Groups: Motion, Symmetry & Puzzles
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Groups as they occur naturally
Lecture - 01
Permutation, symmetry and groups

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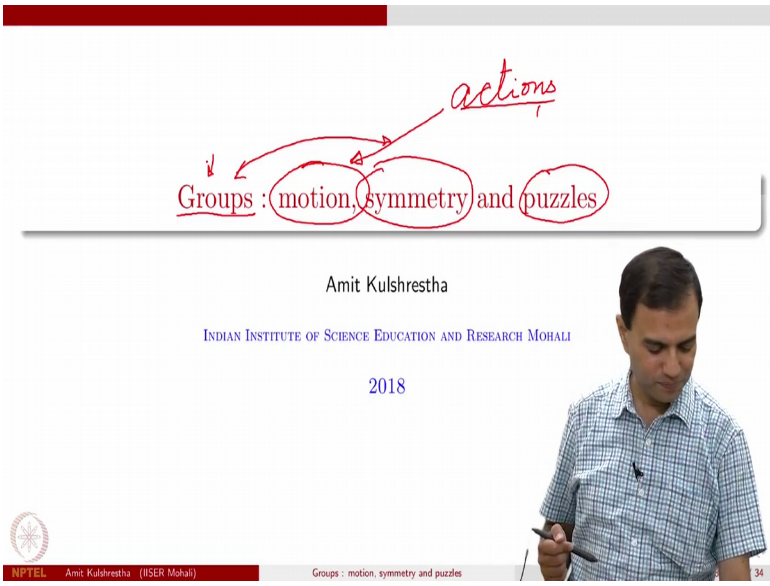

So, welcome to this course. This course is about groups, many of you have seen what groups are, but many of you are also scared when groups were being talk in your class. Nevertheless, we are always happy with puzzles right. There is many interesting puzzles which are associated to groups and we are going to learn in this course.

Symmetry is another aspect which fascinates everyone, which is everywhere in science, which is there in nature and in fact, groups were originally created to understand symmetry and it was actually symmetry of equations. We will see all that in this course. And motion is key word, usually motion is associated to physics but the word motion here essentially it represents what is called action of groups group actions. You might not be aware of what actions are, some of you may be scared of what groups are, does not matter.

In this course we are going to learn groups right from the scratch. We are going to learn why groups were created and what are the applications of groups in other works of life say puzzles or symmetry or many other things. So, for quite long time I am not even going to define what group is. So, if you do not understand what group is does not matter. First we have to see the examples of groups which are there which were there right from the beginning with us.

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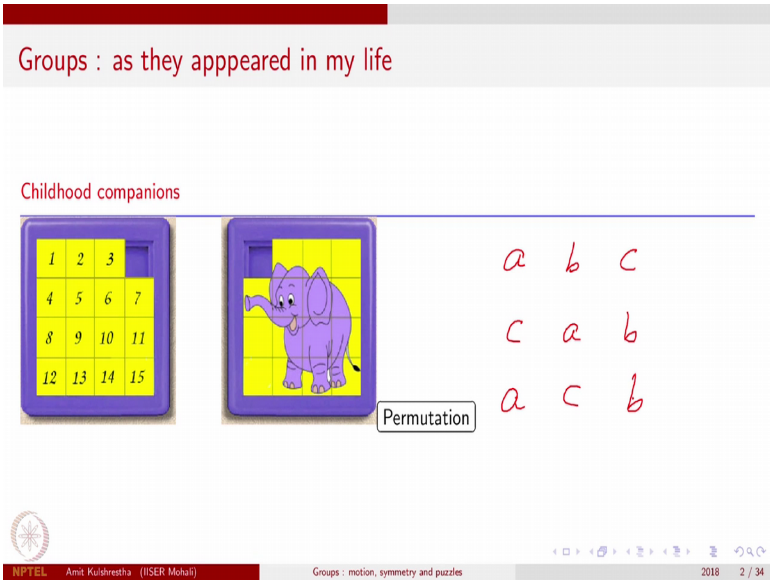

Have you seen this? This is a beautiful puzzle that I used to play when I was a kid and I have one version, this one. You know what we are supposed to do here? This jumbled small a tiles are there jumbled from 1 to 15, I just unjumbled it so that everything is on place, everything is from 1 to 15. What does it have something? What does it have to do with groups? Well, there is something called permutations. Permutations are very basic examples of groups. So, the set of all permutations it has the structure of group.

