Groups: Motion, Symmetry & Puzzles
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More applications of Groups
Lecture – 14
Symmetries of plane and wallpapers

Hello so, last time we saw that one can realize abstract group elements as concrete say matrices rotations and all that and I told you last time that we are going to learn representations in the slash, mathematical concept of representations.

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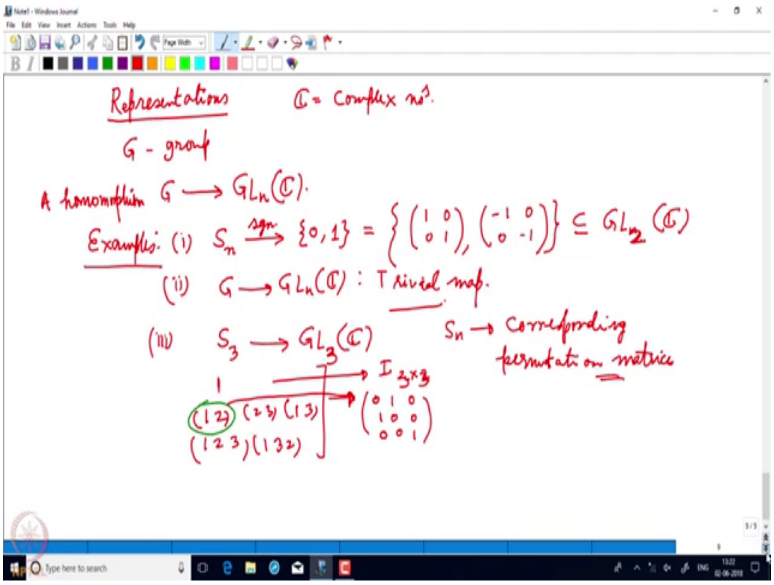

So, what I have is a group and representation is very simple it is nothing, but a homomorphism from G to GL n say complex numbers C is field of complex numbers. So, that is a representation quite simple definition. So, it is a homomorphism which starts from G and which takes venues in GL n C.

So, what is this, this is our attempt to realize any abstract element in the group as a matrix, what is the advantage, we will see after sometime. So, what are the examples, some easy examples we can see. We can think of you remember signature map or the parity map which takes value in I told this is 0 1, 0 is even parity, 1 is odd parity and that can be thought of as a representation because I can think of 0 as say multiplicative identity.

1 as this by minus 1 0 0 1 and that is a sub of GL 2 C, that is very straightforward at some point or simply you could have originating from any group trivial map that is also example of representation when you have Rubik's group for example, and when you map an abstract state of Rubik's group to a concrete matrix which corresponds to the permutation that is also an example of representation.

Few more examples this time let me just concretely take S 3. So, from S 3 to this time I am taking values in GL 2 of C. So, I can think of S 3 as permutation of 3 objects or permutations of or the rotations of rotations in the flippings so, symmetries of triangle. So, there are 6 symmetries are triangle and accordingly you map it to corresponding matrix.

So, let me just write it in this form maybe you have 1, you have 1 2, you have 1 2 3. So, 1 2, 2 3, 1 3 and 1 2 3 and 1 3 2 like this so, I can map it to. So, one I can map to identity matrix 2 by 2 identity matrix 1 2 for example, 1 2 I can map to matrix which swaps firsthand the second rows so, it is just.

So, I am having in GL 3 sorry I am having in GL 3 I 3 I 3. So, this just swapping 2 rows say 0 1 0 1 0 0 and then you have 0 0 1 like that. So, corresponding permutation matrices so, S n goes to in general corresponding permutation matrices. So, that is also a representation there are much more interesting representations.

