Groups: Motion, Symmetry & Puzzles
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Symmetries and GAP exploration
Lecture – 10
Symmetries of plane and wallpapers

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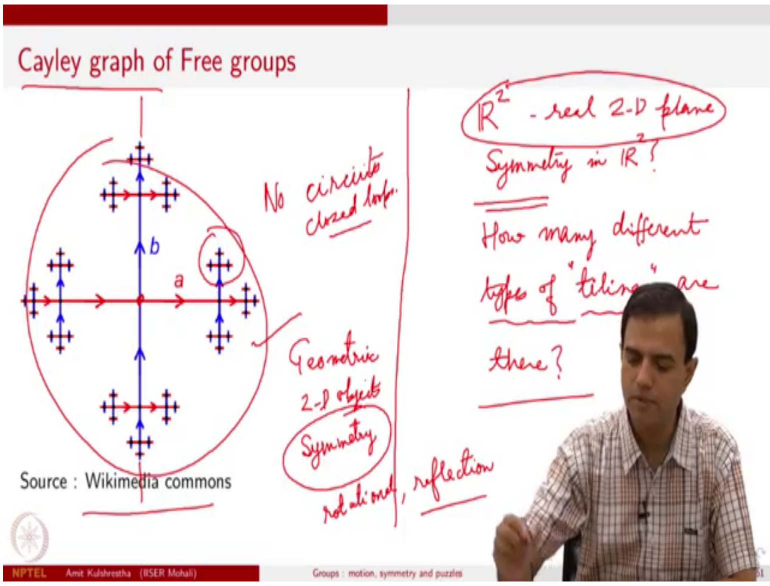

So, we are back last time we did some platonic solids we studied symmetry of the platonic solids. And we also saw certain graphs, Cayley graphs, you remember I tried to draw the Cayley graph of free group with 2 generators. Here is a nice picture I got it from the Wikimedia, and the key thing is that there are no circuits here. There is no closed loops, like this. Just one minute sorry I should have put it should have cut it.

So, there are no closed loops in this we shall play with Cayley graphs after sometime. And as you can see this as an object as a geometric object as a geometry it is a 2D object it has certain symmetry. What kind of symmetries? For example, if you rotate it by 90 degree, there is a symmetry if you reflect it about see one of the axis the y axis, there is a reflection symmetry and there they could be more symmetries in this.

I am going to shows again 2 dimensional objects certain pictures which have beautiful symmetry and see that, right very nice pictures quite fascinating some of them are very classic they are found all across the world in various monuments. Our own Taj Mahal is an is a beautiful example of symmetry see this there are so many, there are so many of them, very simple one.

So, how to understand symmetry in 2 dimension in 2 dimension in R 2? R 2 is plane real 2D plane then how to understand symmetry. How many symmetries are there all those are questions or if I ask you to look at one of these pictures say this one and I asked you how many symmetries does it have. Another interesting question I can ask here is wallpaper, wallpaper symmetry. You have seen all the wallpapers which are there that is tiling with those wallpapers you can also use them as a flooring for the flooring of the surface of the of the floor using certain tiles maybe something like this.

So, what are the different types of tiles one can actually make so that they can complete the white paper or the tiling pattern that is an interesting question. So, question is how many different types of tilings. So, one has to understand what is the meaning of different types of tilings are there interesting question before I answer all these questions let me just talk of certain symmetries in R 2 at the elimination all types of symmetries in R 2 and as you can guess they are infinitely many symmetries, let me see.