So, what is the meaning of permutation? You have various objects say you have 3 things just name them as a, b and c and then you make some shuffling. So, shuffling is the permutation. After shuffling c may come here, maybe a comes here, b comes here or may be after shuffling a remains here, c comes here and b goes there. So, there are various ways in which these 3 points can shuffle themselves shuffling is very much there when we play we play cards that card games. So, shuffling is permutation. So, there is lot about groups already there in this puzzle we will see that in this course.

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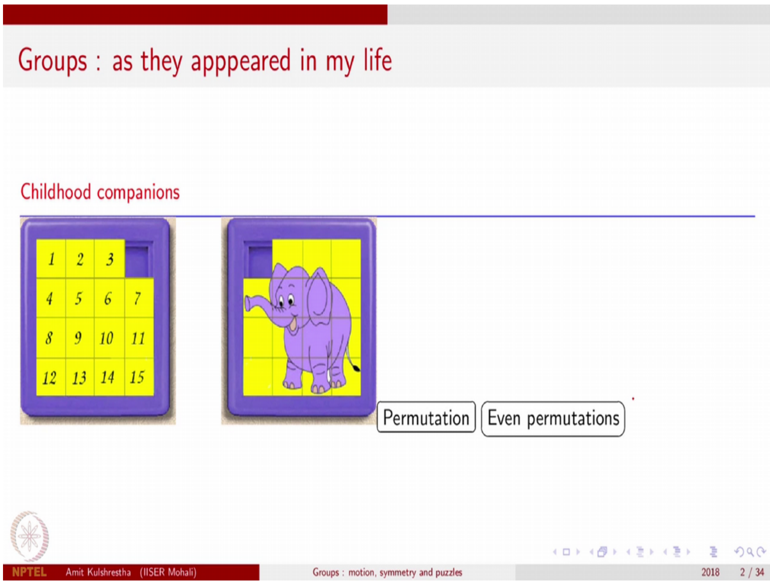

And the key word is actually even permutation, because here tiles all these tiles which are there they are not allowed to move freely because they are there are some restrictions there are 4 sides are there these tiles cannot just go out and no permute themselves freely there are some conditions. In fact, what turns out is that these are what are so called even permutations which are which are relevant here.

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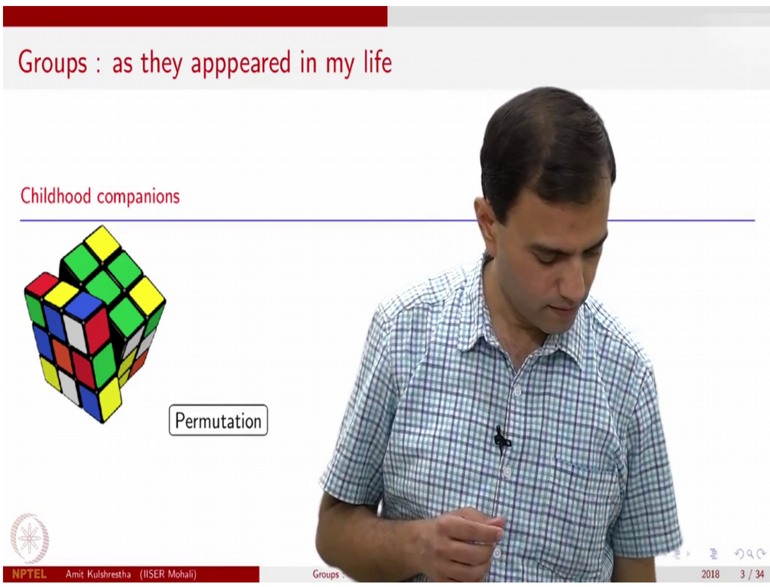

Again this thing, after this kind of thing I came across this which you must have seen you must have played you must have seen videos of these things this is 2 by 2 Rubik’s cube this is 3 by 3 Rubik’s cube. Again what is happening here is permutation right. But again this is restricted permutation, permutation is restricted it is not that these small cubes are just falling apart and any kind of permutation is happening, no, its not in the case. There is some restriction there are some rules by which these permutations can happen.

For example, if you take this one middle one the yellow middle is not going to move at all right it remains exactly the same place. So, there is some restriction to the permutation that we have here.

So in fact, if you consider the salt cube as one state and then I permute this after sometime I permute the again the permuted version. So, what am I doing? I am considering the composition of 2 permutations. The composition of 2 permutations is again a permutation I can do a permutation and I can reverse it back right. So, for a permutation there is a reverse permutation. These are some of the properties which groups have. I not finding groups now those who know groups they will understand, that composition of 2 things is again of the same type that is one of the properties of groups.