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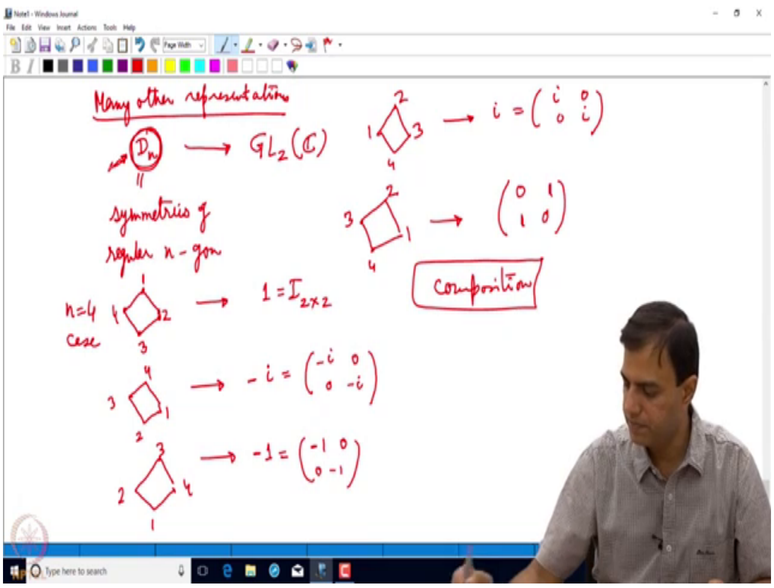

For example, when you have D n which are what symmetries of regular n gon. So, from D n you can give a representation which is a taking values in GL 2 of C. So, what is this representation, it just again it is a kind of permutation so, when you have say for example, the square say n is equal to 4 case there are 4 states. So, suppose this is identity it goes to 1 so, there are for cyclic right.

So, suppose this a state where 1 goes to 2. So, 1 is here, 2 is here, 3 is here, 4 is here. So, it goes to minus of identity. So, this one is actually 2 by 2 identity matrix minus i is this scalar minus i, 0 0 minus i or you could have then it i also (Refer Time:07:42) i or minus i and then now 1 goes to so, 1 2 3 4 like that. So, here it is minus 1 which is minus 1 0 0 minus 1 and similarly you have 1 here 1 2 3 4 and that goes to say i which is i 0 0 i and then the flipping 1 the 1 then this is flipped.

So, I have this so, flipping could be 2 here, 1 here, and 3 here and 4 here and that goes to say flipping off this thing. So, this goes to 0 1 1 0 and then other you can obtain by appropriate composition. So, whatever is actually happening for the definition the way D n is constructed just by taking the regular n gon and considering symmetries first you put all the cyclic orientations all cyclic configurations and then flip it and then once more all cyclic configurations and the way GL is defined using that, you can define a representation.

Here is an important concept for representations, which is that of character of a representation.

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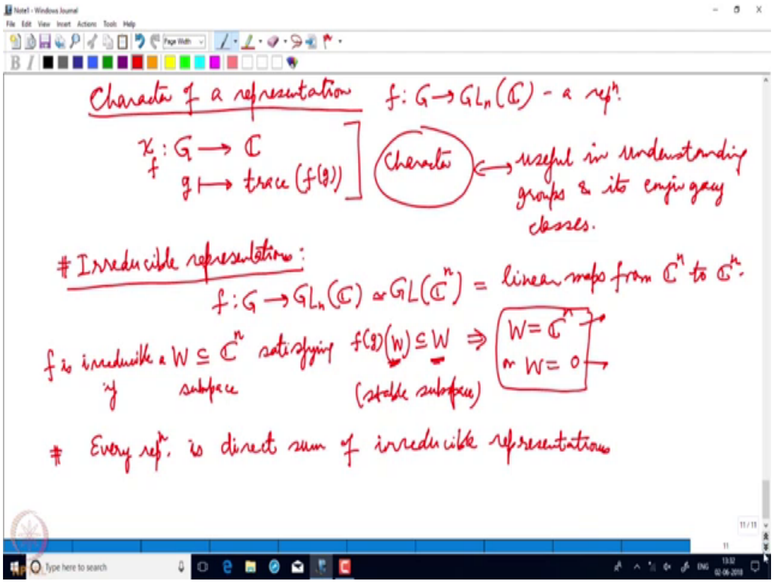

So, what do we mean by character, character is a map from G to C. So, I have a representation let me call it f the representation given this representation I can associate with a map maybe I will call it chi f it is quite character of f what it does is given g to it. So, given g it associates to it so, you can see just say this fg, fg is a matrix. So, I can think of trace of this matrix right and that comes with the various interesting properties.

