Groups: Motion, Symmetry & Puzzles
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More applications of Groups
Lecture – 15
Rotations and quaternions

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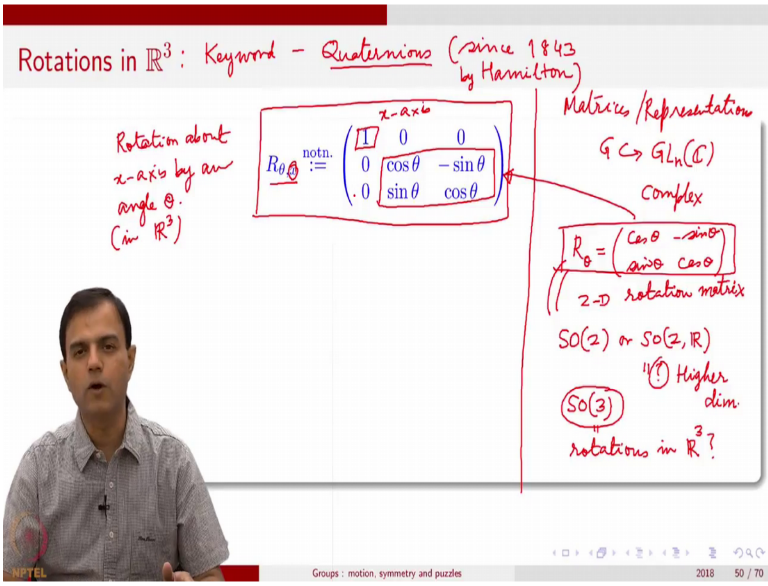

So, last time we were talking about matrices, representations. And I said that various groups can be realized as matrices, suitable say complex matrices. And then I also mentioned that rotations in 2 dimensions they can be understood in terms of matrices. And then there was this rotation matrix rotation by an angle theta, we had deduced quite easily that it is just cos theta minus sin theta sin theta cos theta. That is rotation matrix. But that was in 2 dimension. So, those are rotations in R 2.

So, collection of all rotations was called SO 2 or SO 2 R or its R 2. Now the question is what about higher dimensions. Calculation for this was easy because it was just 2 dimensional case, but what about SO 3, what about rotations in R 3; 3 dimensionally think. So, imagine one sphere and I am rotating this sphere how can I just get all possible rotations of this sphere?

Now there are some tricky things. One tricky thing is that there are. So, many axis about which you can rotate there are infinitely many options for you to pick the access about which you are going to rotate. So, there are issues. And as you would learn matrices are not the best notation, they are not the best representations for expressing rotations in 3 dimensions. And the key word for this talk is; key word for this talk is quaternions. So, quaternions are again objects, I am going to explain in this lecture and they were discovered in the year 1843 by Hamilton. We are going to learn all that in this lecture. And we will also see why quaternions are better options to express rotations in 3 dimensions, ok. So, let us start.

Let us start with rotation matrix in R 3. So, what is happening here? R theta x denotes rotation by angle theta about the x axis; so the x rotation about x axis by an angle theta. So, which direction you take clockwise anticlockwise what you just fixed one dimension; you just take this system x y z and just fix one orientation along which you call the angle positive and other where you call negative; so, just your choice.

You can see this block, right. So, what does the whole matrix do? Whole matrix keeps the x axis fixed. So, this one corresponds to x axis and whatever rest is there it rotates by angle theta. So, it is quite easy for us to write down the matrix of rotation in R 3; so all this rotation in R 3. So, just from the 2D case we could simply write 3D case, but remember the choice of axis is very special; x axis is what we have taken. So, writing in this block form and writing 1 0 0 in the row and 1 0 0 in the column is fine. What about other options?

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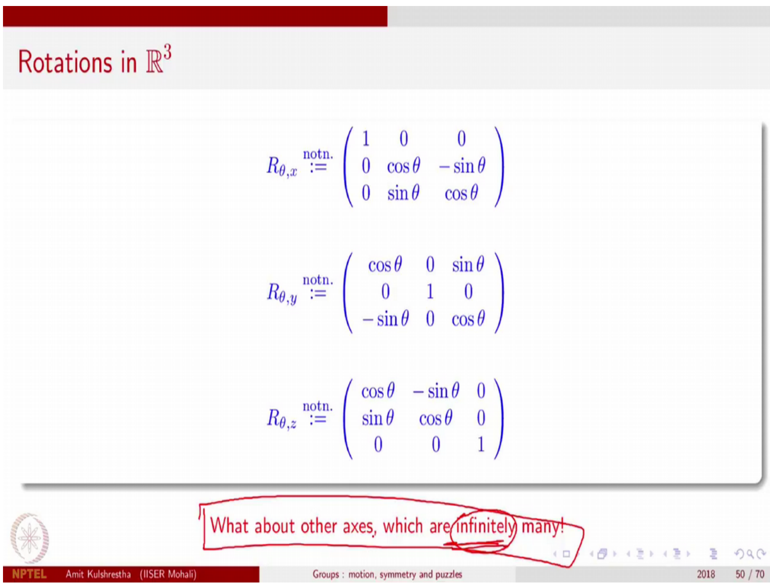

For y for y axis I will have 0 1 0, and 0 1 0 both in column and row, because this is what is going to keep y axis intact. And then cos theta sin theta minus sin theta cos theta; I will have like this I would have taken minus sin theta here and plus sin theta here does not matter, it is just a choice of orientation, it is just a choice for the direction where you will call your angle positive and your angle negative

And similarly you have R theta z here I am fixing z axis. So, this is the this very clear cut very obvious calculation where you do not have to do much just right from the 2 dimensional case you are lifting situation to 3 dimensions. So, for these three is very special axis for x axis, for y axis, and for z axis it is pretty straightforward for us to write rotations in terms of matrices. But, point is how far can we go with this approach how far can we use matrices in order to understand rotations in R 3. That is going to be interesting question.

So, that is what; what about other axis and there are infinitely many axis. So, are we going to write infinitely many matrices or if not. Then given a choice of axis and given your rotation angle what is the expression for the matrix. And as you will you see its quite cumbersome and that is way we are going to give up matrix notation. And you are going to realize the same group, same group as o 3 via quaternions. We are going to see that, ok.

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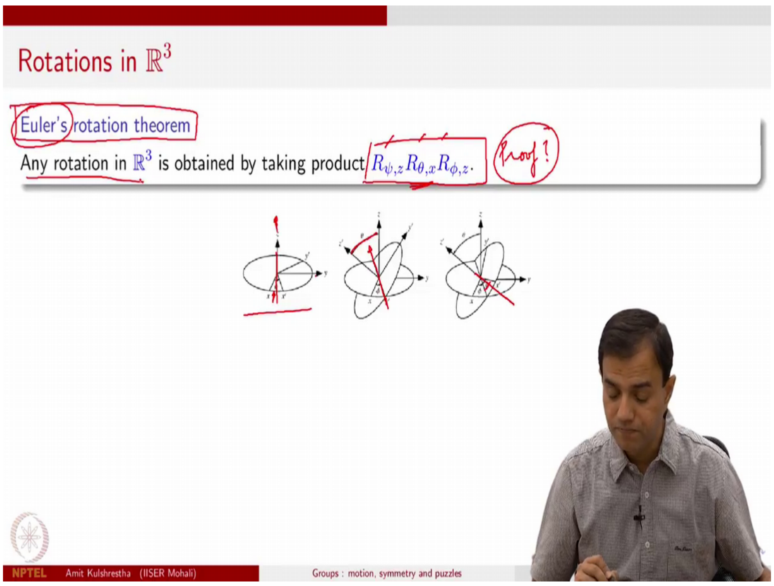

So, here is a theorem which is attributed to famous 18th century mathematician Euler. And which says that: you can combine these basic rotations in order to obtain any other rotation. So what it says is that: any rotation R 3 can be obtained by considering this composition; so some rotation about z axis, some rotation about the x axis, and again some rotation about same z axis. How does it work?

