Groups: Motion, Symmetry & Puzzles
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Symmetries and GAP exploration
Lecture – 12
GAP through Rubik's cube

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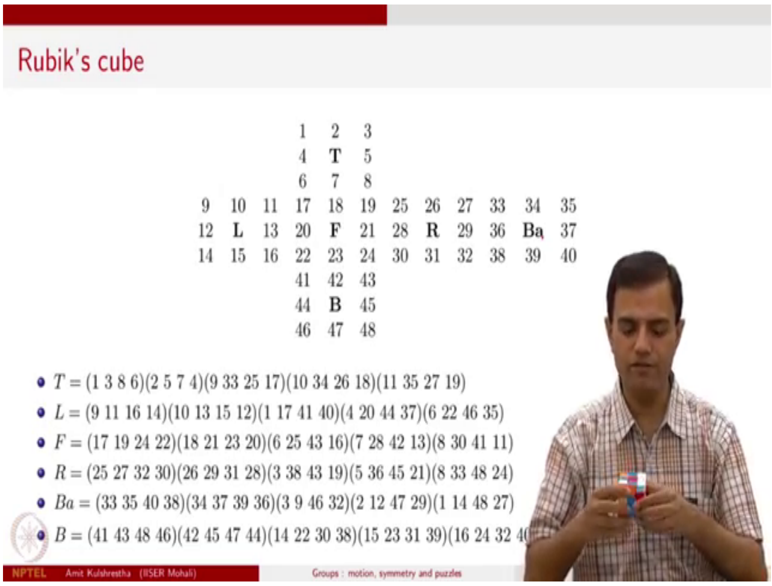

So, as you recall last time what we have done was just putting into in the form of permutation, how all these moves can be written as permutations right. So, this was clear to us a top one and now we are going to see for other phases as well. So, for the left one what is happening. It is quite clear 9 goes to 11, 11 goes to 16, 16 goes to 14 and then 14 comes back.

So, you have like this and then what happens to other one 10 13 15 12 10 10 13 15 12 and back to 10, what about 1 1 goes to this move, so you are moving it. So, this is the thing and the left one is what you are moving right. So, you moving it like this, it from here the clock wise direction right and when you open the cube you realize that what happens to 1. In this move is 1 goes to 17, 17 goes to 41 and 41 goes to well the way cube is packed, this opened cube 41 actually goes to 40.

So, that you have to realize by experimental and similarly what happens to 4 4 goes to 20, 20 goes to 44 and 44 actually goes to 7; again the way cube is packed and then 37 comes back to 4 and similarly 6 6 goes to 22, 22 goes to 46 and 46 goes to 35. So, we can write all this F is like this 17, 19, 24, 22 back to 17 and then 18, 21, 23, 20 and back to 18 and it also affects other sides. So, for that this is being given 6 goes to 25. So, this 6 goes to and we are moving like this, actually goes to 25 and 25 goes to 43, 43 goes to 16 and then it comes back to 6.

So, you have to identify all this. Similarly for write you have this or back you have this. This is back and for bottom also you have this yeah. So, this is what Rubik's cube is made of, Rubik's group is made of all these basic permutations and they have written all these basic permutations. So, if I do the left thing twice, when I move the left face twice I will do this square of this permutation right.

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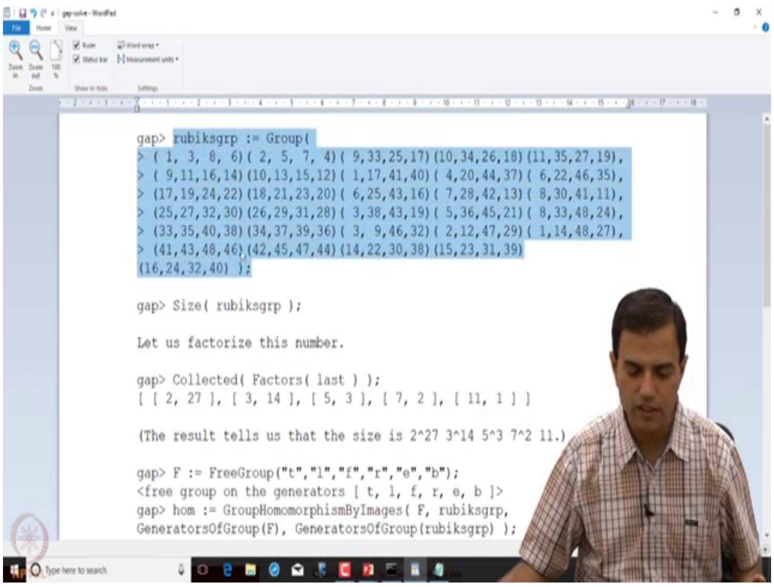