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This is quite interesting some of you might have played this game this is called peg solitaire some people call in brainvita and this permutation which is happening, but there is something quite interesting which is called checking of parity, parity check. So, some question I am going to ask later on concerning this particular game, this particular puzzle and it has not much to do with permutations, but it has something to do with the abstract group law. So, using abstract group law I am going to explain something quite interesting about this puzzle. So, you have to watch.

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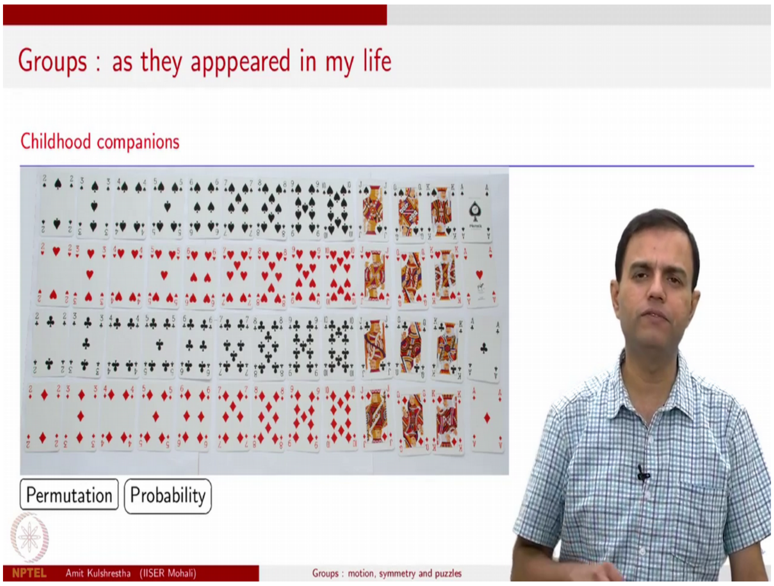

And then these card games we have always played these card games. Again watch the keyboard as you know permutation permutations coming again and again and with card games this probability which is a sign probability which is associated.

So, well groups are associated to groups are connected to probability as well, there is something called a graph of the group and you can move in a random fashion you can do random walks on groups and those are also quite interesting topics. But so far we do not know what groups are I am just giving you examples of groups before actually telling you what groups are.

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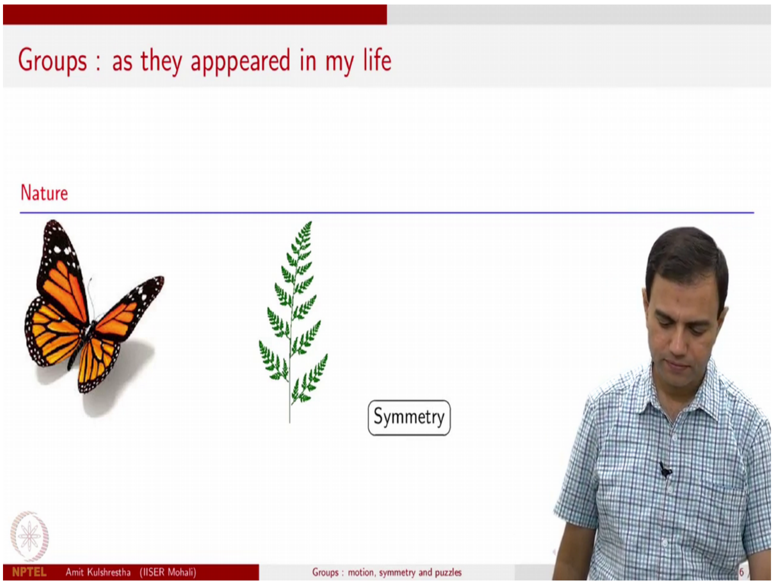

Nature has very much symmetry, butterflies, leaves, trees, so many aspects of nature have symmetry, and as and as I said groups were originally created to understand symmetry of roots of polynomial equations will come to that point; the symmetry.

