Groups: Motion, Symmetry & Puzzles
Prof. Amit Kulshrestha
Department of Mathematical Sciences
Indian Institute of Science Education and Research, Mohali

More applications of Groups
Lecture - 13
Representing abstract groups

Hello. So during previous lectures, we have seen a couple of things.

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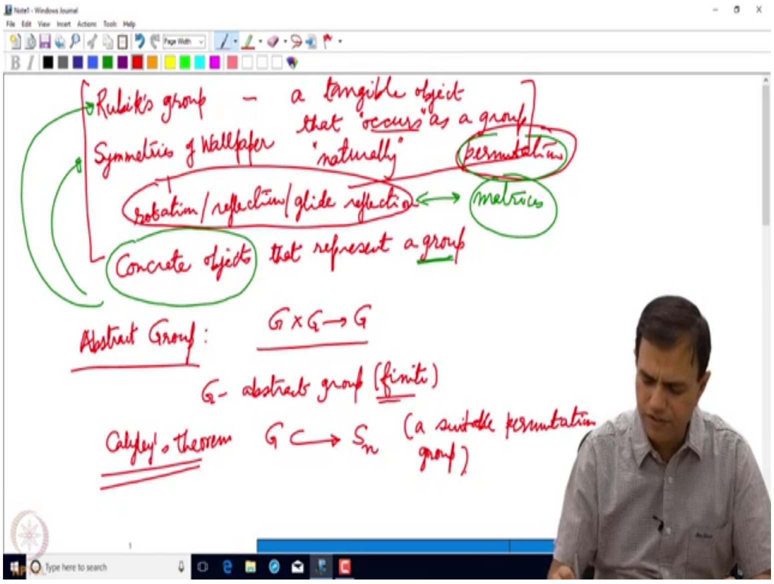

We saw Rubik's group, we saw symmetries of wallpaper. So, what was Rubik's group, it was just group of all permutations, all possible states of Rubik's cube. So, in that sense, Rubik's group or the Rubik's cube is very tangible representation of a group of Rubik's cube.You can touch it, you can feel it, and group is right there in your hand, all possible 4 into 10 to the power 19 states, more than that, they are just in your hand right, or the group of symmetries of wallpaper. Again this picture is representing something it is representing a group. This toy is again representing something that is representing a group, so that is something.

So, Rubik's group is a tangible object that occurs as a group naturally. So, one of course, has to understand what is the meaning of occurs, occurs in the sense that the permutations, which are there all possible permutations, they form a group, but in one single toy, you can see all possible permutations. Similarly, symmetries of wallpapers given a wallpaper pattern given say one tile, you have so many symmetries.

So, for example, rotation, reflection, and other things that I mentioned last time, glide reflection all these things are there, and just through one picture you can understand about a group. So, these are concrete examples of group. So, concrete objects, these are concrete objects that represent a group, so concrete objects there. And group as we know is an abstract structure.

Through these examples, through the example of wallpaper symmetry, through the example of a Rubik's group that abstract group gets a meaning, it gets it gets represented in a where if you say, real life situation. So, for us it is always good, if you could represent a given abstract group, in some meaningful fashion. So, here the meaningful fashion was permutation, and here the meaningful representation was all this rotation, reflection, glide reflection, all these things.

So, here is something quite interesting. Any abstract group can be understood in terms of permutations, any abstract group can be understood terms of matrices, I hope you recall that or rotation and reflection, yesterday we are talking about matrices. So, these two key words, permutations and matrices, they are quite concrete right. And the statement is that any abstract group, what is abstract group, any group that satisfies all those properties in the definition of group, associated key existence of identity, existence of inverse.

So, any abstract group actually can be thought of as group of permutations, any abstract group can be thought of as group of matrices, and that is quite relieving that is quite interesting thing. So, statement is, when you have an abstract group, then it can be thought of as if it is sitting inside S n a suitable permutation group, and that is called Calyley's theorem.

