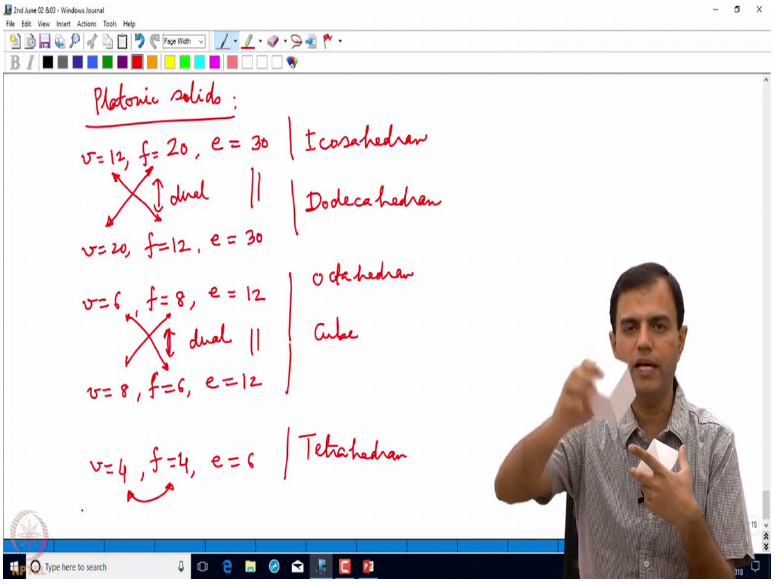

I will show again this one this has 20 faces. So, f is 20 number of vertices is 12 and number of edges is 30 and this is called sorry not tetrahedron, icosahedrons this one, every face is a triangle. And then we have this thing which is dual to it and this is called dodecahedron. And how many faces does it have? It has 12 faces and how many verses it has it has let us count 5 up, 5 down, 10 and then 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20 verses and number of edges is 30.

That is dodecahedron and this one is octahedron. For this number of faces is 8 octa number of words is 6 and number of edges is 12 I guess, so 1 2 3 4 up, 4 down and 3 in the middle 12. And then we have cube which is dual to it the cube has 6 faces and 8 vertices and 12 ages for up for bottom for vertical and this one tetrahedron where number of faces is 4 number of vertices is also 4 and number of edges is 6, 3 here, 3 vertical, 3 in the bottom 6 and this is dual to itself.

So, what is the duality? Duality it is between faces and vertices faces and vertices that is duality, here again faces and vertices and number of edges is equal number of edges is equal and this is dual to itself. So, for example, I can take this I can consider middle of the face middle of this face and opposite face I hold like this and then I hold it like this.

So, here the vertex, vertex is you see if I rotate about this about this axis this is order 4 rotation. So, I can rotate by 90 degree, 90 degree, 90 degree, 90 degree 4 times I am back and similarly here or therefore, rotation 90 degree each. So, the midpoint of the phase is acting as x midpoint of phases I am taking it joining them opposite ones and those are acting as axis.

So, this is phase vertex duality and how the call them dual do any key. And now the question is how they interesting in sub in determining subgroups of S 3.

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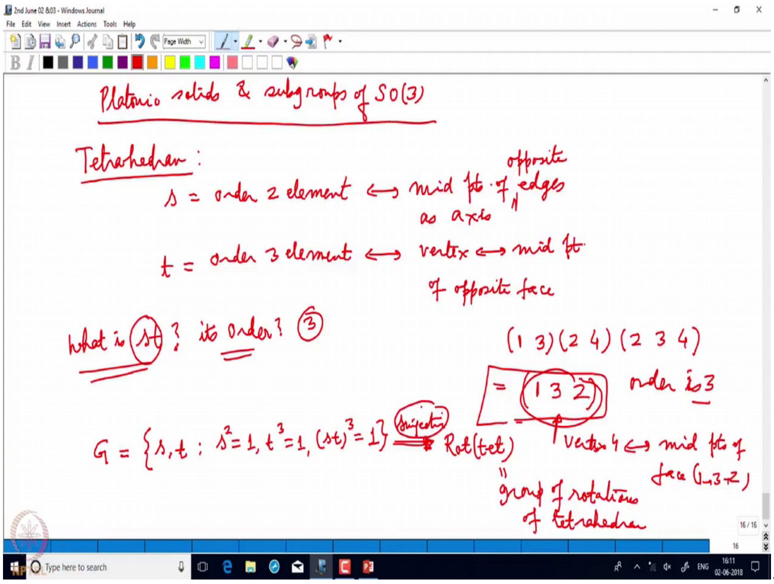

So, platonic solids and sub groups of SO 3, SO 3 R. For that let us first determine what are the groups of symmetries of that objectives, and it is not very hard everything is just about observation and putting in the right mathematical language.