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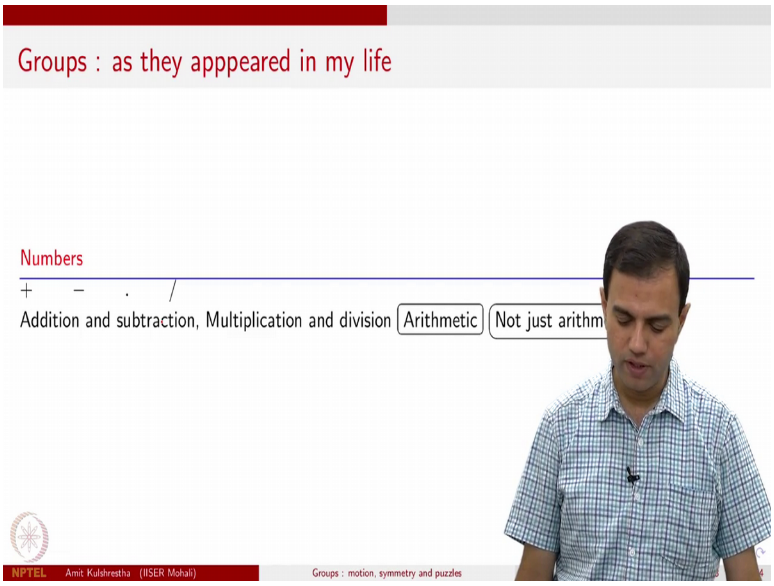

And then we grew maybe when we are in primary school we saw these operations addition, subtraction, multiplication and division. And what is there is a arithmetic. So, if you have 2 numbers you add those numbers it is again a number.

So, let us start let us take those that in the set of numbers to be say the set of integers. If you add 2 integers it remains an integer if you subtract one integer from another integer it remains an integer. Similarly if you take nonzero rational numbers we can multiply 2 nonzero rational numbers and you can divide by nonzero rational numbers as well. So, arithmetic is there any arithmetic has lot to do with groups lots of examples of groups come from arithmetic we will see that, but groups are beyond that.

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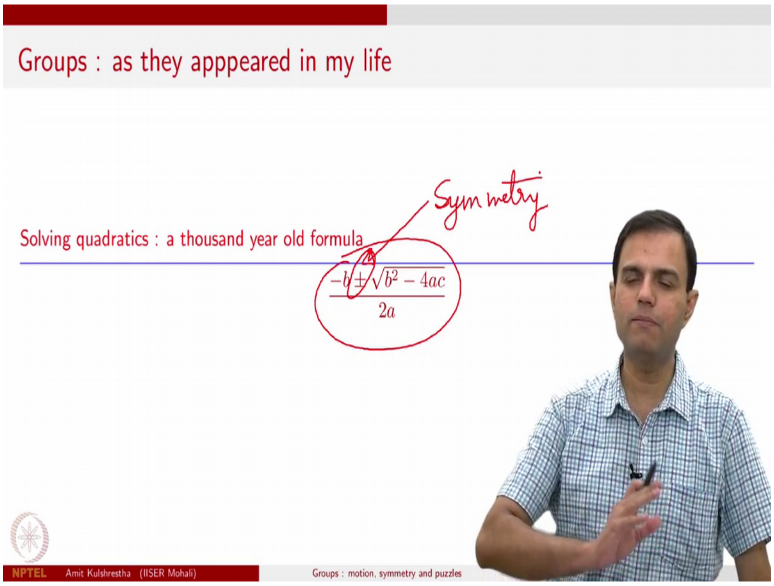

This we have seen right from our high school this formula this is to solve quadratic equations minus b plus minus square root of b square minus 4 ac divided by 2 a and there is some symmetry which is here and that is why I am saying that there is some group which is there in the picture. The symmetry, symmetry of foods this symmetry around minus b by 2a there is plus and the same amount is there in minus this is symmetry in the expression. So, symmetry of expressions has something to do with groups any kind of symmetry has something to do with groups.

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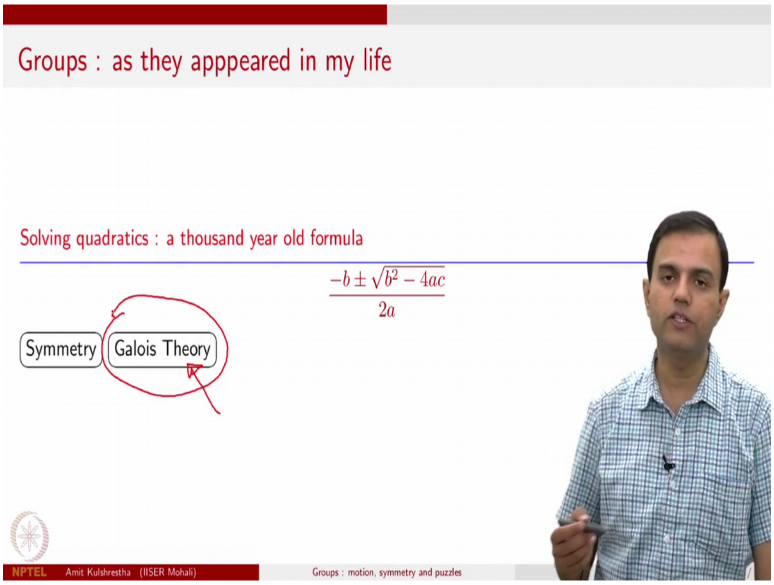