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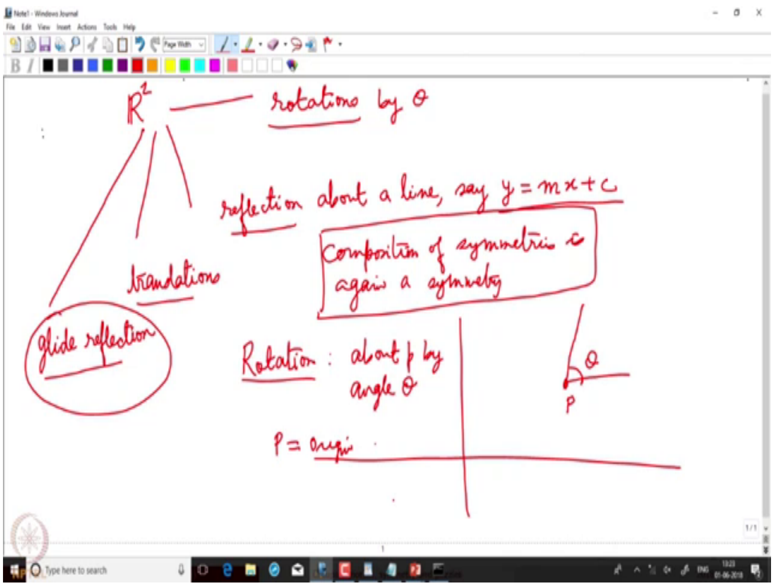

Let me first define certain types of symmetries you are aware of rotations, see rotations by any degree theta and then reflection about a line say y is equal to mx plus c. I hope you remember equation of line m is the slope. What else? Translations, the translations I am going to talk about translations. And then there is very interesting type of symmetry which is there quite loud in nature which is called glide deflection. So, I am going to talk about all these 4 types of switches rotations refraction, translations and glide reflections. And as you would recall in one of the earlier lectures I had mentioned that composition of 2 symmetries is again a symmetry, composition of symmetries is again a symmetry.

So, what should be all possibles which is a R 2? They should be compositions of what I have written here, yeah I am going to explain one by one what all these switches are particularly glide reflection which totally you might or not have heard off before I start I would like to show you some interesting artwork. This one, it is a beautiful artwork on the plate the Russian style of painting and can you observe certain symmetry here. Well, if I just rotate by 180 degree there is a symmetry can you observe any other symmetry in this well that is all that is all these symmetry it is having rotation by 180 degree quite beautiful and making such artwork as I am sure quite difficult thing. What you see again? This and I just rotate by 180 degree same.

You remember once I told it for the symmetry here I am showing you and then on my back I am doing something and if I show you and it looks like same. So, that you cannot detect what was done what was not done that is symmetry. So, doing something that you can detect is symmetry.

Rotation as you know, I can this R 2 this is just for difference every point is same as any other point when we talk of R 2 as a geometric object any 2 points are same there is no reason for us to distinguish between distinguish 2 points. So, what is rotation I pick a point p and then I decide an angle theta certain angle theta say and they rotate everything by an angle theta about p, about p by angle theta. So, one can express all this in terms of material oh in terms of matrix. So, if p is origin then I can express rotation as a matrix.

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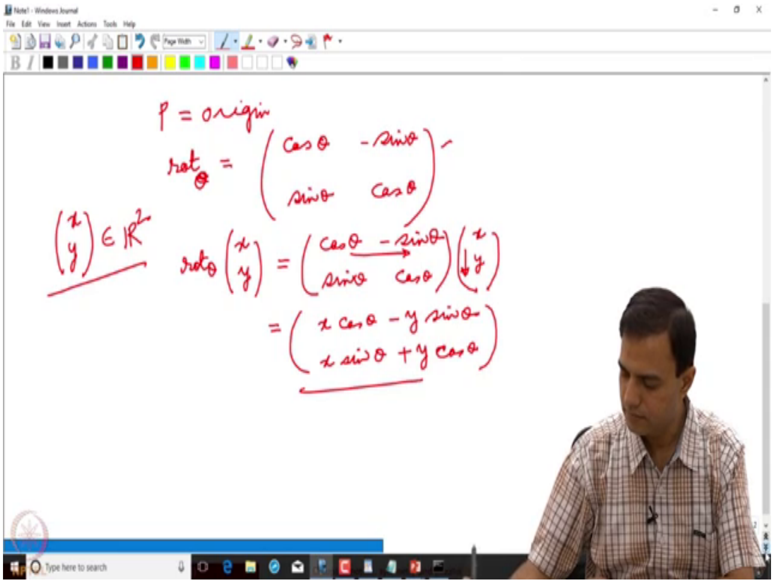

So, if p is origin then rotation matrix rotation by theta is having very nice expression. It is equivalent it is multiplication by this matrix cos theta minus sin theta sin theta cos theta. So, what is the meaning? Meaning is if I have a point xy, so I am writing it as a column as a point in R 2 then we rotate this point about origin by angle theta the answer is simply matches multiplication of cos theta minus sin theta sin theta cos theta with this column matrix, that is why as you recalled matrix multiplication I multiplied this row with this column. So, this is x cos theta minus y sin theta and then I have x sin theta plus y cos theta. So, that is this new coordinate. So, rotation matrices can be used to express rotations.