So, you do that and after that we are going to learn how can we determine all subgroups of SO 3 in terms of platonic solids. So, let us take this one, this tetrahedron, and what are all the symmetries. In fact, we wrote all the symmetries in terms of permutations in terms of even permutations that we have already done couple of lectures ago where I said 1 3 and then 2 4 that kind of thing and then 2 3 4 those all are 3 and all are 2 are elements in a 4.

Nevertheless, let us have a look at it again. So, first job is to identify order 2 elements in the symmetry group. So, how do you find order 2 elements in this? So, here I take midpoint of edge and opposite side, right. I am holding it from the midpoint of this edge and when I hold it naturally the other side my thumb is also on the midpoint of opposite edge.

And now I can rotate by 180 degree. So, if I rotate by 180 degree I am like this back, to the similar kind of the symmetry motion, this is the symmetric rotation. So, there was ordered 2 element in this. So, the order 2 element is; let me call it s order 2 element. And which is obtained by taking midpoints of edges as axis, midpoints of say opposite edges.

Now, what else is very natural order which occurs here? In a group of this group of symmetry of this, right: triangle. So, there has to be order 3 element naturally what which one, I take vertex and opposite phase midpoint of the opposite phase I have and then I simply rotate, I rotate by R 2 pi by 3 120 degree each time. And that is how I get order 3 element in this. I am just rotating with that as an axis, right I am just rotating that as an axis.

So, here is t which is order 3 element. And how do I get that order 3 element? So, I take a vertex and I join it with midpoint of opposite face. And that is how I get order 3 element. What else? What if I compose two, what if I compose these 2 what would happen? And that is going to be interesting.

So, I take first I rotate like this and then I take midpoints and then rotate again. was it What is it going to be? That is interesting. And for this let us try to do a competition. Although, what you can do one way to see it is. So, question is what is s t, right. So, one way is to actually do the experiment and realize what is s t, what is the order of s t. Or the other way as we did some lectures ago. So, other 2 element would be something like 1 3 2 4 something like this. And what is t? T let us say t is, so I am writing s here. So, t say for example, 2 3 4. Let us compute what it is.

So, this is edge joining. So, this is the edge which is joining vertex 1 with 3 and the other is vertex 2 and 4. And here the vertex 1 is having 1 end of the axis and midpoint of the phase 2 3 4 is having midpoint of the he is having other point of that same axis. So, what happens to 1? Here nothing happens, here nothing happens, and here 1 goes to 3; 1 goes to 3. What happens to 3? 3 goes to 4 and 4 goes to 2 and that is all. So, 3 eventually goes to 2. What happens to 2? 2 goes to 3 and 3 goes to 1, this is over and what about 4? 4 goes to 2 and 2 comes back to 4. So, this is it. So, order is 3; the order is 3.

So, when they are actually compose like this order is 3. So, you should be realizing that, therefore s t is actually rotation and this time this the vertex is 4th vertex. So, this is a different vertex about which you are having this rotation. So, the one end of this is vertex 4 and you are having other end of the axis as midpoint of is 2 3 4.

So, whatever is that face 2 3 4. No here in this case 1 3 2; in this case its face 1 3 that face yeah. So, s t has order 3. So now, I have group consider the group G which is generated by 2 elements s and t, such that square of s is 1; these are relations, remember generators and relations t cube is 1 and s t cube is 1. And by doing all this what we have obtained is actually a homomorphism from this group to the group of rotations of tetrahedral.

So, here is this homomorphism and it is a surjective homomorphism; surjective because, we have observed all possible symmetries in this. So, that is how we are came here to be surjective.