And the symmetry of these expressions these polynomial equations these roots comes under the branch of mathematics which is called Galois theory. A Galois theory is beta advanced topic we are not going to touch it, but nevertheless it is worth mentioning that, Galois theory is essentially the origin is it is there in the origin of groups.

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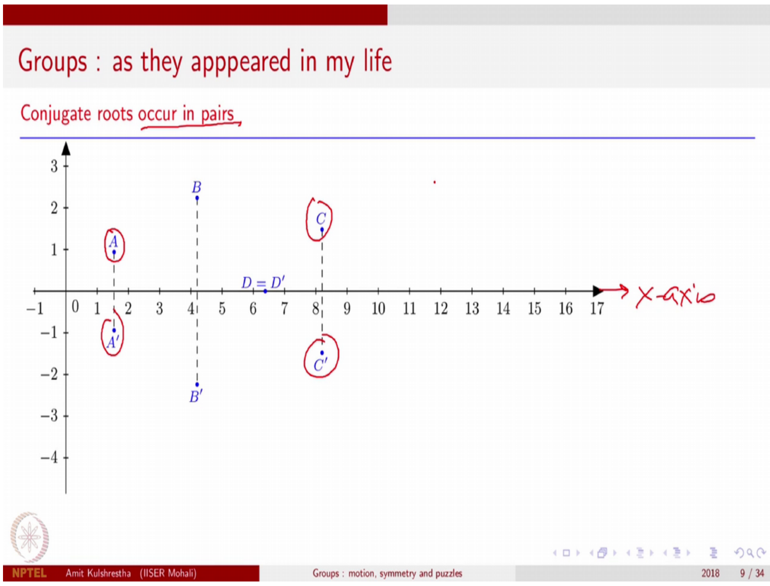

And we remember about complex roots right. So, complex roots occur in pairs we know that. If iota is a root then minus iota is also a root for polynomial equation which is having real coefficients the real coefficients are there the conjugate roots will occur in pairs. So, this is a root, then this is conjugate, this is root, then this is conjugate. What is there? There is symmetry across this axis we call it x axis, this is symmetry right. So, that symmetry is again captured in groups.

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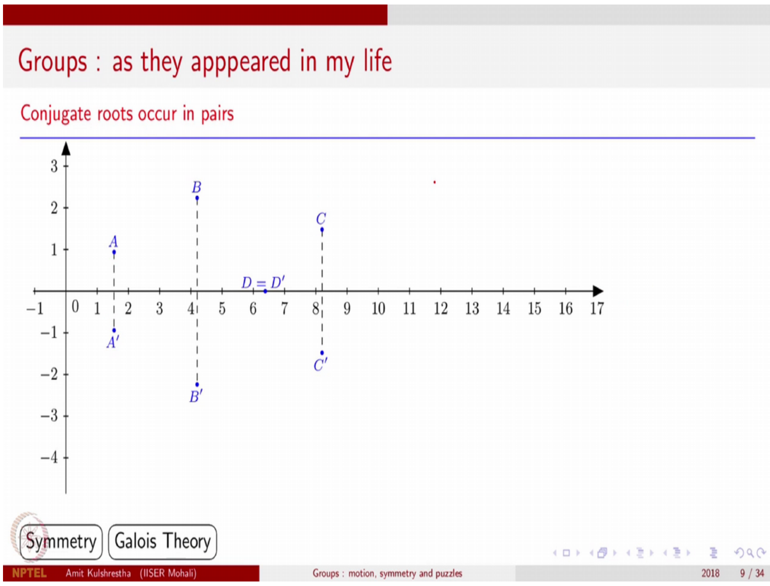

So, symmetry and of course Galois theory as I said.