So, let us see. This is the initial position, this is the initial sphere and we are going to rotate it. So, first I am rotating about z axis. So, z is fixed z axis and I am having an angle phi; I am having angle phi and z axis is fixed. So, x comes to x dash y goes to y dash, nothing happens to say because we are rotating about z axis.

So, x dash has come here. And then I am rotating about x axis by an angle theta. So, I am rotating about x axis, this x is going to be fixed now and the angle which I am taking so this is going to be axis and the angle by which I am going is theta. So, z will go to some z dash and y will go to some y dash. And then I keep this fixed, this x is fixed. So, here this was the axis, here this is the axis, and here in the third step this is the this is the axis. And how much is the angle? Angle is psi, I am just rotating by angle psi.

It is not very straightforward to prove this, but it would be a good idea if I could work out a proof of it or look for a proof somewhere. It is not an easy theorem as such, but it is interesting fact to know that every rotation is a product of these three basic ones. You notice y is not appearing anywhere. So, I think pick two axis two types of matrices and then see these two types of matrices I can take this one and this one, and I can combine these types of matrices to get all possible rotations in R 3.

What is the problem with this approach? Well, there are many not just one.

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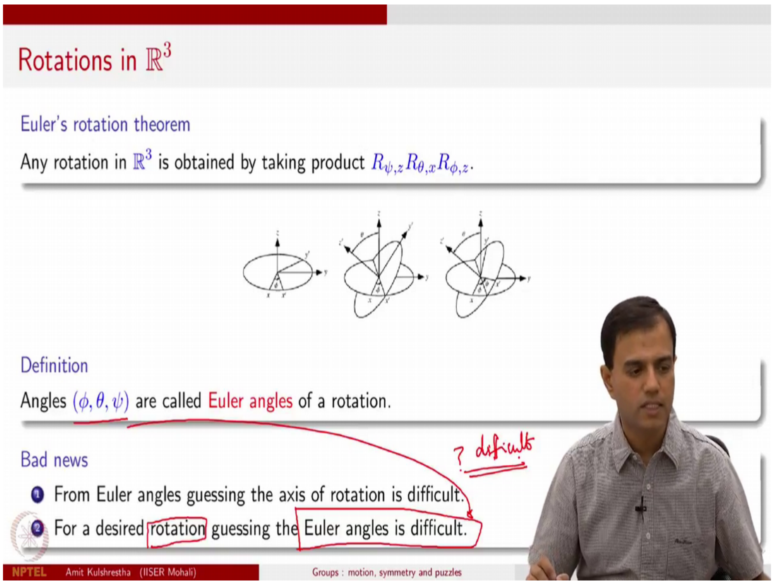

So, if my rotation is expressible in this form then these three angles phi theta and psi phi theta and psi they are called Euler angles of the rotation. That is very classic way of understanding rotations. However, if you want to do calculations it does not go too far. So, that is the bad thing, right.

From Euler angles guessing the axis of rotation is very difficult. If I can do three other angles which you are going to combine this session even guessing what is going to be the final rotation, because product of rotations is going to be rotations; what is the final rotation and for that final rotation which is composed composition of all these three what you going to be the axis even guessing that much is very difficult. And then in a converse election: if I give me rotation in R 3 and guessing what is going to be choice for Euler angles, that is very difficult.

So, there is a difficulty in both the directions. So, how do you resolve such issues?

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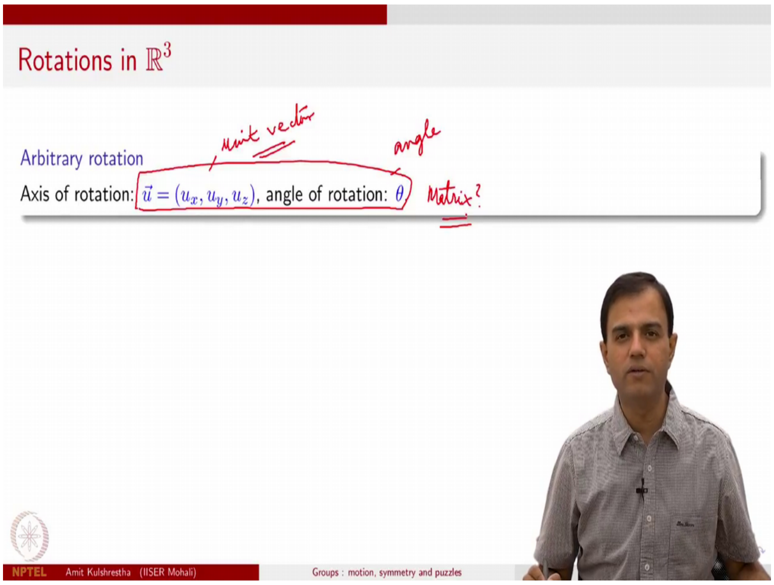

So, axis of rotation I take as unit vector and angle of rotation is this angle by which I am going to rotate. So, I have a sphere. I pick an axis there are so many choices or axis and I rotate it by angle theta. What would be the matrix of this? What would be the matrix of rotation for this? And I am going to show you, you can work it out this very complicated.

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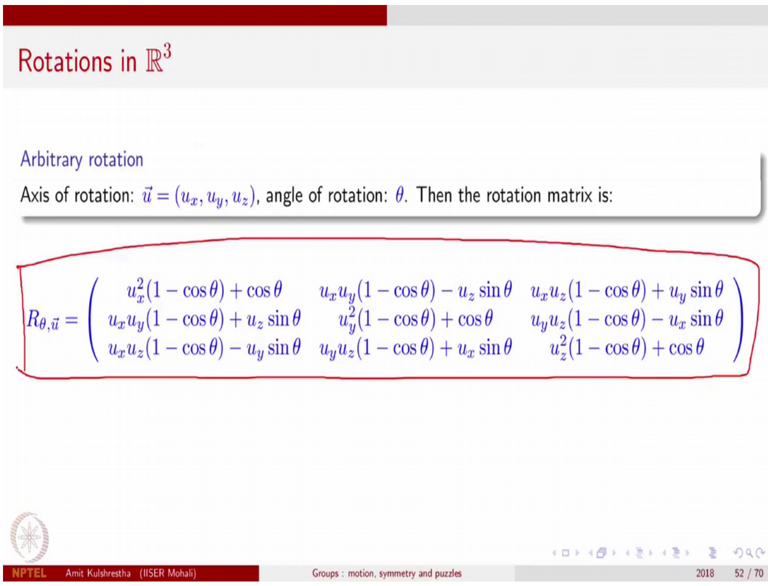

So, if I give you angle of rotation and the axis this is what it is going to be, very cumbersome expression. So, certainly this is not the way to go with calculations of rotations either.