So, rotation is simply as I am saying you have this you fix a point just rotate as simple as that, you are already aware of that. And then we have reflections, right.

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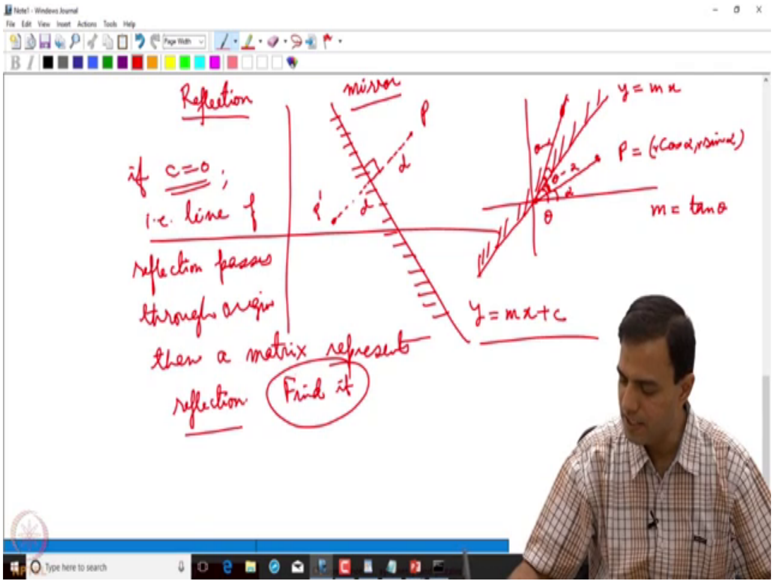

So, for reflections what do we do we pick a line and we, so here is R 2 just for reference. I pick any line say y is equal to mx plus c and then they reflect about this. If there is a point p here I just and suppose this is 90 degree distances d, I further go by distance of d that is all there is a point p dash.

So, I can reflect my points in R 2 and again you can use certain kind of matrix to understand reflections if the point line passes through origin. So, if c is 0 yes that is line of reflection line of reflection is just as mirror you can think of it as a mirror, right. If c is 0 then that is the line of reflection passes through origin, then one can actually write then the matrix actually represents reflection, it is not very difficult to find out again in terms of cos theta and sin theta. So, for example, suppose this is our mirror this is a mirror line through which we are supposed to find reflections and suppose this is a point say p let me write this point p as say cos alpha sine, if I just taking the point to be on a unit circle or maybe let me simply write it R cos alpha R sine alpha.

So, that this is alpha angle and the distance of point p from origin is R and suppose this is like y is equal to mx, then this angle is theta so m is actually just tan theta and then for reflection what we are supposed to do this, this is angle theta minus alpha. So, we have to go further by angle theta minus alpha and see what happens to this point because for reflection this angle theta minus alpha will be same as this angle theta minus alpha and then see what happens, that is not very difficult find this and this matrix is actually going to be in terms of cos 2 theta and sine 2 theta. So, matrix of reflection is also there.

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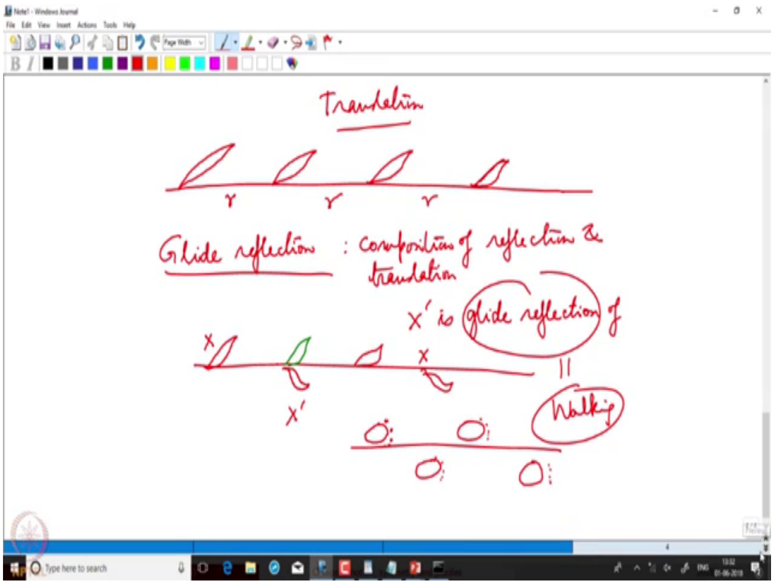