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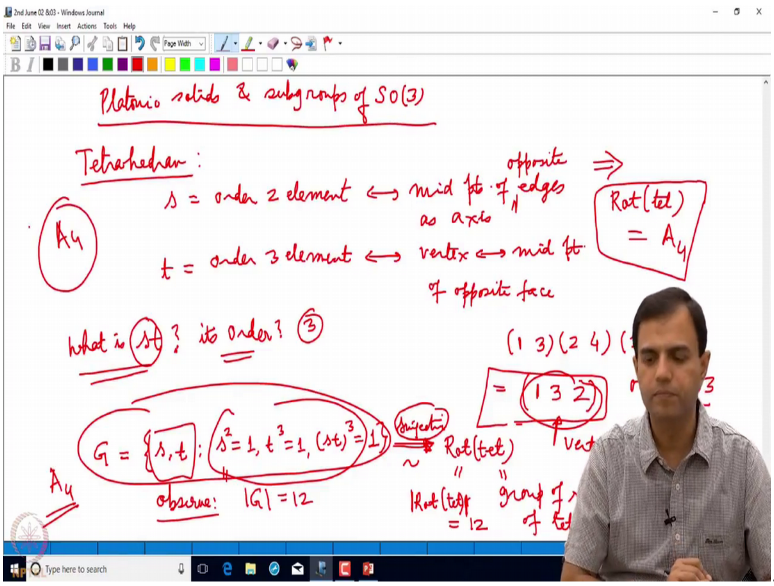

And what is the order of this? So, observe by some work that order of G is 12 and the symmetries of tetrahedron that order is also 12. And I have this subjection. So, this has to be isomorphism. So, what we have is that rotations of tetrahedron, the group is this and it is not hard to see that this group is actually A 4. The group is A 4, you can actually in longhand you can do some calculations and obtain that this group is A 4. You just have to observe these relations and these generator the this kind of relations on these two generators.

So, tetrahedral symmetry is group A 4. Let us see few more, let me take my favorite one Icosahedron.

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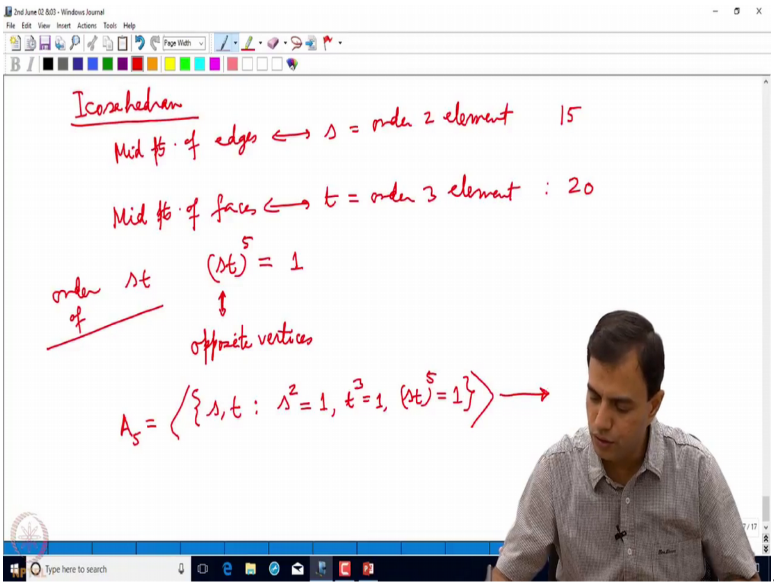

What to pick first, let us see take vertices. If I get vertices and rotate how much is the angle which is admissible for symmetry? You can see 5. So here is an element of order 5, before that let us see element of order 2. So, again the element of order 2 is obtained by a midpoint of edges.. So, this is a what you say order 2 element. So, order 2 element is present there what about order 3 elements, since triangles are there; since triangles are there midpoints of triangles you rotate by 2 pi by 3 and that is what is going to give me element of order 3.

So, I take midpoints of faces these triangles and this is order 3 anyway. So, how many order 2 elements are here? So, there are total of are 30 edges, there are 15 pairs of edges, right. So, since this is 15 pairs of edges are there, I have 15 of these. So, one is this position other is that position, 15 of them. And order 3 elements, how many order 3 elements will be there? Again, there are 20 faces. So, I will have one identity position. So, there are total 10 possible choices and each thing is giving you 2 nonidentity rotations, so what you have is 20 of them.

What about s t? Again the same question: what is the order of s t?. So, for that maybe you have to do some certain experiment, you make your own icosahedron and try to do some experiment. And what I can tell you is that s t to the power 5 is 1. So, s t is of order 5 and that order 5 element is actually obtained like this by taking vertices.

So, this is obtained by taking midpoints of that, not midpoints but taking opposite vertices. So, when you take opposite vertices you get order 5 element. And again you can think of this group which is generated by 2 elements s and t. And the relations are s square is 1, t cube is 1, and s t to the power 5 is 1; this group. And this group it is not hard to see some work is required that this is actually A 5. So, group generated by this is A 5.