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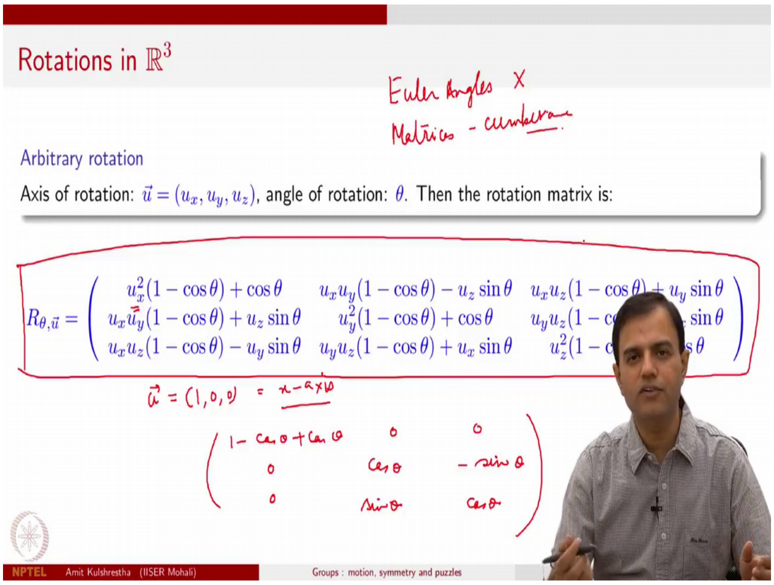

So, I Euler angles as I said there is difficulty and with matrices again there is difficulty it is cumbersome to use matrices as well. Let us see here. So, if you have x axis, so suppose u is 1 0 0 that is x axis, then what happens this matrix should reduced to the one that I had mentioned in the very beginning of this lecture. So, u; so, this would be 1 minus cos theta plus cos theta, because u x is 1 and then this term u x u y will be 0, and then minus u z again is 0. And then this term u x u z again this is going to be 0 and this u y, and then it is going to be 0; again the u x into u y 0 and u z 0 this term u x u z 0 u y 0; so you get 1 0 0 in a column as well as row.

What about other terms let us see u y square. So, this term is going to be 0. So, what is going to remain is cos theta. And here the only term that is going to survive is minus your sin theta, so you have just minus sin theta. Here u y u z, so this term will go away what will remain is u x sin theta u x is 1, so you have sin theta and u z square u z is 0. So, what you have is cos theta. So, you indeed have the matrix that I had shown you in the beginning of this lecture.

But again working with these kinds of matrices is very cumbersome, and it is not the approach we should be taking. So, what is the point of all this? Here is a group right, the group of rotations of sphere or the rotation group in R 3. The group is there, group is well defined the point is how to express that group, how to express, how did you know how to make calculations for this group. We realize that Euler angles are not the best way to understand the group operation, because guessing in both the directions for a given rotation what are Euler's angles and for a given Euler's angle for a given combination theta find psi; what is that even axis of rotation and that is not clear, and matrices as you can see here they are also quite cumbersome.

So, it is very important for us that given a group, how we are going to express it, how we are going to extract mathematical information out of it.

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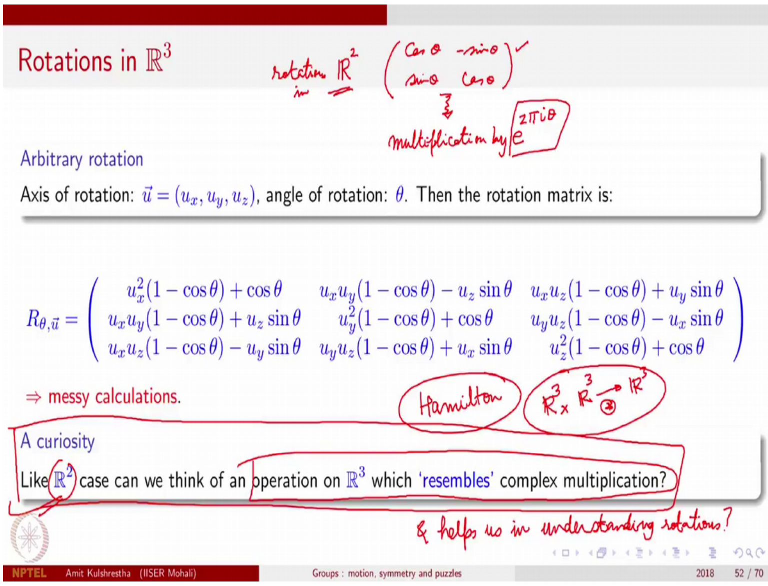

So, as I said calculations are going to be messy if we keep this notation. So, this is something quite important. You remember for R 2 for rotations in R 2, I had mentioned in earlier lecture that cos theta minus sin theta sin theta cos theta is matrix. And one can also realize this as; one can also realize it as multiplication by e to the power 2 pi i theta.

So, complex multiplication is a complex multiplication and rotation they have something to do with each other. And in terms of complex multiplication understanding rotations in 2 dimensions is quite easy. So, that is a point. Like what we had in case of R 2 can we think of an operation on R 3 which resembles complex multiplication and helps us in understanding rotations; that is a rotation. And that is an interesting curiosity, and that is what led Hamilton to the discovery of quaternions.

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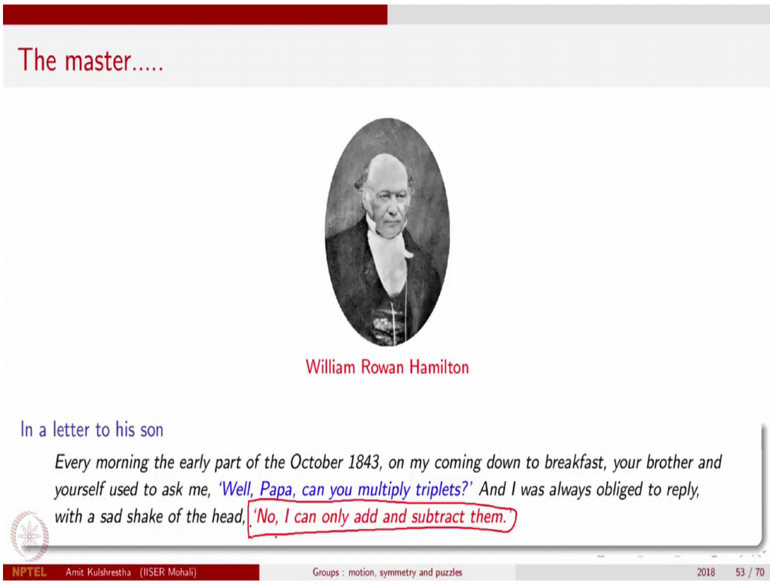

There is a person William Rowan Hamilton, and the point is here to find some operation of R 3. So, I am thinking of some way to combine two vectors in R 3, so that answer is again R 3. And this rule whatever I define is actually resembling complex multiplication then also it is helping us in understanding rotations. So, that is what is a curiosity for us.

So, Hamilton wrote after his discovery of quaternions: in every morning in October 1843 my son used to come to me and during the breakfast you asked to he used to ask along with his brother - “papa can you multiply triplet” So, that is what is the reference. Triplets is multiplication in R 3.

And I saw was always obliged to reply with a sad shake of the head. No, I can only add and subtract them. This is about some multiplication in R 3 which resembles complex multiplication.

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And later actually the same year a October 183 October 1843 he came up with this nice idea for which there is a postal stamp as well as Ireland. So, he was actually walking through walking on a bridge in Dublin and just decide the occurred to him and he wrote it on the stone there. So, this is apparently what he wrote on the stone. So, we are going to understand this.

So, did he actually managed to multiply triplets, did he actually managed to get an operation from R 3. It was R 3 to R 3 which helped in understanding rotations; not quite, not at all actually, ok.