So, this is point character so, g going to trace of the corresponding matrix character is quite useful in understanding groups in understanding conjugacy classes of groups, useful in understanding groups and it is conjugacy classes and in fact, this notion of what are called irreducible representations is very central to our theory of a characters to the theory of representations

So, what is irreducible representation, so, I have say this representation. So, I can think of GL n C as matrix or I can think of it as automorphisms of the vector space C n. So, C n is a vector space I could have said r n, but theory over r is slightly complicated theory over complex numbers is quite straightforward. So, you can think of it as a linear map from C n to C n. So, I can say that when do I say there are representation this f is irreducible, if you consider a subspace in this. So, f is irreducible, if a subspace which satisfies that f g W is contained in W, so called stable subspace.

So, if stable subspace is either C n or t will 1. So, it is very clear that if I have a subspace 0 subspace then any matrix over 0 is going to be 0 any matrix over the whole thing is again going to take it to C n within C n, but if you can get some non trivial element non trivial subspace which satisfies this property. So, non trivial stable sub space then the representation is not going to be irreducible.

So, it is irreducible if whenever you have stable subspace then there are only 2 options for stable subspace either this or this. So, it is so turns out they (Refer Time: 14:34) representation is made of these irreducible representations, every representation is actually what is called a direct sum of irreducible representations.

So, I am not going to say all that in detail every representation is direct sum of irreducible representations.

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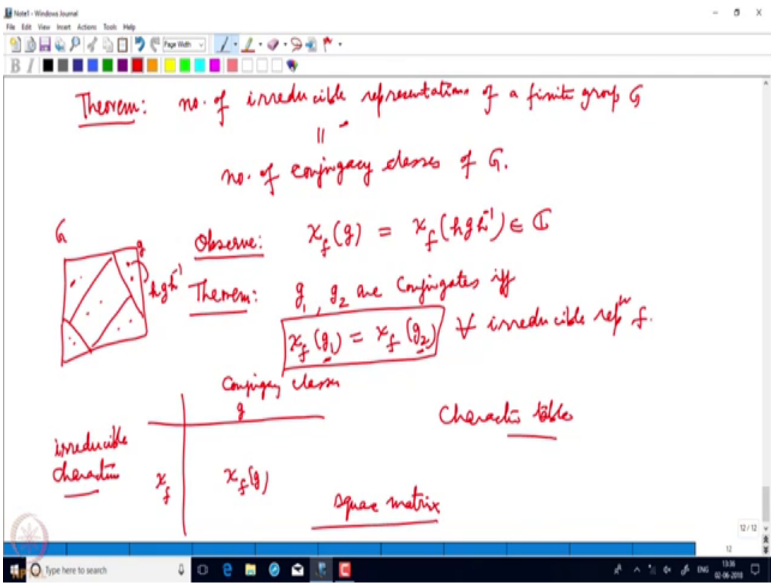

And here is very interesting thing theorem quite interesting thing, which says that number of irreducible representations of finite group G equals number of conjugacy classes of G, what are conjugacy classes, you have a group into break it into partitions on what basis 2 elements are in same partition if they can be obtained from each other by conjugation.

So, any 2 elements here are conjugate to each other and any 2 elements here are conjugate to each other and like that with that kind of equivalence relation you partition your group and each partition is conjugacy class. So, how many conjugacy classes are there is, actually same as number of irreducible representations and in fact, easy thing for us is to realize that if you have a character evaluate it on the element or if you evaluate it on a conjugate element.

Answer is same, same complex number is there because if you have 2 different matrices if they are conjugate to each other than their traces are also same, chi f is just about take the trace or theorem is that 2 elements g 1 and g 2, they are conjugate to each other if and only if chi f g 1 is same as chi f g 2 for every irreducible representation here. So, for all characters all irreducible characters if you are having same value of characters for on these 2 elements g 1 and g 2 then they have to be conjugates.

So, one can think of what is called character table and what does character table do. So, here you have characters, here you have conjugacy classes, say here is a conjugacy classes of g, here is a character corresponding to representation irreducible representation g. So, these are irreducible characters, which are corresponding to irreducible representations. So, here your entry will be chi f g in like that you will make a square matrix what why is it square matrix? The square matrix because of this equality number of rows the same as number of columns, number of rows is number of irreducible characters and number of columns is number of conjugacy classes.