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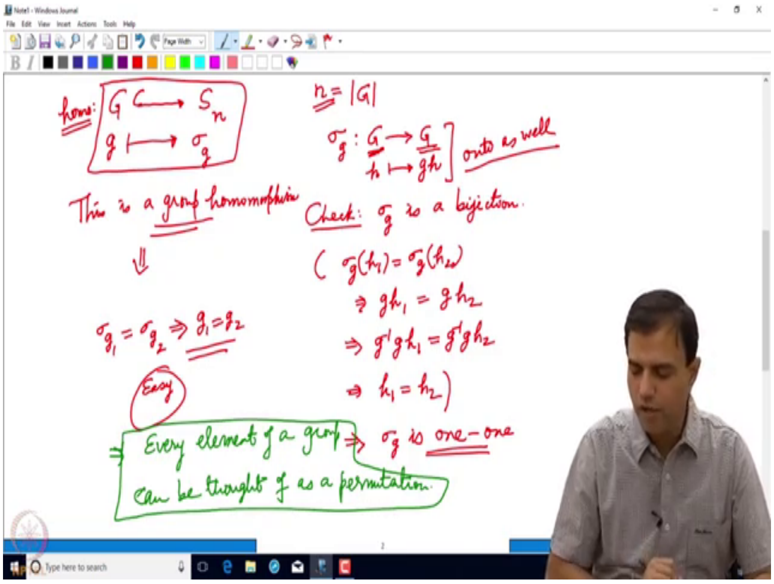

So, what is that how can we realize. And in group as if it is sitting inside, S n. So, here is this idea you have a group, and to this group I want to associate some permutation S n. So, first question, what n, should we take, so here I take n, which is same as the cardinality of G, number of elements of G.

So, here I am taking finite group. So, here in this previous slide, it should have been finite group. All the for infinite groups also one can makes in your statement, but the actually finite, because you want to get S n. So, how do I associate a permutation to an abstract element to a given element of group g that is not very difficult, let me just say that sigma goes g goes to a permutation sigma g, a permutation is of n symbols.

So, what do I do how do I define sigma. So, sigma g is going to be a permutation that is a bijection on G. And how do you give bijection on group using an element, well one way is you just permute in this fashion, you just multiply g from the left side, and it is clear that as a set is set map, sigma g is a bijection. How do we see that is just cancellation property, just left cancellation? So, if I have a sigma g say h 1, and sigma g h 2, that means I am having g h 1, g h 2, both are equal. Existence of inverse allows me to cancel these, so g inverse g h 1 is same as g inverse g h 2. And therefore h 1 is h 2, and that means, sigma g is a one-one map.

So, here is a one-one map, which goes from a set of finite set to itself. So, it has to be onto as well, there is a one-one onto map that means, the permutation right, any one-one onto map; so bijection, bijection precisely permutation. So, this sigma g is actually permutation.

And you can see that this is a group of a morphism, this kind of assignment is a group homomorphism, not very difficult to see. So, therefore, this is a homomorphism, and I came that this is a injection. So, why should it be injection? So, just check that if g 1 is so if sigma g 1 is same as sigma g 2, then you just need to check that g 1 is equal to g 2 that is not very difficult to make out, so that's all. So, what is the conclusion, conclusion is that every group, every abstract group can be thought of as a group of permutations.

So secondly, for n equal to size of g you can, but then in some cases, you can also optimize this, you can also reduce this n. So, this is just one algorithm, one way to see group as a subgroup of permutations. So, everything therefore, can be thought of as permutation, so that is the message. Message is every element of a group can be thought of as a permutation.

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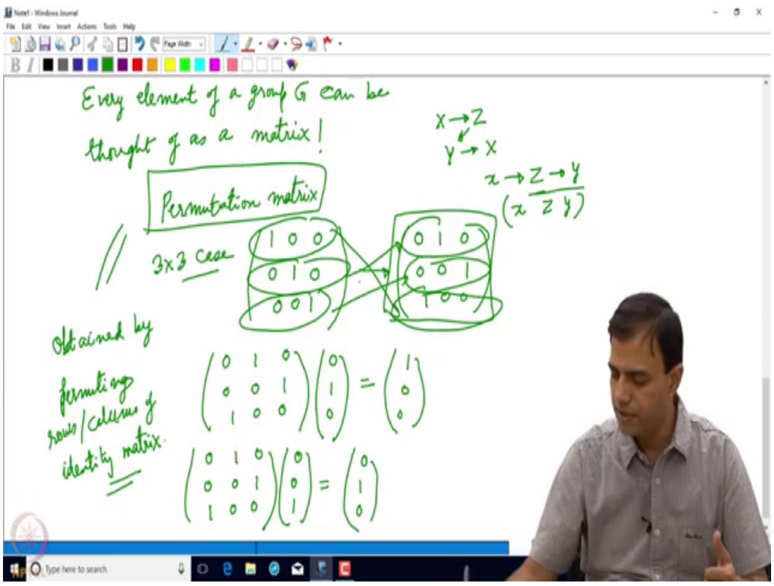

And now, I am making another statement namely, every element of a group can be thought of as a matrix. So, how to see that not very difficult? So, statement is every element of a group, can be thought of as a matrix. Not very difficult when even think of an element as a permutation, you can also think of it as permutation matrix.