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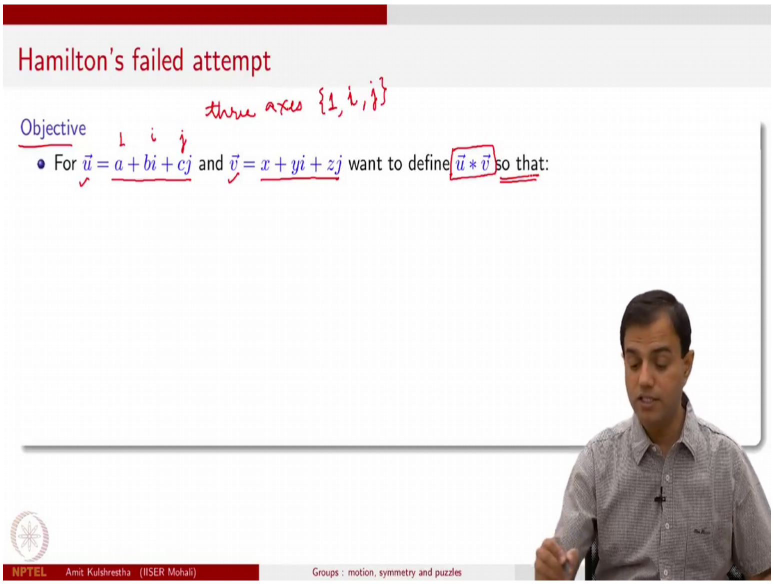

So, let us see what all your failed attempts of Hamilton. So, what is the purpose? I have 1 vector 3 dimensional another vector 3 dimension. So, I am denoting this as scalar axis i axis j axis just a notation. So, there are 3 axis. Those axes rather than calling xyz, I am calling scalar axis i-th axis and j-th axis. So, I will denote vectors in terms of u and v like this a plus bi plus cj and x plus yi plus zj. And I want to define this. And what are my demands, what am I demand so that this resembles quite early in this resembles complex multiplication.

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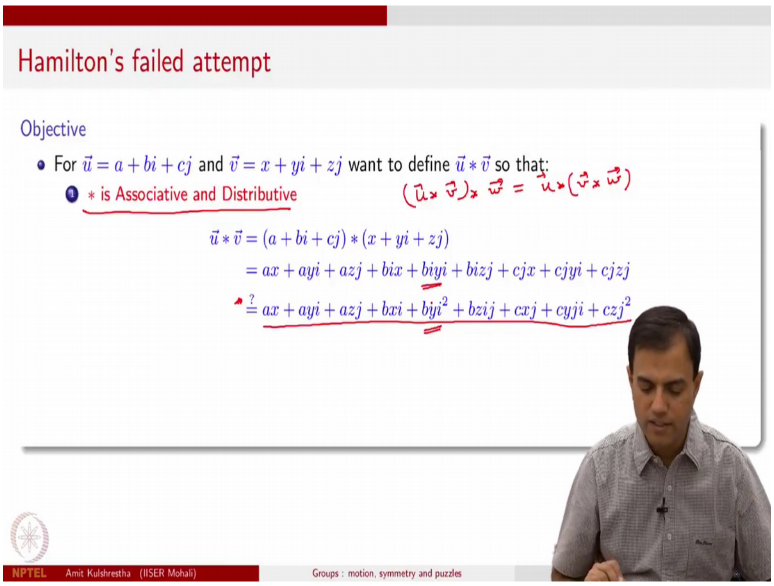

My demands are that this is associative and distributive. So, I want u star v when I multiply with w what I get is u star v star w that is associativity. So, if I force it then I should be getting something like this. So u star v, if I put the distributivity then I should be getting something like this. And if you allow scalars to commute that is b and y to commute with symbols i and j with axis i and j in the multiplication then you get this.

So, this I am writing question mark, because if you allow this kind of commutativity. So, you get this kind of expressions.

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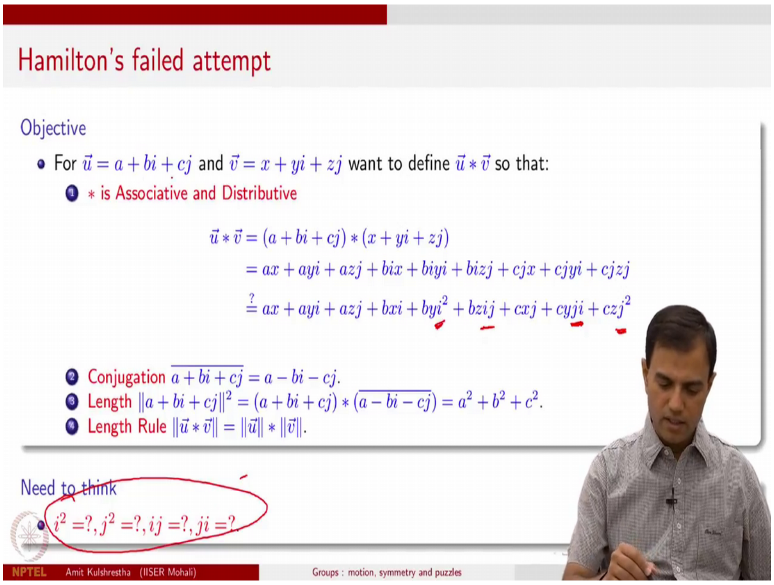

And what are other demands. So, this is first demand associative and distributive, other demand is there should be conjugation; this complex conjugation, right. Similarly we are expecting a conjugation for 3 dimensional case. So, a plus bi plus cj should have conjugation a minus bi minus cj.

And like what we have in case of R 2 in case of 2 dimensions the concept of length is there which is just an element times its conjugate. And that should be equal to a square plus b square plus c square. And then we also want length rule, when we have two vectors, when we have 2 complex numbers z 1 and z 2 the product of them and then you take the norm is same as product of individual norms; individual norms of z 1 and z 2.

So, this is all what we are expecting from our multiplication. So, if you if you want to define this you really have to understand what is i square, because i square is a new symbol here. What is j square and what is ij 1, what is ji and what is ij. So, one needs to understand this that what is it going to be in terms of i and j. And that was the crucial part of all this.

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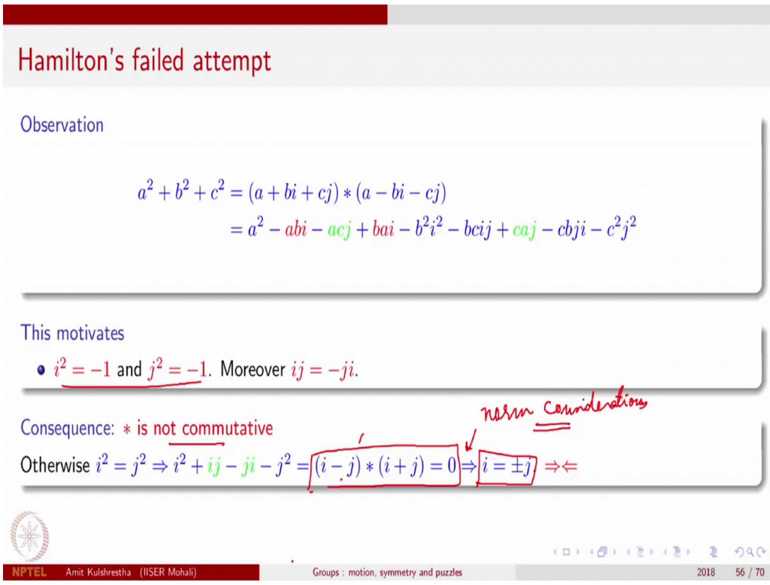

So, some observations are there. So, you if you want a square plus b square or c square to be this element times its conjugate, then forcing the associativity and all that you get this. So, what does it motivate? So, if you make this with this, if you compare then it motivates it. If you cannot quite conclude from this, but some kind of motivation you can get that probably here I have i square probably I should be comparing this with this, probably I should be comparing this with this and so on.