And translation is easier thing I have say I am just I have something here I am just translating it, a similar shape just assumed that the (Refer Time: 15:56), is this ok. So, this is suppose distance R, further distance R, further distance R like this, the same pattern is being passed on to via distance R that is our translation. So, I can do translation by any number, any distance R, any R. So, there are infinitely many translations.

And then there is quite interesting one which is glide reflection. What is that? It is a composition of reflection and translation, ok. So, here is some object let us take this one some object I translate it and then I reflect it. Let me take some other color I translate it and then I reflect it something.

So, suppose this object is x x dash. So, x dash is glide reflection of x. This kind of pattern is observed it leaves and then it goes like this like this and so on, it is quite natural. Actually have you seen the footprints when you walk suppose you are walking on a muddy road the way your footprints are there see they actually form a glide reflection pattern. So, left leg and, right leg their imprint their footprints are actually glide reflections of each other. So, as you are walking your one leg is here other leg is here, and then next leg is here, next leg is here like this. So, that pattern this glide reflection, so very naturally.

So, when you walk then you have glide deflection. So, 4 types of symmetry is we have seen glide reflection, translation, reflection and rotation and then you can compose them, right and there are so many possibilities, compose them all.

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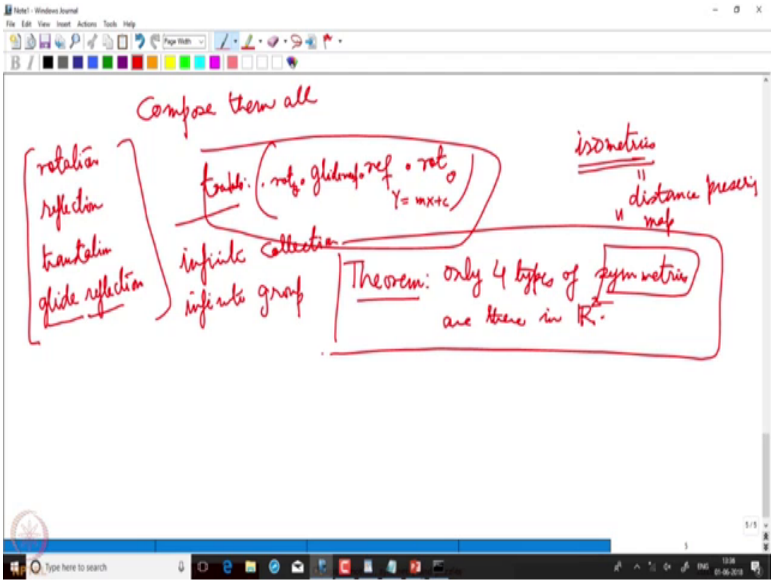

So, which ones rotation, reflection, translation, and glide reflection, compose them all. So, there could be some more complicating things. So, how many of these are there? Rotation you can do by any angle reflection you can again reflect by any line translation, by any amount you can do glide reflection again you can pick any axis of reflection and any amount by which you will be translating.