So, now what we can do is, we can try to write it as a group. So, I want to group which is generated by these elements. So, I want to, I want a gap software to remember a group and then group is generated by. These are the basic moves right, this is the moves we know our 1 3 8 6 2 5 7 4, these 1 3 8 6 2 5 7 4 and so on. So, I just cut and paste all these in the software gap software.

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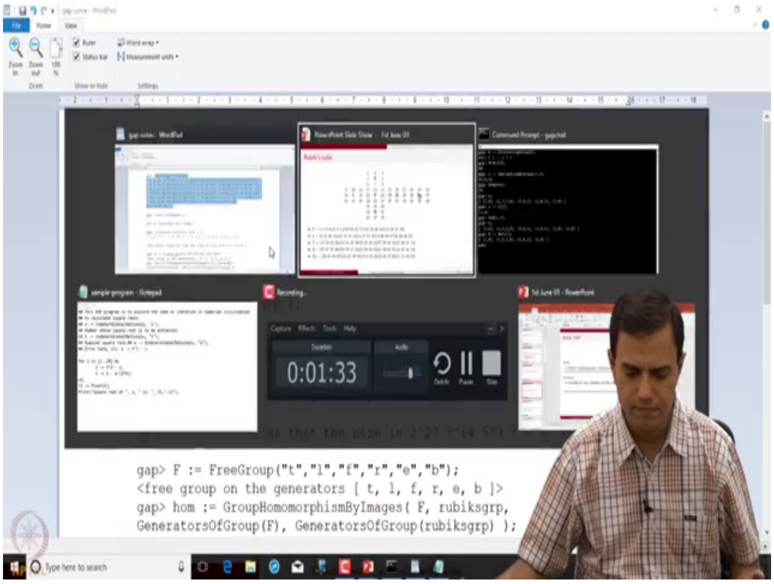

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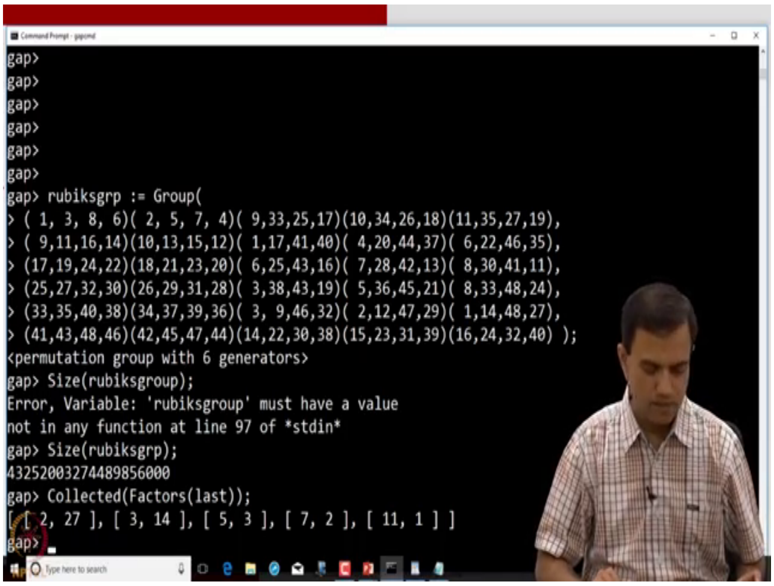

So, I have put all these, and I enter what it says is that I have created a permutation group with 6 generators and you recall what are those 6 generators are, repressively T L F R Ba and B all these things. Those are 6 generators of the Rubik's cube.

What the first thing you would do? Well we will like to determine the size of this, how much you think would it be millions, trillions. Let us see, oh I gave the wrong name this is rubiks grp and this was the size (Refer Time: 05:4) grid, its beyond billions beyond trillions. So, how to rectify this? I will try to write it in the readable form. So, factors of this and I write it collected. I hope you remember last time I mention this command.