So, what is permutation matrix? Well, you take identity matrix, just I am giving an illustration of 3 by 3 case, and then what do you do, just permute rows or columns. So, here let us permute rows, so I have this is 0 1 0, 0 0 1, 1 0 0. So, what happens here? If I multiply this matrix to x axis, so I am taking x, y, z, three axis; so x axis corresponds to 1 0 0, what happens you know, so when I multiply this, I get 0, I multiply this, I get 0, I multiply this 1 last one, I get 1. So, here x axis goes to z axis, what happens, when I do the same calculation for y axis; so x axis goes to z axis, and for y axis what happens, so I get 1 here, and I get 0 here, I get 0 here, so matrix multiplication. So, y axis goes to x axis.

And what about z axis, 0 1 0, 0 0 1, 1 0 0, so I am looking at z axis, 0 0 1, and when I multiply what I get, I get 0 here, I get 1 here, and I get 0 here. So, z axis goes to y axis. So, z axis goes to y axis. So, it is a permutation, x goes to z goes to y. So, x z y, and then y comes back to x. So, this matrix is actually representing a permutation. What is happening, first row first row of this is going to z last row, third second row is going to first row, and then third row is going to second row, third row is going to second row, it is z is going to y. So, this matrix is actually representing a permutation. So, these such matrices, which are obtained after permuting rows or say columns of identity matrix, they are called permutation matrices. So, they are obtained by permuting rows or columns of identity matrix. So, you can apply a given permutation on rows or columns.

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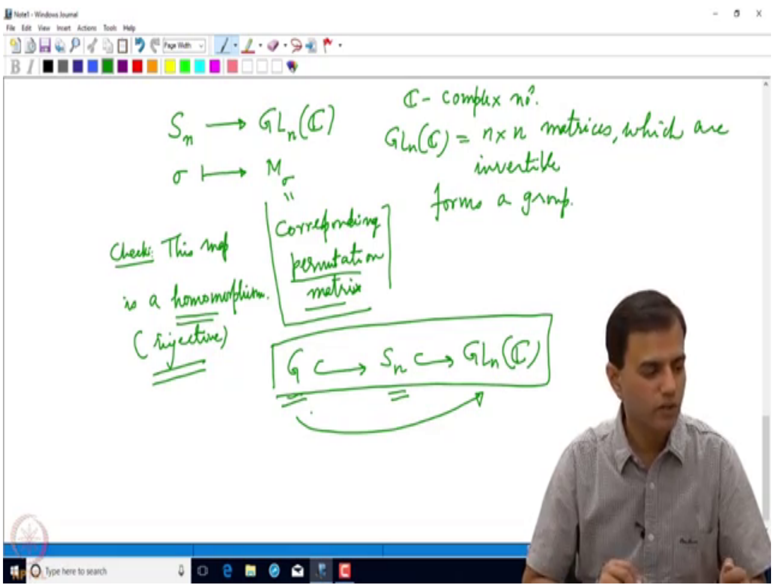

So, here is thing, I can therefore, give a map from S n to group of matrices over a field say complex numbers. So, C or C is the set of a field of complex numbers. And G L n C is n by n matrices, with which are invertible. Those n by n matrices, which are invertible, and it forms a group; so all permutation matrices being permutation of identity matrix. So, permutation of rows or columns, they determinant is either plus 1 or minus 1, because you will be doing that permutation like swapping of rows or columns, either even number of times or odd number of times. So, since the determinant is 1, matrix is invertible.

So, any sigma, any permutation can be performed on say rows of identity matrix, and therefore you get let me just call it M sigma, M sigma is corresponding permutation matrix. And therefore, this map is there, and you have to check that this map this map is a homomorphism. This is a homomorphism, you have to check, whether it is homomorphism. When sigma is permuted, sigma permutes n identity element as columns or rows, so that is the question ok. So, for what notion of permutation matrices, whether it is in terms of rows, or in terms of columns, this is a homomorphism nevertheless.

So, you have a homomorphism from here to here, it is actually injected homomorphism. So, any group sits inside S n, in S n sits inside G L n over a field, say complex numbers. Therefore, every abstract group can be thought of as a permutation, and every abstract group can also be thought of as a matrix. So, matrices and permutations, they are very concrete examples of elements of a group, so that is how these things are useful. So, I am representing every abstract element as a permutation, I am representing every abstract element as a matrix, so that is quite possible.