So, some kind of motivation is there. And there is no term of ij here. So, there is a term of ij here, this is a term of ij here, this term and this term. So, probably they should cancel. In other words ij should be minus ji. But then the then the problem is that what is ij. One thing which is very clear is that such an operation if we define it cannot be commutative, because if it is commutative and then i square that; since i square is going to be j squared which is same as minus 1 this is going to hold i minus j times i plus j. So, i squared minus j square is going to be this, if the operation is commutative and this being 0 meaning either i is j or i is minus j.

So, I can conclude from this that one of them is 0, because of norm considerations. So, this implication comes because of norm considerations, because of because product of norms is same as norm of the product. So, that is a contradiction, because i and j they are two different axis they are independent axis they cannot be scalar multiple of each other.

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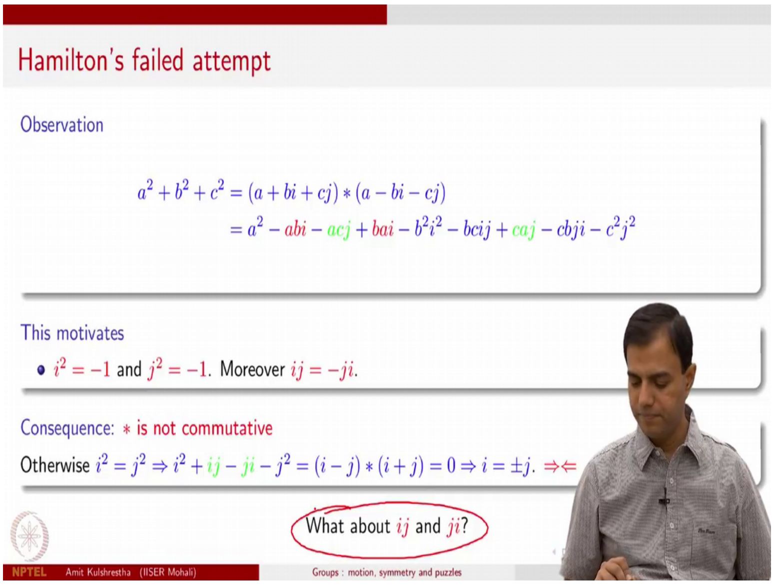

So that means, the question remains what is ij and what is ji.

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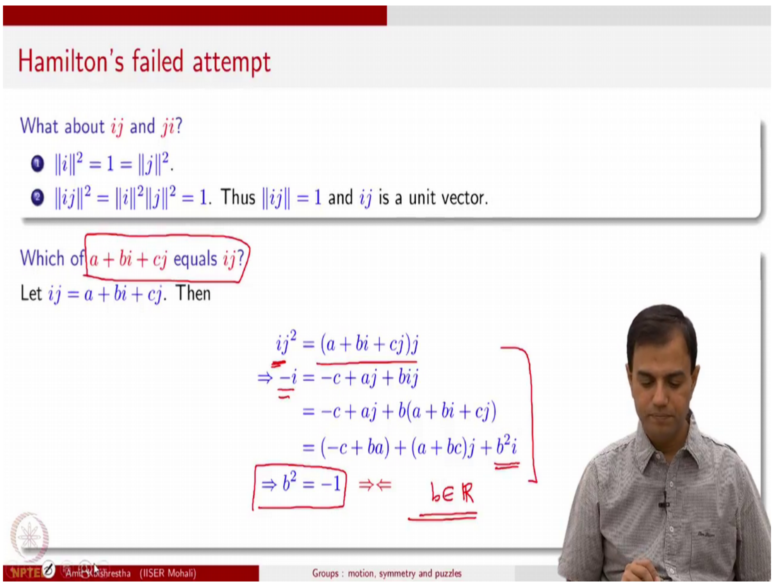

We can easily conclude that ij is a unit vector. Why is that? Because norm of i is 1, so is norm of j, and norm of ij square is product of individual norms which is 1. And therefore, norm of ij is 1, norm of ij cannot be minus 1 because norm is always positive. So, this is what we get from this; so ij certainly a unit vector. But the point is to find what unit vector, what choice of unit vector will serve our purpose. So, which of the vectors is equal to ij?

So, all these are failed attempts of Hamilton. He is trying all the way, all possible algebraic manipulations with the hope that he gets some nice multiplication on R 3 which resembles complex multiplication; no problem.

So, let ij be this and then what are the choices for a b and c. So, there ij times j which is ij square is this and with some manipulation, so because j square is minus 1 so i minus I is this and making all these calculations it is not very difficult to conclude that b square is minus 1. Because here I am just comparing the coefficients of this, and other coefficients I have to put 0. So, b square is minus 1. So, that is also giving some kind of contradiction, because b is a real number and real squares can be positive or 0 only; there is no possibility of a negative number being the square and real.

So, the question remains probably we probably we should discard the idea that i square is minus 1 and j square is minus 1.

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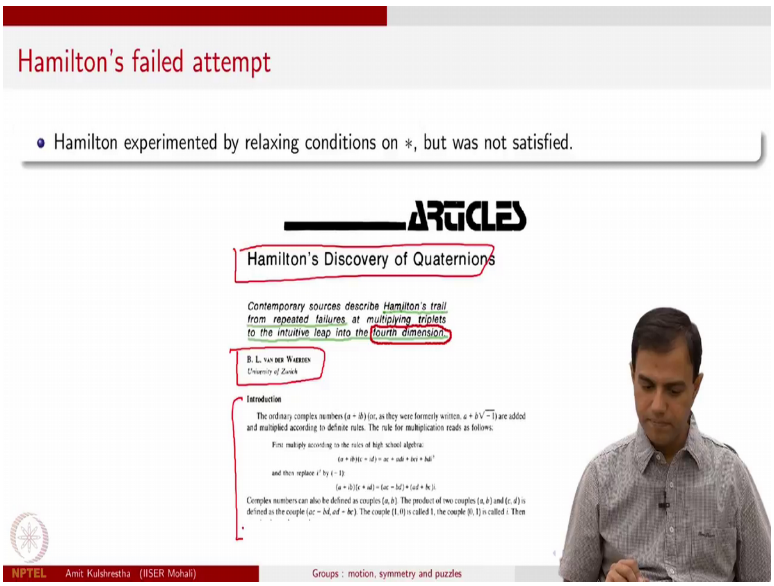

So, here is interesting piece of history on Hamilton's discovery of quaternions van der Waerden from University of Zwick. Since, again he wrote very nice article which is about failed attempts of Hamilton to find some operation on R 3. So, a very nice article it is worth reading.

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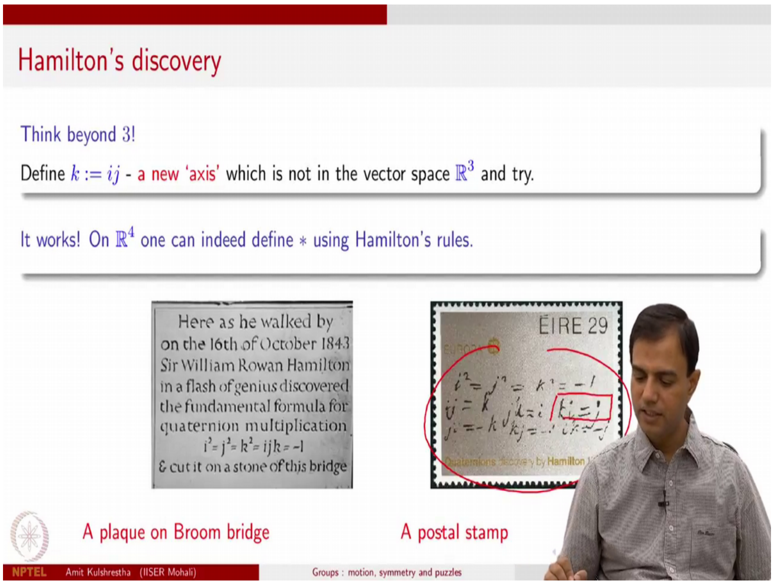

And that is what Hamilton realized after a series of failed attempts. You probably canot do it in 3 dimensions. Maybe you should define ij as a new axis you should expand your dimension and you should go to 4 dimensional space and not in R 3. So, you define a new symbol which is product of ij. So, this is one more iteration; that is what he is imagining and i into j should be that dimension, and certainly k is not in the vector space.