So, this is the number 2 to the power 27 times 3 to the 14 times 5 to the power 3 and 7 to the power 2 and 11, multiplied by 11, it is a huge number huge number. So, I would show you and just in I do not.

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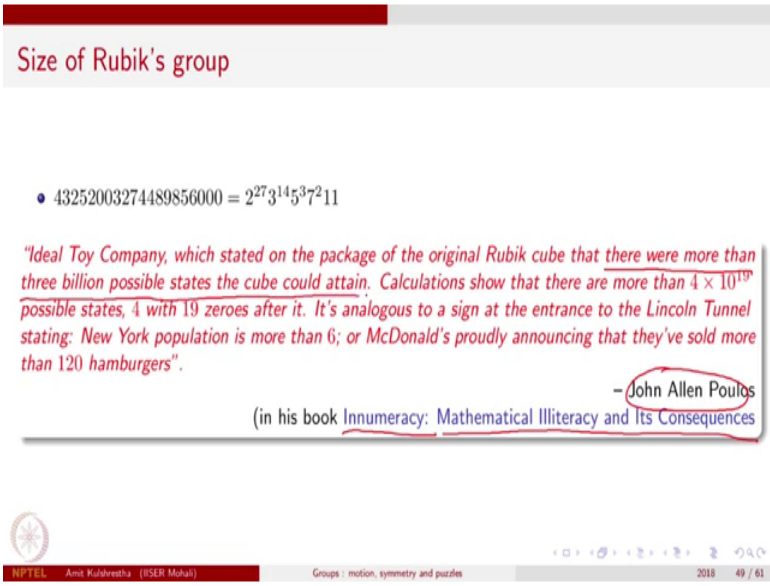

So, this is what we have got right, size of Rubik's group is this much 4 3 2 5 and this huge number the size of the Rubik's group. I will mention interesting quotation from this book called Innumeracy Mathematical illiteracy and its consequences, quite interesting book by John Allen Poulos. And it quotes and this is the beginning days when Rubik's cube was made with this Toy Company claimed, that there were more than 3 billion possible states the Rubik's cube could attain.

They claim with a beginning there are around 3 millions different states. So, this is 1 state; that is 1 state, that is 1 state, exactly 1 of the states is solid states so, the claim that it has more than 3 billion states; however, calculations show.

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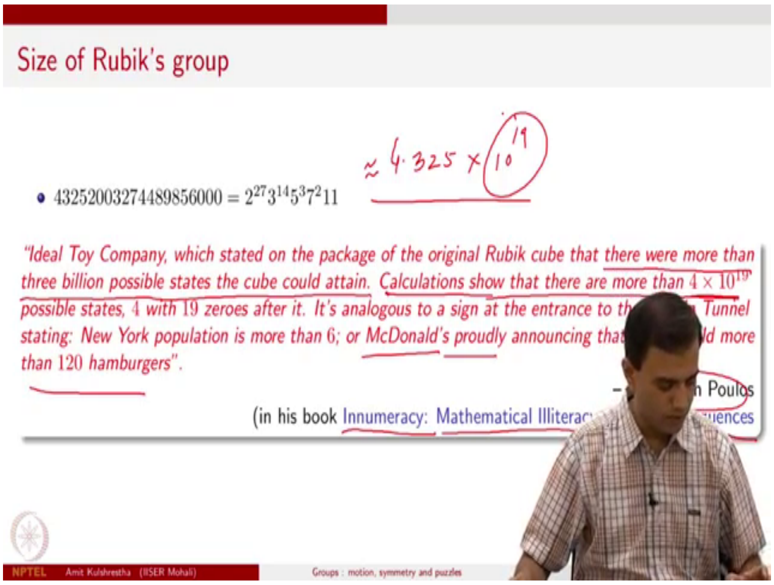

So, the calculation that we did just now using gap, it show that there are more than 4 into 10 to the power 19 right. So, this something like 4.325 into 10 to the power 19, the more than this possible states, 4 with 19 0's after it. And then he gives beautiful analogy; say this analogous to a sign at the entrance of Lincoln Tunnel stating New York population is more than 6, its analogy; or McDonald proudly announcing that they have sold, they have sold more than 120 hamburgers.

So, this is the huge in a accuracy, this is the orders of magnitudes of a inaccuracy that was there in the beginning. Anyway let us come back, let us come back to our gap program by realizing that there are so many of the order of 10 to the power 19 different states that Rubik's cube could attain. So, we have, we have already factorize this, the huge numbers, so this is the size and now is the crucial thing.