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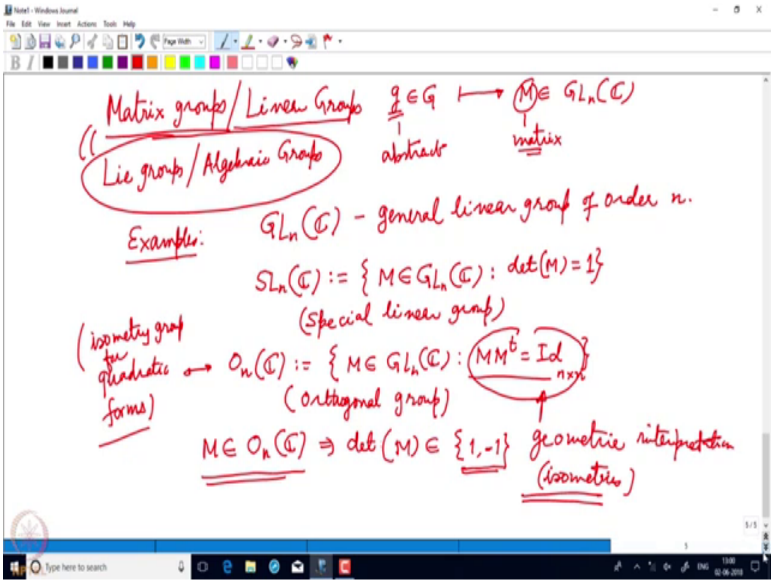

So, with that what becomes important are matrix groups. What is the advantage of considering elements as the matrices, elements of abstract group as matrices? When I have an abstract group, there is no structure to this element G, G is atomic. But, when I think of a matrix, so this is abstract, this is matrix. I can perform various operations of matrix itself. For example, I can compute its determinant, I can compute its trace, I can add up all the entries of it, so many things that I can do with matrix, but within abstract element you cannot.

So, in some cases, it might be useful to think of an abstract group as actually matrix groups. So, what are the examples of matrix groups? There are plenty of them, and in fact, it is a whole theory of matrix groups, they can be thought of as what are called continuous groups, or say lie groups, or in terms of algebraic geometry, or in terms of polynomial operations, you want to understand them, what are called algebraic groups, these are very very rich theory of mathematics having their own special case in mathematics. We are not going to talk about them, but we are going to just give an give some examples of linear groups.

So, what are the examples, I am saying first example that you already know of G L n C, this called general linear group of order n. And then S L n C, what is the definition of S L n C, is those matrices, those elements in general linear group, whose determinant is 1. So, elements have determinant 1 form a group. So, you take 2 2 matrices, whose determinant is 1, the product is again determinant 1, and inverse of matrix whose determinant is 1 is again having determinant 1, so that forms a group called special linear group.

Few more groups, this group is O n C. What is this group, this group is all those matrices in G L n C, for which M M transpose is identity matrix n by n identity matrix. So, here is an observation about such elements. Observation is that if you pick an element from O n C, then its determinant is either plus 1 or minus 1, this is because determinant of transpose of matrix is same as determinant of matrix, and therefore determinant of M square is just 1 ok.

So, this group is called orthogonal group. And in fact, there is a geometric interpretation of this, in terms of isometrics. I am not going to say all this in detail, but this group can also be thought of and in fact, general version of this is closely related to what is called isometric group for quadratic forms. It is quite interesting object has its own theory ok.

So, elements of M elements of O n C elements of orthogonal group, therefore can be categorized in two types, the ones which have determinant plus 1, and once which have determinant minus 1. So, the ones, which have determinant plus 1 are called special the collection is called special orthogonal group.