So, this is an infinite collection. So, what you have is actually infinite group. And you may expect there are some complicated patterns out there say for example, I can rotate by certain angle say theta then I can compose by certain reflection by certain line and then I can do. So, I can glide reflection then I can rotate again I can make life more complicated. So, life could be more complicated, right I can have lots of compositions rotation reflection glide reflection and rotation and then further maybe some translation. What is interesting is do not get anything new. What is the output of this is either are single irradiation or a single reflection or single translation or single glide reflection that is amazing

So, only 2 therefore, there are only 4 types of symmetries which I have already mentioned and that is a theorem, a serious theorem. So, it is a theorem only 4 types of symmetries are there in R 2. So, when I talk of symmetries here I mean isometries, right. So, here isometries meaning, distance preserving maps. If I have 2 points x and y after performing one of these symmetric operations the distance should be preserved. So, if x goes to x dash, y goes to y dash then the distance of x dash and y dash distance of x dash and y dash is same as distance of x and y distance between x and y. So, there is a theorem there are only 4 types of symmetries in R 2. And now let us have some fun I have shown you all these cards, right. There are so many of them, what you can do is you can try to identify what symmetries what symmetric patterns do they exhibit.

Let me show you one, this one would you like this as one of the wallpapers their room. Does it exhibit extraordinary symmetry? Well, not quite probably you would like something like this as a wallpaper pattern as your wallpaper, this has more symmetry. So, what do I mean by that? So in fact, on the basis of symmetry one can classify all these possibilities whatever are the wallpaper patterns into certain numbers and that number I am going to tell you later. Let us discuss what kind of symmetries are there in.

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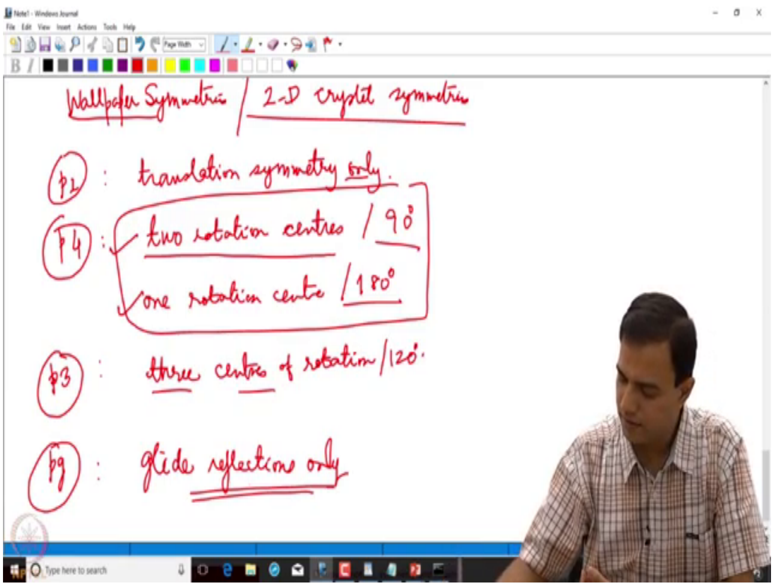

So, there are wallpaper symmetries or 2D crystal symmetries take this one. I mentioned 4 types of symmetries which one you observe here only translational, right.

So, there are various names of these symmetries. So, this is called p 1, p 1 is the pattern where only translational symmetries are there, not much interest. Here is another one let me show you this one, this is called p 4. What does p 4 has? p 4 has 2 rotation axis or maybe I should call center, this 90 degree. So, what are the rotation centers here? 90 degree. If I pick the center of this and then I rotate that is one or I can actually pick the center of this one and then I can rotate now you imagine not just this square getting rotated, but assume that this is a wallpaper. So, you have this or there is a tiling you have this and then the similar tile, here similar tile everywhere and then what is being rotated it is the whole pattern, symmetry operations happening over the whole system the whole wallpaper.

So, I can take this and I can rotate the whole wallpaper system about this again you get a symmetry. So, there are 2 centers this one and this one both are 2 rotation centers 90 degree and then there is one rotation center which is a 180 degree. Can you identify that 180 degree center? Well, that is somewhere here on the edge you take this rotate the whole wallpaper cycle, rotate the whole a wallpaper pattern by 180 degree.

So, when wallpaper or single tile observes this kind of property this kind of phenomena that with similar kind of tile you exhibit you just complete the wallpaper pattern and then observe there are 2 rotation x centers of 90 degree in one rotation centers of 180 degree and there is no other symmetries then we call this type p 4. The one with only translational symmetry is called p 1. Let me show you few more, this one.