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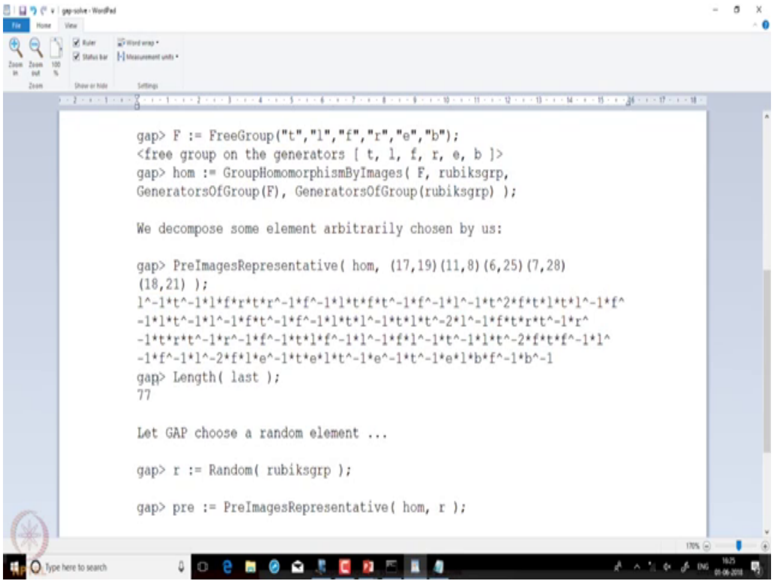

So, we have realized that there are 6 generators of it. So, how you actually solve? So, first I construct a free group with 6 generators. So, I just call those generators as t l f are e, just because I do not want to use b twice and I am just using the denomination e. So, I am just having free group on these 6 generators. So, let me just do that.

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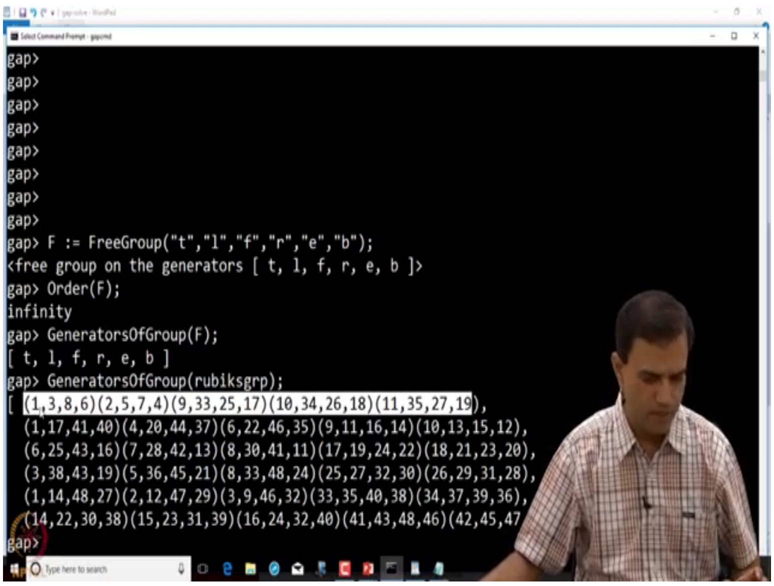

So, free group on 6 generators. What is the order of it? Well free group is infinite just for fun let us see, order of a F infinity, it is a quite smart ok. So, I have constructed free group which is 6 generators, and now I construct homomorphism from the free group two Rubik's group; that is crucial part. So, free group is something where operations, all the symbols are getting multiplied with themselves freely.

So, if I do it four times, I know I am going back to identity position, but in the free group anything raised to the power is non identity, if the original thing is non identity. So, there are no relations in a free group. So, I want use free group to understand this. So, for that I will construct a homomorphism from free groups to Rubik's group and how to I do this. Here is command, so I group homomorphism by images. So, I specify the this element goes to that element, this elements goes to that element like that, I would specify the homomorphism. So, let us see what are the generators of group F and what are the generators of group Rubik's cube.

So, first generators of, this is a command, I will phase like command here. So, generators of group f r these, and then generators of the group Rubik's group are all these elements right. So, I will map t to this, t which is the element of free group to this. Then I will wrap l to this and then I will map f to this like that, and then I will generate a homomorphism. Homomorphism is supposed to preserve relations, but then there are no relations here. So, there is no not much of the worry.