So, with this basic idea he started thinking and then it was quite easy it works. So, in R 4 not R 3 on R 4 one can indeed define a very nice product formula, very nice multiplication which resembles complex multiplication; resembles in what sense all the conditions that I had mentioned earlier.

So, whatever are the Hamilton's rules, what were those. It should be distributive, and associative, and the concept of conjugation should be there, and then the concept of norm should be there which is precisely an element times x norm; so element times its conjugation. So, norm square is an element times its conjugation.

So, this is what he is writing ij is k. So, he is writing k i is equal to j like this, ok.

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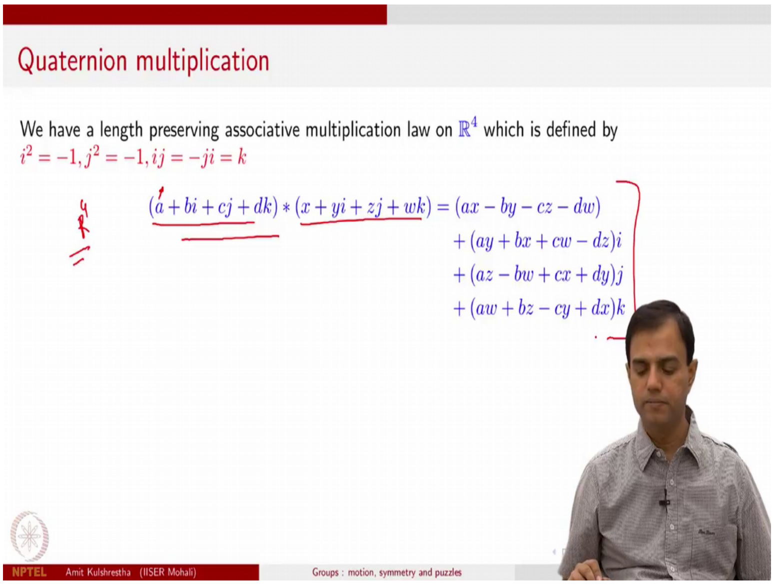

So, let see what does this give. I have i square minus 1, j square minus 1, and ij is k. And with some small manipulations you can make work out that ij is minus ji, in that I had already said it while deducing i square is minus 1 j square is minus 1; I already said that probably ij is minus ji.

So, let us try with this, let us try with this new idea. The only idea which is new here is that you are assuming that there is one more axis which is precisely the direction of ij ok. So, I have this, I have this. I have two vectors and this time these two vectors are in R 4, I have one scalar axis and these three are if you want to say vector axis.

So, using this actually what you can make out that this is how it should be. If I start putting i squared to minus 1 j square to be minus 1 and ij to be minus ji after making all the computation this is what we come up with.

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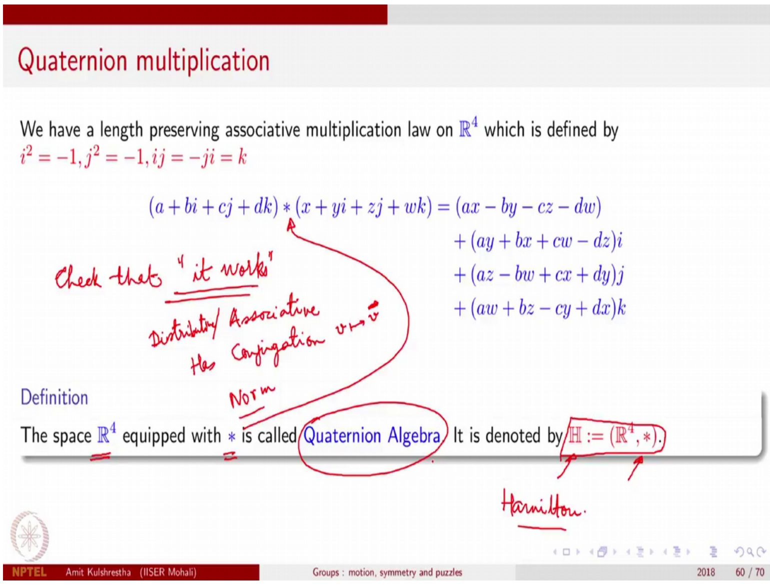

And then it is easy to check. So, that is maybe an exercise for you. Check that it works, that is it is associative has concept of conjugation, associative as well as distributive as conjugations or say some vector v goes to v bar and has concept of norm; so these conditions.

So, the space R 4; so the dimensional space which is equipped with this operation, this way of multiplying vectors that is called quaternion algebra. And as we are going to see in few minutes quaternary algebra is going to be quite handy, while understanding rotations. So, your quaternions space a quaternions algebra is denoted by H; H in the honor of Hamilton as a 4 dimensional space with this operation. So, let us see what kind of things can be can be achieved by all this.

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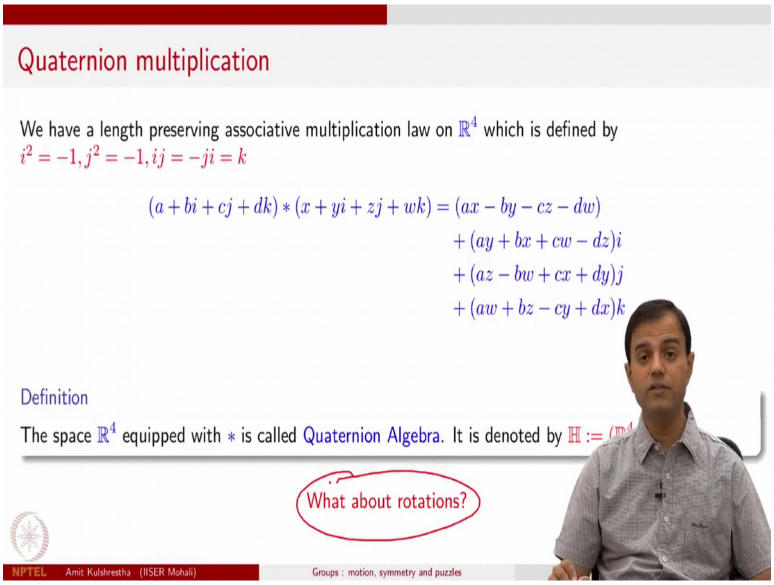

And can we; actually answer our original question which was about rotations.