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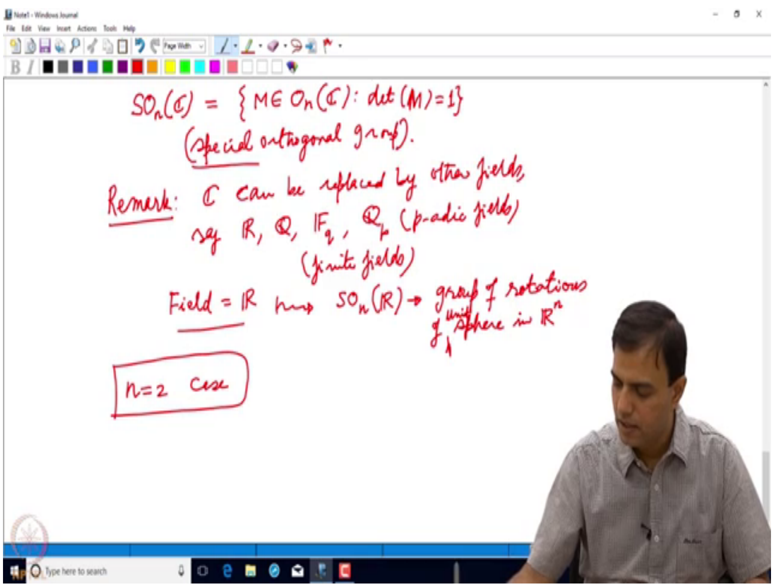

So, S O n c, this is all those elements, in O n, for which determinant, determinant is 1, and this is special orthogonal group There is a reason behind calling them special, I would come to that in a minute. In fact, one particular group, which is S O n, over reals is what we are going to understand in later lectures in much detail. And that is going to be very nice application of group actions. Set a remark, there is this remark, and quite often C is not important, C can be replaced by other fields, say reals, or rationals, or what are called finite fields. Fields, which have finitely many elements, or there are some fields, which have a number theoretic significance, say p-adic fields, or local fields, there is so many fields. So, you can replace them by different fields.

And particular interest is there, if field is a like a field numbers, then say in this case you have say S O n R that has particular significance you can geometrically see this again, and it occurs as group of rotations of sphere in R n. So, what is sphere in R 3, the usually sphere that we imagine. What is sphere in R 2, circle the usual circle, say of sphere or does not matter, you can say units sphere, which has radius 1, So, that is of particular importance. Let me try to illustrate it for n is equal to 2 case that is very easy illustration

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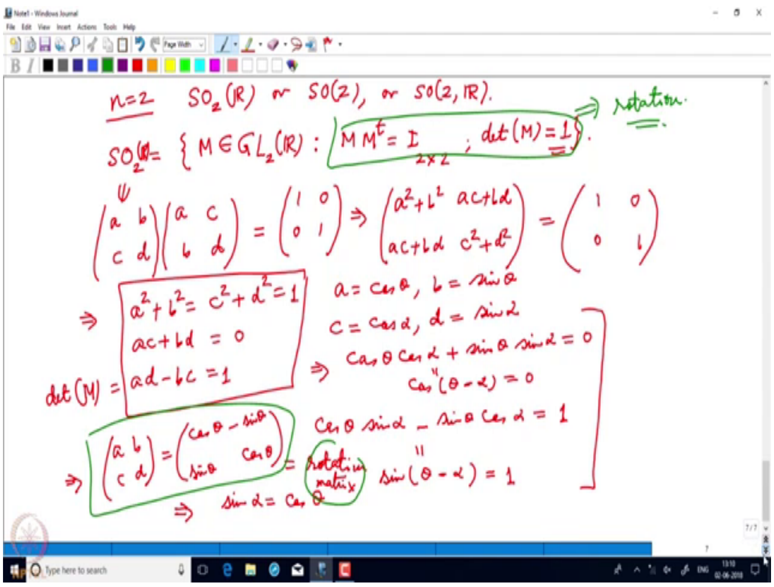

So, what is the condition there? S O 2 R or another notation is simply call it S O 2, or you just call it S O 2 comma R. So, what is the condition? So, condition is S O 2 is collection of those elements, so M in let me write this name G L 2, and I am having, S O

2 R, G L 2 R, the field is that of reals such that M M transpose is 2 by 2 identity matrix, and determinant of M is 1.

So, let us try to understand, what are all such matrices? So, suppose I take a matrix a b c d, which is lying here. So, what is the property of that property is at a b c d, when I multiply with its transpose a c b d, it is going to have determine, it is going to have product to be 2 by 2 identity matrix. So, what conditions do I get from this. So, I multiply, I get a square plus b square here, and here I get c square plus d square, therefore both the elements are a square plus b square as well as c square plus b square are 1, cos square theta plus sin square theta. Maybe in those terms you try to understand, and that is how rotations come into picture. What are what are other entries? So, here I have got ac plus bd, which has to be 0, and again here I get ac plus bd.