So, if t goes to this, the t square will be going to square of this. So, in that fashion, I am going to have my homomorphism and that homomorphism I can define in gap using this command; group homomorphism by images. And I am giving homomorphism from the free group f to Rubik's group and generators of f which are these t l f r e b, they are going to generators of the Rubik's group, which are precisely the basic moves. So, those, those basic moves are being mapped too ok.

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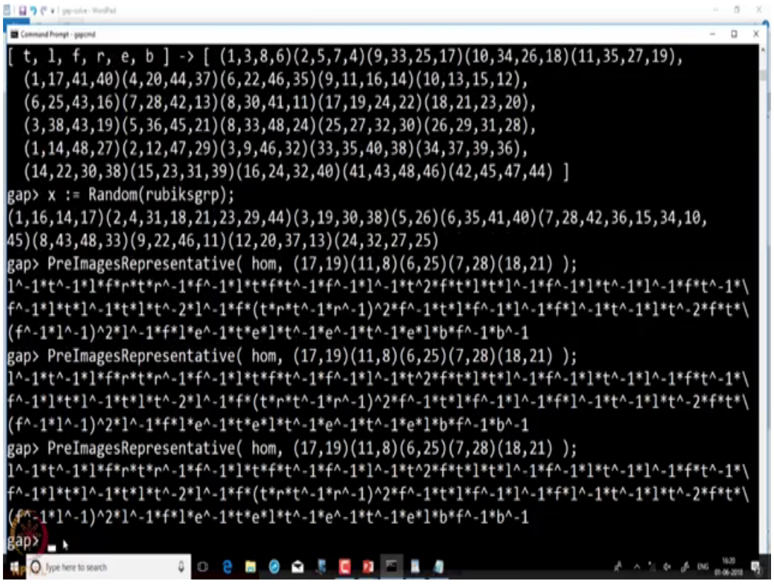

So, let me define this, copy it. So, this is phase stage. So, homomorphism is created. So, homomorphism this goes to this. So, first key goes to that and l goes to that like that, l is left and f is front and all that. So, I defined homomorphism ok. So, here is a free group, the homomorphism is defined to the Rubik's group. So, I will not worry about relations, because free group is not having relations and then let, let us just arbitrarily picks some element in the Rubik's group and look at its pre image.

So, this element I have just written, but here let us see, let us pick some arbitrary element. So, I am taking, say x is a random element of Rubik's group, some element x and now what, what do I do? I consider pre image of the of that element. So, here I just consider pre image of that element or the some other element, but let me just take

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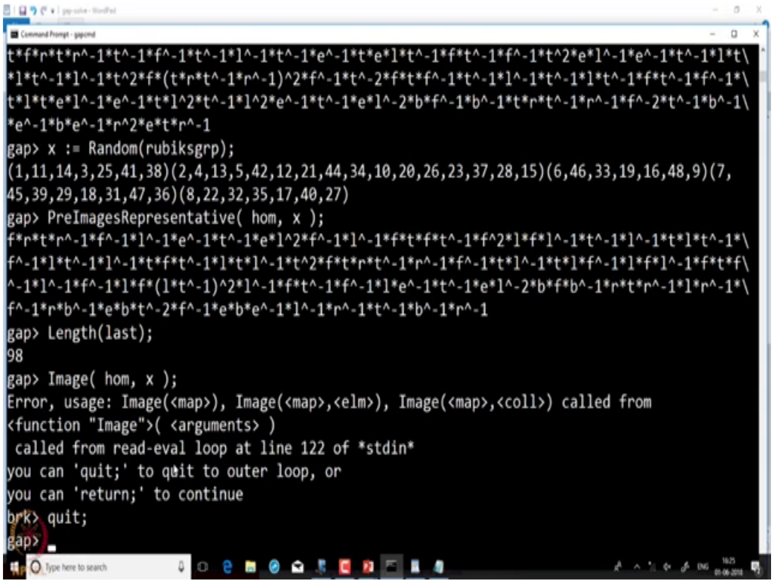

So, in case of that I am takes some elements; say x, the some element x and the pre image of a that element in the Rubik's group. So, let me explain what I have done so far.