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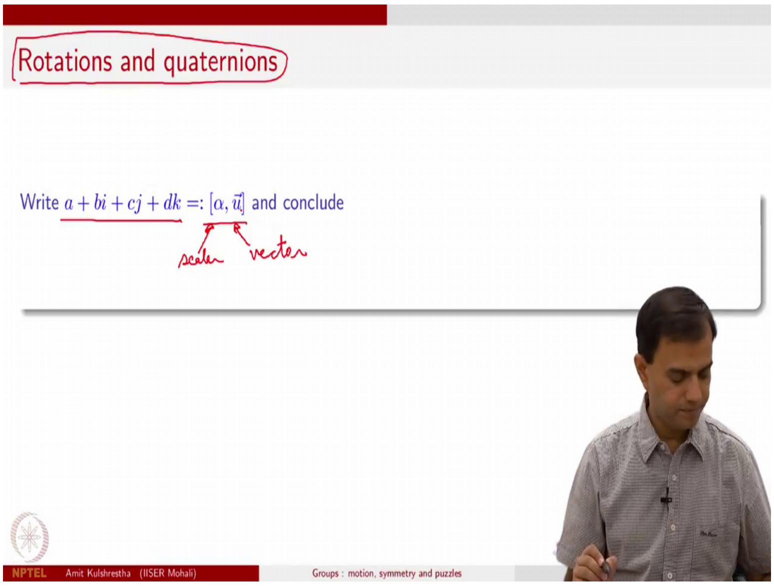

So, here is how we can connect rotations with quaternion. So, whenever I have vector I just write explicitly like this, I keep the scalar part here, and back vector here, 1 dimensional part this is 3 dimensional part.

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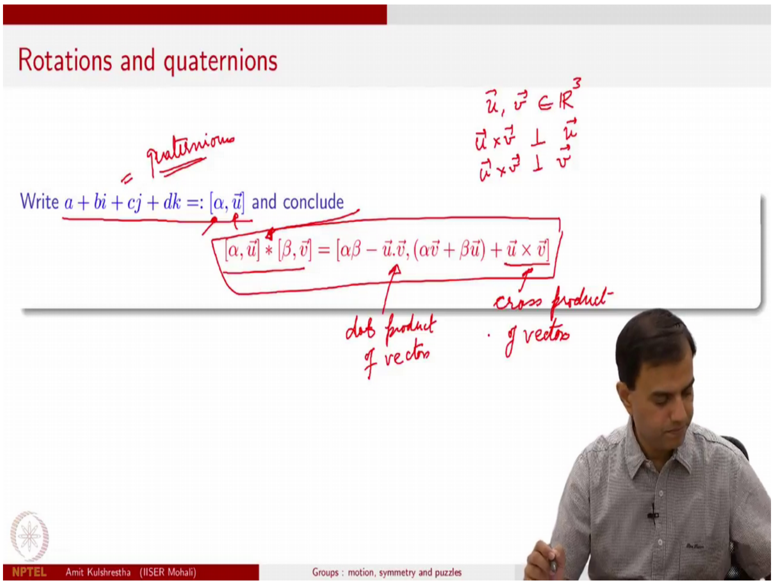

And this is the thing, this is. So, how rotations and quaternions are related? So, we have written this a plus bi plus cj plus dk quaternion, so these elements are called quaternions. So, I have written the scalar part and vector part and when I multiply the star that I had defined earlier; the star that I had defined here is precisely this. So, this is. So, operation of quarter range is precisely this. So, when I multiply in the earlier the fashion in this function what I get is: accompanying that involves scalar product as well as a component that involves cross product cross product of vectors. And this is what dot product of vectors.

I hope you remember what cross product of vector is. So, when you have two vectors. So, u and v they are vectors in R 3. So, I have one vector u, one vector v then the cross product is having direction which is orthogonal to both of them. So, whatever the plane generated by u and b the cross product is going to be orthogonal to that plane.So, u cross v is going to be orthogonal to u perpendicular to u and also its going to be perpendicular to v and its magnitude is going to be magnitude of u times magnitude of v times sin theta where theta is angle between u and v.

So, cross product of vectors is also appearing in this expression.

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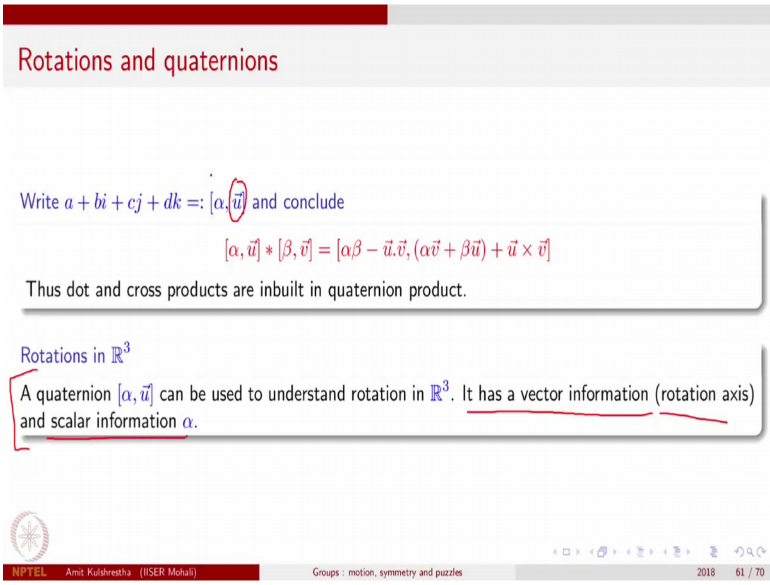

So, as I said dot product and cross product both are inbuilt in a quaternion product. And now we are going to see how a quaternion can be used to understand rotations. It has a vector information vector is rotation axis, and scalar information which is angle.

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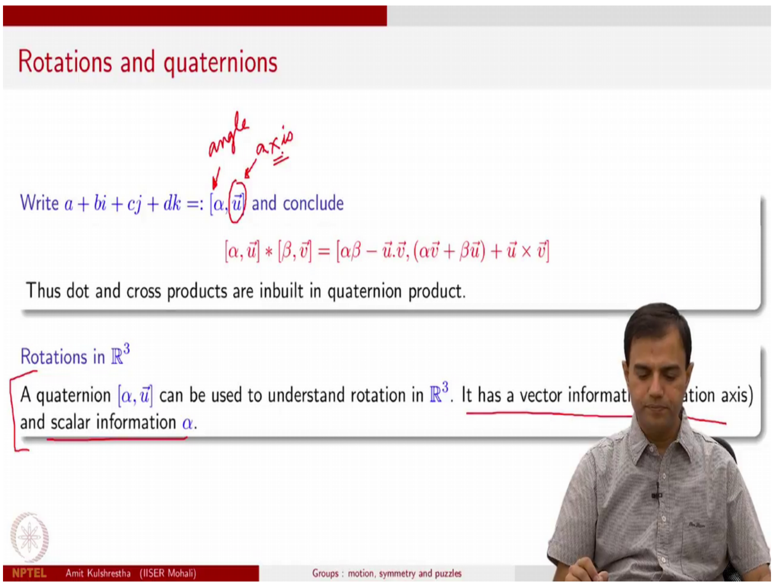

So, this is angle and this is axis. So, both the informations are there in this.

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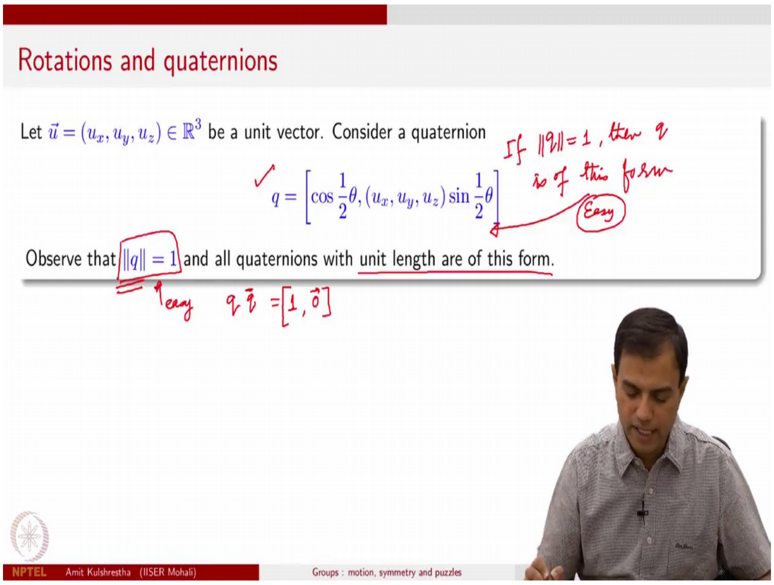

So, this is how rotations can be understood. Via a quaternion you have vector and you have an angle theta. And I am asking you to consider this quaternion, this the vector part, soin this unit vector in this unit vector you just multiply by sin theta by 2, take that as a vector part and consider scalar part to be simply cos of theta by 2.