So, therefore, to each group you can associate this data which is called character table there are various interesting properties of character table for example, character table is actually an invertible matrix you can think of it as a complex matrix right and this is a invertible matrix quite interesting property. A important thing is that you can read lot about a group that understand various properties of groups via character tables in via in general theory of characters.

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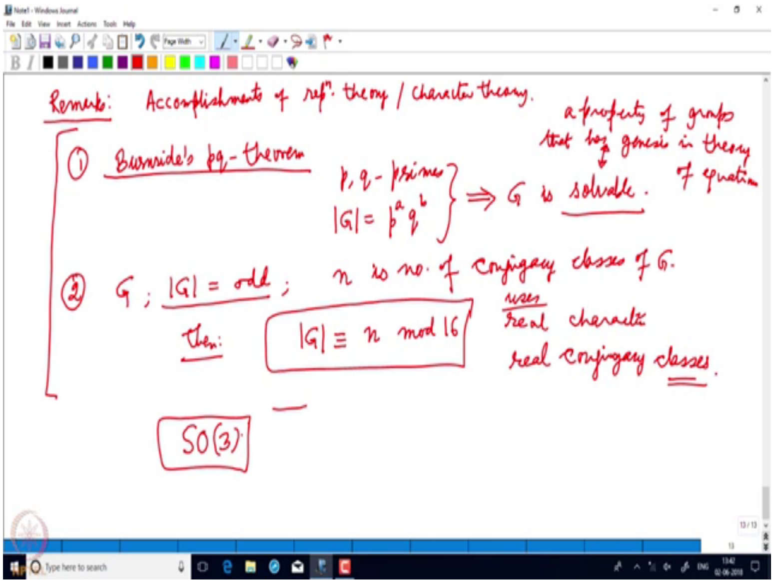

So, some remarks I would mention, these are accomplishments of representation theory or character theory. So, there are various problems in group theory which are otherwise difficult to deal with, but using the concept of representations of characters one can easily deal with those problems and then just tell some historical example what is called Burnside's pq theorem, what it says that if you have 2 primes and you have a group whose order is p to the power a, q to the power b; that means, in the factorization of group they are not more than 2 distinct primes which are occurring.

So, this is just information of the order of the group and nothing else what is surprising is that from this you can conclude that your group is having very special property of being what is called soluble group, soluble groups can be understood in terms of what is called a chain a descending chain of groups the derived chain of group. But let me just say that this, the solubility has something do with in the solubility of certain polynomial.

So, this is the property of groups, solubility is the property of groups, important property of groups that has genesis in theory of equations, theory of polynomial equations quite important property also. So, once side p p q theorem actually this is statement proving in group theory, but one can prove it using corrected theory. Another interesting problem which I will mention is quite curious, we take G group such that the order of the group is odd and then you say n is number of conjugacy classes.

In general computing number of conjugacy classes is not straightforward for a group, statement is that if you consider G and you consider n this is a relation G is actually congruent to n mod 16 quite curious number of conjugacy classes and number of elements the group the difference is always divisible by 16 provided your order the group is odd. This makes use of water called real representations rather real characters and so, called real conjugacy classes.

I would not say much in detailed, but the message that I want to put forth is that you have a group, it is abstract group you find ways to understand it you find different representations of it sometimes those representations could be tangible, they could be there in the real life all the examples I have given earlier through puzzles and toys and some symmetry considerations and sometimes you can realize you can represent those groups in terms of matrices and if you analyze those matrices close enough then you can that then it can reveal lots of interesting properties or groups.

Representation theory is quite a wide subject I just try to give a flavor of that and in fact, not just of groups, but for other objects mathematics as well is the representation theory and it has it is own ways of characterizing reducible representations and all that and purpose of any kind of representation theory association to understand your original object much better and quite often in some combinatorial way or in a way were making computations is quite handy.

So, I hope you put all your somewhat in coming lectures what we are going to understand is something more about this particular group and as I said this is a group of rotations in r 3 some interesting things are coming up and those things will have a some they are coming up as a nice application of group actions so, be with me.

Thank you.