So, this is example of what is called p 3, p 3 wallpaper symmetry and what is going to keep it symmetric or keep the whole wallpaper pattern symmetric. There are actually 3 centers of rotation, and they are by 120 degrees which one are those can you identify. Well, you take the hexagon take the center of the hexagon rotate or you take centre of the right triangle rotate or you take the center of the green triangle and then rotate. So, there are 3 types of centers of rotation in their by 120 degree. So, that is called p 3 symmetry.

And then there is this one and name is pg just named. What does it have? Can you observe carefully? So, this has show you guide reflections only. Just carefully look at it glide reflection as the record is a combination its composition of reflection and a translation it has only drive reflections and (Refer Time: 29:28) pg. Now, I have so many of these right, I have shown you I have so many of these all these and we have seen that there are infinitely many symmetries for R 2, how many symmetries will there be for wallpaper patterns, that sounds interesting.

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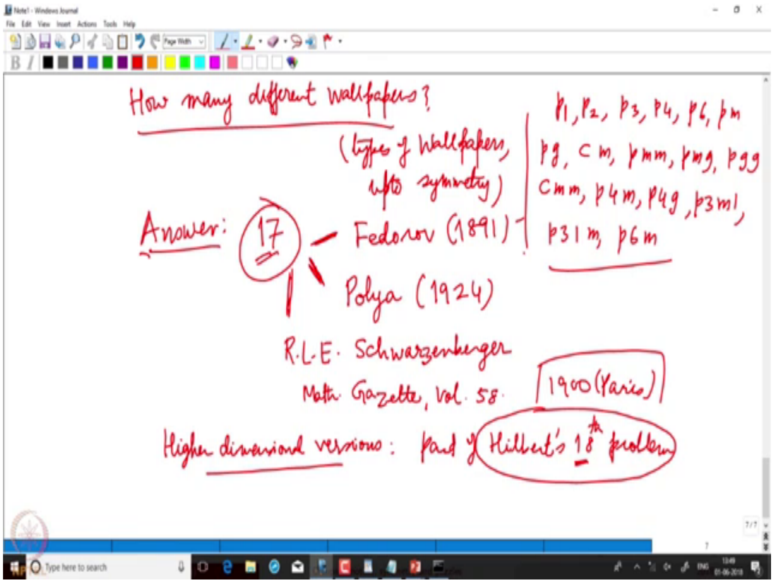

So, how many different wallpapers matrix, different wallpapers within types of? The answer is quite interesting and it is not very easy, 17 very curious. So, there are 17 different wallpaper pattern; there are 17 different tiles types of tiles that you can make up to symmetry, up to symmetry considerations.

So, who found all this and how did define it. So, this is Fedorov in 1891 he had observed all this and then much famous person many of you must have heard of him Polya, George Polya who 1924 wrote article on wallpaper symmetries. And very nice reference for this is an article by R.L.E Schwarzenberger and this article appeared in mathematical Gazette, mathematical Gazette volume 58. In fact, there are some higher dimension versions of this and in higher dimensions higher than 4 the question is still open and how many different wallpaper patterns are there, and high dimension versions were kind of part of Hilbert's 18th problem.

Those who know what Hilbert's problems are in the year 1900 in Paris, Hilbert proposed 23 problems and that was kind of his expectation from mathematicians for last for next 100 years what he would like people into work upon. So, he proposed 23 problems and problem 18th has a portion where he is asking about higher dimension versions of this theorem of Fedorov which says that there are only 17 wallpaper patterns. And there are certain names, I can just mention some of those names here you are seen p 1, p 3, p 4, p g and then there are many in fact, I just write all those names, p 1, p 1, p 2, p 3, p 4, p 6, p m, p g, c m, p m m, p m g, p g g, c m m, p 4 m, p 4 g, p 3 m 1, p 3 1 m, p 6 m; I hope here 16, 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 here and this nomenclature is according to what kind of symmetry is do they exhibit. In fact, one can divides algorithms on how to determine what kind of wallpaper symmetry is there is one of the wallpapers and one of the tiles, it is not very difficulty device once you understand what all these symmetries are.

So, this time we do not talked about symmetry next time we are going to talk about one interesting software that is used to understand symmetry that is useful understand groups and we would make that software solve Rubik's cube for us. So, keep watching.

Thank you.