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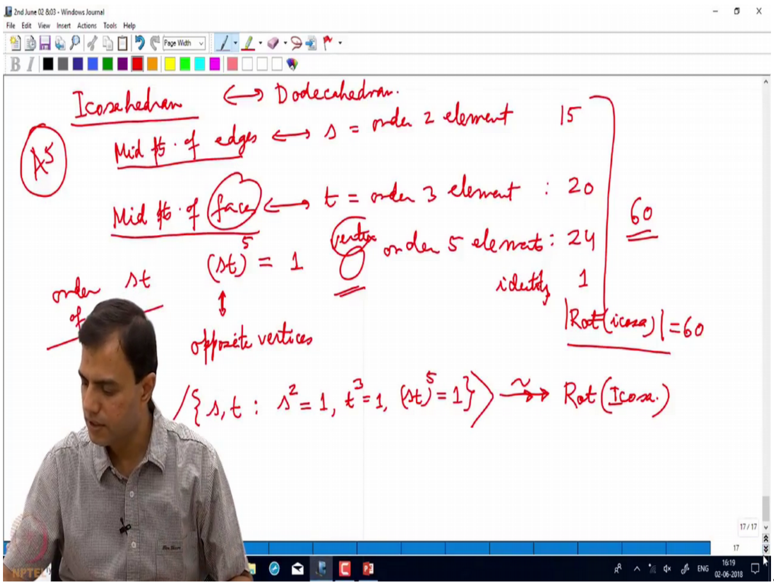

So, from here to groups of rotations of icosahedron you have again surjective map. And you know what this symbol as does it corresponds to the midpoint of edges rotation, and similarly t corresponds to midpoints of faces rotation; that kind of rotation. And here it is again not very difficult to see that order of this is 60.

So, here actually if you look at order 3 elements and they are 20, and order 5 elements they are 24; because there are 6 options there are 12 vertices, so there are 6 options 6 and each pair of vertex when you take as axis and keep (Refer Time: 26:43) opposite ones then you get 4 nonidentity positions. So, 6 into 4 there are 24. And then there is one order one element which is identity.

So, there is a total of 20 plus 15 - 35; 35 plus 1 - 36 total of 60. So, group of rotations of icosahedron has size 60 and if you look at these generators and the subjective map these generators again have; again generate a group which is a for your 60. So, here is an isomorphism. So, A 5 is group of symmetries of this object which is icosahedron.

Now, for this dodecahedron is the same thing, it is all about vertex and face duality. So, rather than saying all this, I will write face, in case of face I will write vertex here, in case of vertex which I will write face here and that all and so for dodecahedron as well I get same symmetry group which is A 5.

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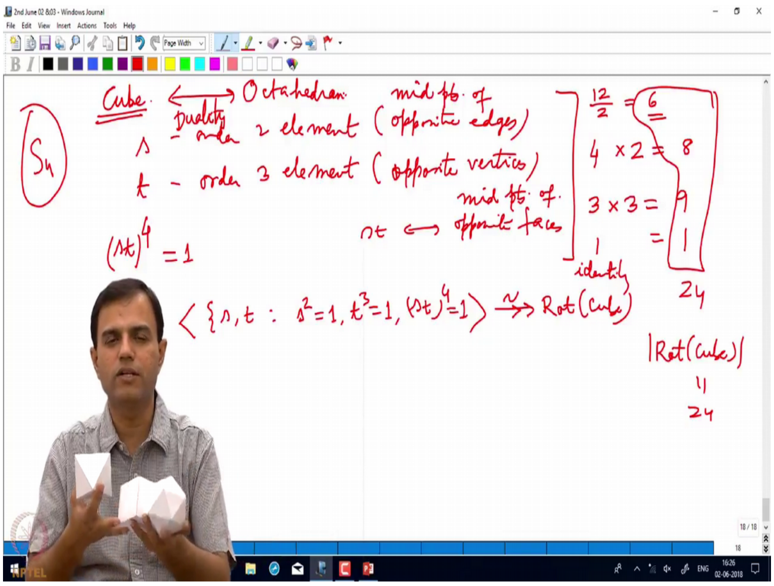

Now, let us look at cube or octahedron. Same thing, yes let us take this cube. So, what would be symmetry is here? So, first of all order 2 element, and order 2 element is again obtained as midpoints of edges. So, I take midpoint of edge opposite edge take the midpoint and I get order 2 element; the same every time. And what if I take opposite vertices; so t. So, order 2 element by taking midpoints of opposite edges, and t which is order 2 element sorry order 3 element I can obtain by considering opposite with or rather yeah; so opposite vertices.