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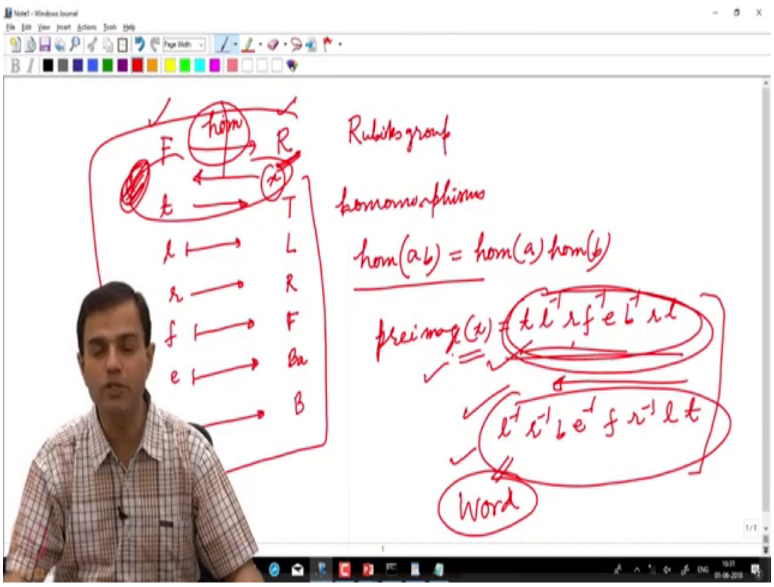

Here is Rubik's group, let me call it the huge one, let me call it R and I have recognize generators of it, top, left, right, front, back, bottom and then I have define a homomorphism, some 6 symbols for this there is no symbol back I did not want to use B again. So, like that I have defined a homomorphism.

And as we would recall, as I am saying this suppose, this is called this is being denoted by hom hom, and hom hom of a b is same as hom a hom b. So, for that reason automatically the t square will mapped to t square like that. So, I have defined this. And now what I do? I pick an element here. What is an element here? An element here is the state of the Rubik's cube, some random state of the Rubik's cube that random element

So, that is where I had picked random element here you remember, this one I have I have this random element and then I am looking at the pre image of this random element. So, pre image of random element in this. Suppose x is random element, it is going to be huge, there is no one single element in f. So, this is not 1 1 right. This is after well in finite group, this is finite group. So, the huge sub set of f which is there in the pre image of x. So, I am just picking one representative from it. So, that representative will; of course, mapped to x. Anything that I have take in the x at is going to mapped to x that is in the pre image.

So, I take a representative in the pre image and that is a representative, this just is f into r into t into r in inverse into f inverse into l inverse and e inverse and so on like that, there is pre there is representative in the pre image. So, I do that and how much is the length of that. Well let us see, length is also a command, length of whatever is the output from previous command, last is k it is 98. In some earlier program that I had written, that I had done calculation form, it was 77 ok.

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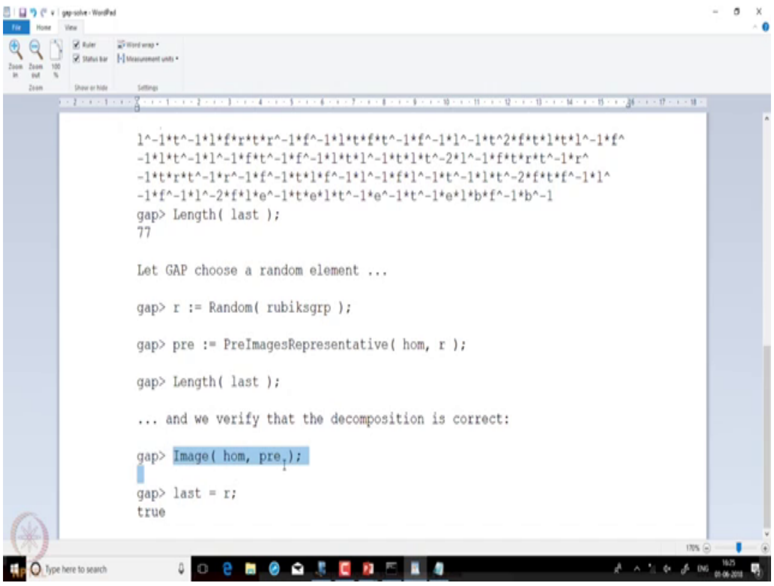

Now, I have already asked gap to choose a random element that is what we did the calculation, and we can actually look at the image of this. So, what you are (Refer Time: 18:19) x we can look at the image of that x, not pre, the is x, that is the image.