So, what are the conditions? Conditions are a square plus b square is equal to c square plus d square is the identity, ac plus bd is 0, and there is one more quantity, which is ad minus bc is a determinant right. Determinant of this matrix, which is ad minus bc that is 1. So, this is what I get. So, this is a condition for a b c d. So, one can think of a as sin of something, or cos of something, something cos of theta, so b will be sin theta, and let me think of c as a cos alpha, so then d will be same sin alpha and what about other conditions.

So, when there are ac plus bd, so ac plus bd is cos theta cos alpha plus sin theta sin alpha, and that is 0. And what about other thing, ad minus bc that is cos theta sin alpha minus, so ac ad minus bc is sin theta cos alpha that is also 0 right. So, what is this, that means cos of theta minus alpha is 0. And from here, I get that sin of theta minus alpha oh sorry this is 1, I just erase this, this is 1 the determinant. This is 1, so sin of theta minus alpha is 1.

And when I make all these calculations I eventually, when I saw you all this, what I get is that sin alpha is cos theta. And eventually what I would get is that your matrix a b c d is essentially cos theta, and d is again cos theta something like minus sin theta sin theta, and that is rotation matrix. So, these elements can be thought of as rotation. So, this condition, these conditions imply rotation, and even in the higher dimension, these conditions indeed imply rotation; so this rotation, which is happening.

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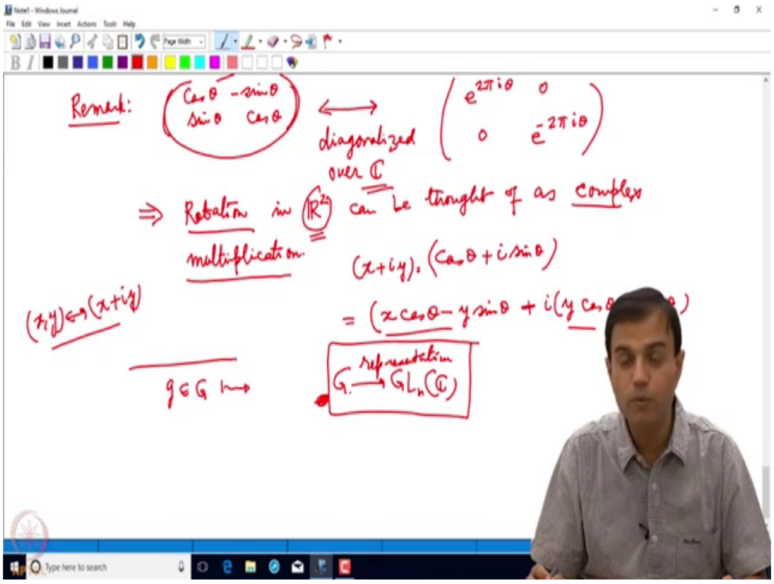

So, in fact, you can think of rotation in higher dimensions as well. And the remark like this rotation matrix, cos theta minus sin theta sin theta cos theta, this matrix can be diagonalized over c complex numbers, just treat them as complex entries. And what you have is e to the power 2 pi i theta e to the power minus 2 pi i theta 0 0. That means, these two matrices are conjugate to each other, provided you allow conjugation by complex numbers, conjugation by entries, by vertices, which have entries in complex numbers.

So, in fact, rotation in R 2 can be thought of as complex multiplication. How is that I can take say cos theta plus i sin theta. And then I can think of x plus i y, for R 3 element, in R 2 x comma y is being identified with x plus i y.

So, when I multiply, what do I get, I get for the real entry x cos theta minus y sin theta plus the complex entry is y cos theta plus x sin theta, which is precisely what this matrix does. So, complex multiplication can be thought of as rotation. We are will talk about higher dimension analog of complex multiplication in next lecture. And that is the case of quarter re-unique multiplication quarter (Refer Time: 38:45) are certain objects, and you are going to learn all that in next lecture. And those quarter means are helpful in the in the understanding rotation in not in R 2, but R 3.

So, what we have seen today that abstract elements in group can be can be thought of as matrices as permutations, they are useful in other things like rotations, and all that into the connection with complex multiplication. Next time, we are going to talk about what is called a representation in the formal way, which is an attempt to make every element as a matrix. And there are certain applications, certain advantages of representations. We are going to talk about mathematical representations next time. But, as of now I am sure, you have understood that groups can be represented in various forms, in forms of puzzles, in forms of some permutation games, in form of some symmetric considerations symmetric considerations, I will show you all those symmetric objects.

So, to know more about representations, watch for the next video.

Thank you.