Now, you can see from here I; so from top if I look at there are 3 edges which are going so just rotate these edges among themselves, just rotate these edges among themselves. And that is how you will get order 3 element and now the question is what is s t.. So, again you can actually do experiment if you want you can enable them so that you remember what is identity position.

So if you do order 2 element, if you do this and after that you do this what actually you obtain is order 4 element. So, s to the s t to the power 4 is actually 1. And s t is obtained by considering opposite faces rather midpoints of them; midpoints of opposite faces you take and that is how you get s 2 to the power 4. And again you consider the group which is on 2 generators s and t for which relations are s square is 1, t cube is 1, and s t to the power 4 is 1. And from here to group of rotations of cube you give a map.

So, it is not very difficult to realize here that groups of rotations of cube has how many elements let us count it. So, here how many pairs of edges are there? There are 12 edges. So, 6 elements you will get this way, right. And here order 3 elements you will get 2 times pairs of opposite vertices. So, there are 8 vertices. So, you will get 4 pairs and each pair is going to be of 2. So, you get 4 into 2 6 of them.

And what about midpoints of faces, so how many faces are there? So, you have 3 pairs. So, 3 pairs of faces and each phase rotation is game going to give you 3; 1 2 3 4 its 9, yes wait a minute sorry some editing will be required I have written something wrong. Mohan, some editing will be required.

Then else one we start from the (Refer Time: 32:04).

And this; so here in this cube let us count how many order 2 elements are there, how many order 3 elements are there, and how many order 4 elements are there.. So, for order 2 elements I will see how many different edges are there. So, there are total of 12 edges. So, they are total 12 by 2 6 pairs, so there are 6 elements of order 2. What about order 3 elements? So, when I do order 3 elements I am considering opposite vertices. And in each rotation I am getting 2 non trivial 2 nonidentity positions. So, I will multiply with not with 3, but with 2. And how many pairs of opposite of vertices are there? So, there are total of 8 vertices: 1 2 3 4 4 up 4 down, so I will get total of 4 4 into 2 is 8.

And what about opposite phases; so how many opposite phases are there. There are 3 pairs and each of them is going to give me three different three nonidentity positions, so 3 into 3 9. And then you have one identity element. So, how much is this? This is 24. So, size of rotation group of cube is therefore 24, it is not hard to see that this is also of order 24 and in fact this is as for permutation group on 4 letters. And then again this surjective map becomes isomorphism as in earlier cases. So, for cube the group of rotational symmetries is S 4. And again by duality, for octahedron as well I have a group of rotational symmetries which is same as S 4.

So, what we did in this lecture today? We just saw all possibilities of groups of rotations, rotational symmetries of all these platonic solids. And as you recall I had already proved that there are only 5 platonic solids, and these are right in front.

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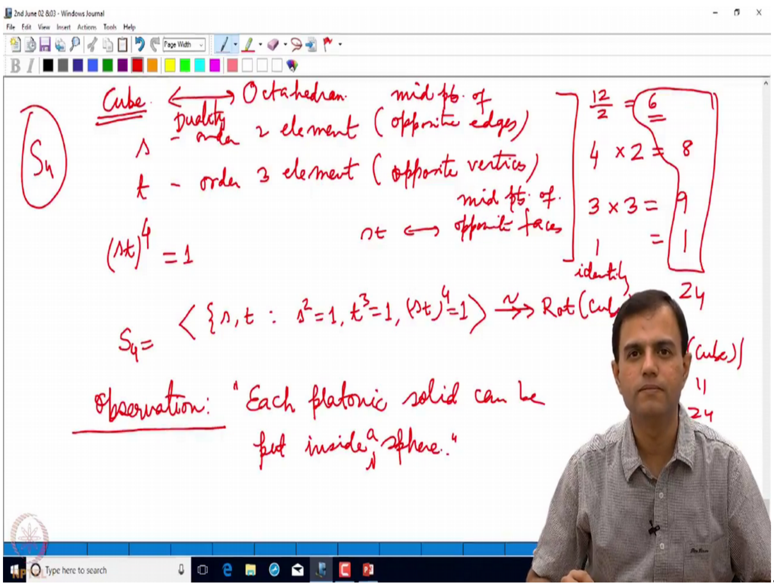

And observation is: that each of these objects can be put inside a sphere, each platonic solid can be put inside sphere. So, a unit sphere or whatever is the radius of these objects inside a sphere. And that is going to help us in determining what are all subgroups of; what are all finite sub groups of SO 3.

So, in next lecture we are going to see all that, it is quite a fun is very interesting application of group actions. See you next time.