So, what are the properties of this? Norm of q is 1. So that is easy, easy to see that q q bar is 1. So, when I say 1 maybe you should be thinking like 1 comma 0, so vector part is 0. So, this is. So, you can think of q q bar therefore as a scalar. So, norm of q is 1 and all the quaternion which have this property. So in fact, if you have a quaternion whose norm is 1, then q is of this form, this form q is of this form; that is easy.

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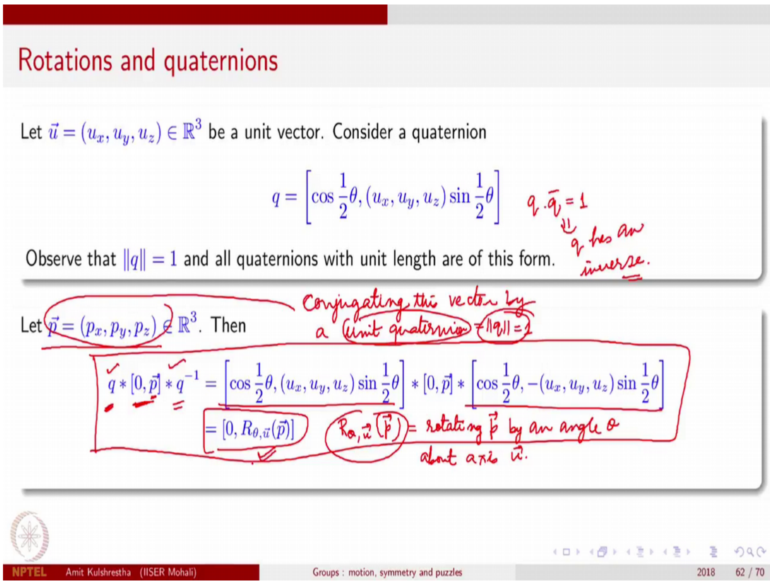

So, all unit quaternion can be written like this. And this is very relevant calculation. So if you take vector in R 3, I am calling 0 p I am considering this vector as a quaternion by putting the scalar part to be 0 and I am conjugating by unit quaternion. So, I am conjugating this vector by a unit quaternion; unit quaternion meaning norm of the quaternion is 1. That is the meaning of unit quaternion.

So, here is unit quaternion, here is unit quaternion. And what is the meaning of conjugation? Conjugation meaning I am multiplying with q in the left and they are multiplying with q inverse in the right. So, since q q bar is 1 it is easy to check that q is invertible q has an inverse. So, q inverse does make sense.. So, I have this and after doing all this calculation through all those matrix notations that I had mentioned earlier one has to realize that this is the scalar part of this calculation is 0 while the vector part is precisely the vector which you obtain after rotating your original vector p by an angle of theta with u as axis.

So, I will just write that: R theta u p is what is rotating p by an angle theta about axis which is determined by u. So, after rotating whatever vector you get that is this. So, this shows that the quaternion unit quaternion and rotations are related.

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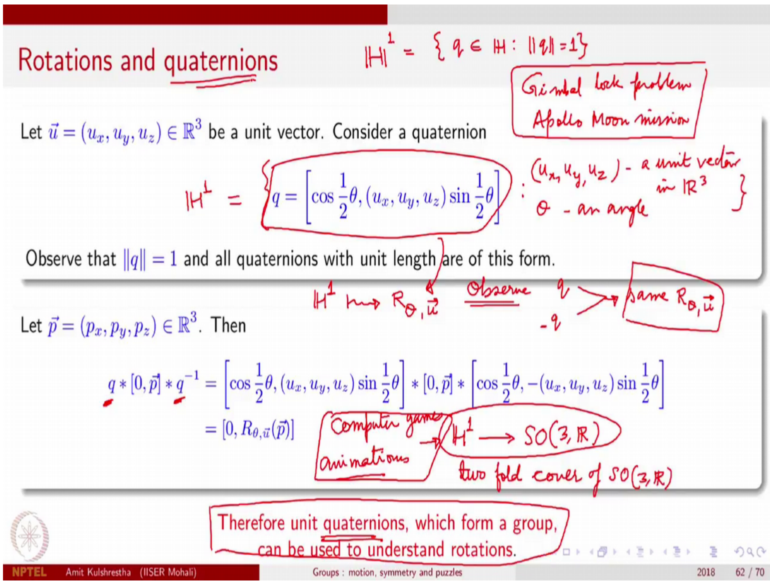

So, with all this we conclude that unit quaternion they form a group and therefore they can be used to understand rotations. So I have H which is quaternions, I call this group H 1 which is all those quaternion whose norm is 1. And I realize that I can write H 1 as this collection here u x u y u z is a unique vector, and theta is an angle. So, H 1 is precisely this collection; sorry.

And as I have shown given H 1 you have R theta u. So, given an element of H 1 maybe I should be saying that given an element this. So, given this q, I have associated to this one rotation. One thing that I should observe is that if I have q, I have minus q both the elements actually give same R theta u. So, q and minus u are going to give you, the same thing because here I am multiplying with minus q here I am multiplying with minus q inverse. So, minus 1 times minus 1 is 1, so that you just go away. So, what we have is same R theta u.

So, therefore, from H 1 I can think of a map to the group of rotations SO 3. And what is the property of this map? Property is that two different elements are going to same. So, this is what is called twofold cover of SO. In fact, this approach of using quaternions is very much used in computer games and animations. There is something called quaternion interpolation. So, that interpolation is very much used in animations, you have one frame here another frame here. Those positions of those frames, those orientations can be understood in terms of quaternions and then it is easier to interpolate what is happening between those two frames.

There have been other advantages of quaternions as well. So, use of quaternions as rotations you should maybe look for what is called Gimbal Lock Problem and Apollo Moon Mission in order to appreciate by our by the representation by our rotation should be expressed in terms of quaternions.

So, what did we learn from this lecture. We learned that the same group which is a group of rotations of sphere, we are denoting it by SO 3. The same group has different ways of interpretations we can interpret the same groups elements in various different ways rotations; not only the notations are different the mathematical treatment of those notations is also different. We had matrix notation which was quite cumbersome, we had Euler angles which were also very difficult to manage, but when we had quaternions it just turns out that everything is about this multiplication.

So, rotations are just about quaternion multiplication. On the way we also see how Hamilton made all those attempts the many of those attempts were failed attempts, but eventually he exceeded in defining something which is analogous to complex multiplication. And thus, he achieved quaternions and then we could use unit quaternions to produce rotations. And in fact, we could obtain a twofold cover of group of rotations in R 3.

There are other situations also where, one type of notation or one type of mathematical treatment is better than other, although as a group both the ways of understanding things are same. So, quaternions are quite good example of that. And in coming lectures we are going to explore more about groups and we are going to see why we should appreciate groups even more.

Thank you.