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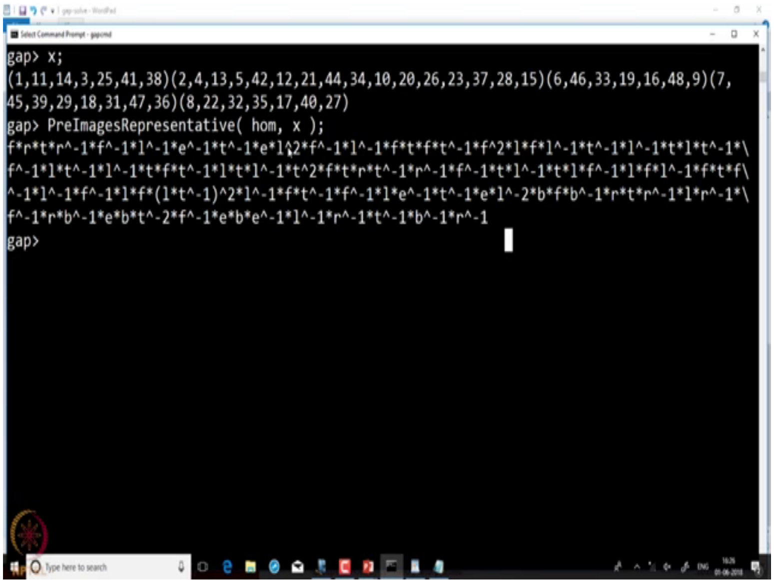

So, let me just see. So, pre image is this one, and what we do with this pre image, we just that all our moves. So, we get that x is. So, our x has some, this random element x is certain combination; say I am just for illustration I am taking say pre image of x to be, say just for illustration; t times l inverse r f inverse e b inverse r l something.

So, that is x right, that is the state. So, in order to get the identity element what do I do? I just start with from here other side right, l inverse r inverse b e inverse f r inverse l t. So, when I apply this move then I will be solving my cube.

So, this tells you how I have obtain this state from the identity right, how I have reached this state from identity, but that is reading in the free group. But if I do it in the reverse direction in the free group, what it does is actually solves the cube for me. So, I hope things are clear to you. So, if certain identity position is there, I am is unscrambling the, I am scrambling the cube and unscrambling is precisely taking inverse in the free group in that is what I am doing.

So, although this might not be the most efficient way, because the pre image that I have paid, is just one of the pre images. I am not taking any, I am not taking the pre image which is shortest in length. So, that is the drawback, but nevertheless through all this we can understand that, we can store first of all the Rubik's cube in gap and then we can solve using gap using the idea of homomorphisms, at free groups are playing important role. And as you can see here all these moves, the sequence of all these moves is nothing, but a word right. This expression as you recall is a word, word in the free group. So, that is all.

So, everything is a origin of free group, all these moves that are doing. You should try to realize them as elements in a free group right. So, we can you, we can use this these simple ideas by constructing Rubik's cube group and then by considering homomorphisms. And then just taking the inverse image of any random, inverse image of any random element and then taking a the inverse of that word and that is, that is it that is precisely one can easily solve Rubik's cube. Although, I am not caving the x going to be efficient way, because the pre image which is going to come is not the shortest one, is not of the shortest length, that is a small drawback.

Similarly what we did for 2 1 by, what we are did for 3 by 3 cube. We can also do it for 2 by 2 cube and this time situation is much simpler and the group which is there. Earlier the group was s 48 and this time everything is moving the 6 faces so, 4 into 6 minus 0. So, as 24 is going to be the group in which all the permutations are going to lie, it would be interesting exercise for you to find out how much is the size of the Rubik's group for 2 by 2, and it wonderful if some of you, those who can easily download cap software then those people can solve Rubik's cube using this method.

So, now, I hope you understand why we introduced free groups and homomorphism, the purpose was to understand how gaps stores all these things and how we can use those things to solve cube, to Rubik's cube in this case. So, it is permutation puzzle. So, all the permutations puzzle, which concern scrambling and unscrambling something, they can be understood using these ideas. So, I hope you could follow, I hope you could enjoy. If there is any difficulty you can always contact us.

Thank you.