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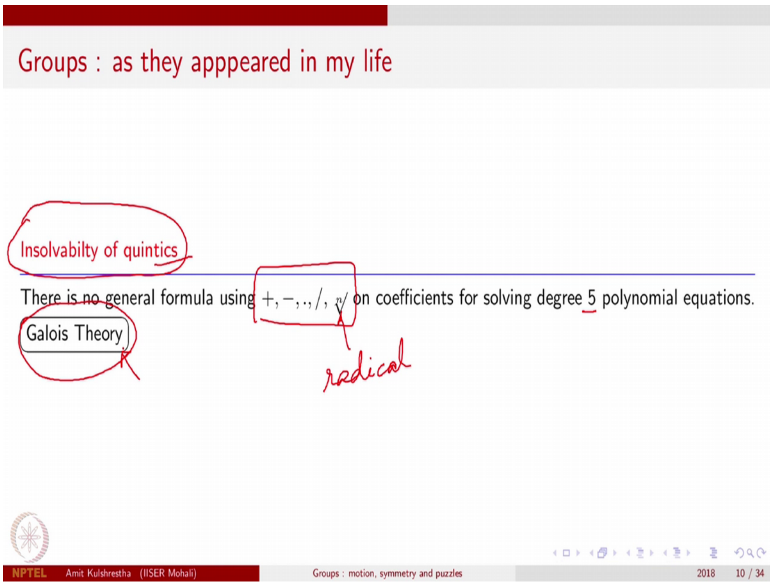

One very important aspects of Galois theory is what is called insolvability of quintics. It says that you cannot have a formula which uses these symbols, this is addition subtraction multiplication and division and this called radical. Radical is square root cube root and nth root. So, using these symbols you cannot create a general formula which can give you solutions of degree 5 polynomial.

Although, we know that if the degree is 2 and not 5 then you have a formula as I showed you some time ago a couple of slides ago, but in general there is no formula for this and ideas which are involved improving this big statement insolvability of quintics. They come from Galois theory and they come from what is called Galois group.

So, group theory is there. Group theory is there improving things which are which are not directly related to group theory it seems the group theory as we see during our first course, but then it has some connections with other things in mathematics, ok.

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At much higher level groups are also there. One very important use of groups is in distinguishing topological spaces. It is just an example I am mentioning in topology there is something called fundamental group. There are certain cohomology homology groups which can be used to distinguish various topological spaces to each topological space you associate some group and if and you do this assignment in such a fashion that if these topological spaces are homeomorphic, then the associated groups are isomorphic. So, if the associative groups are not isomorphic then easily conclude that the given topological space are not homeomorphic.

So, there also one uses concept of groups. So, group theory as I am saying it is there in abstract mathematics, it is there in physics, it is there in many puzzles and the interesting part is going to be there in group actions that I am going to explain after some time.

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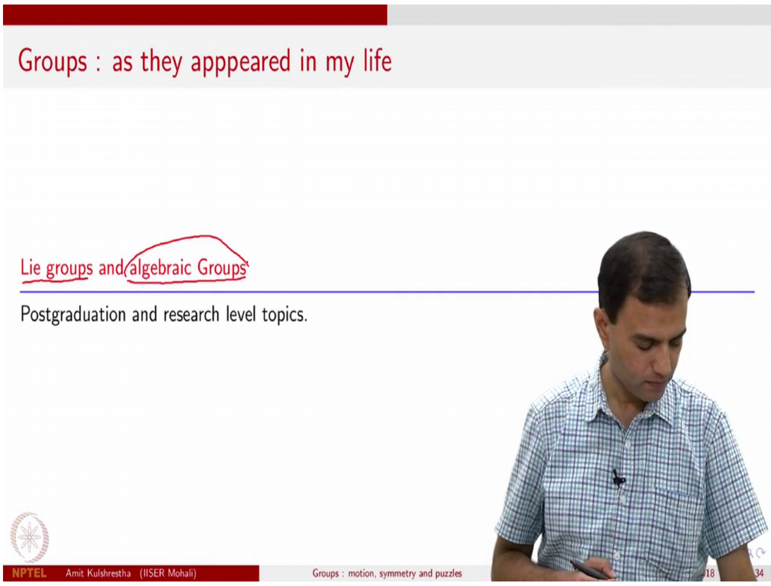

And then in the higher mathematics I can search level in postgraduate level mathematics we learn what are called lie groups, and algebraic groups. Lie groups are quite interesting for physicists and people who work in group theory or algebraic geometry and maybe may be and just changing polynomial equations, they are working they have algebraic groups as topic of interest for them, ok.

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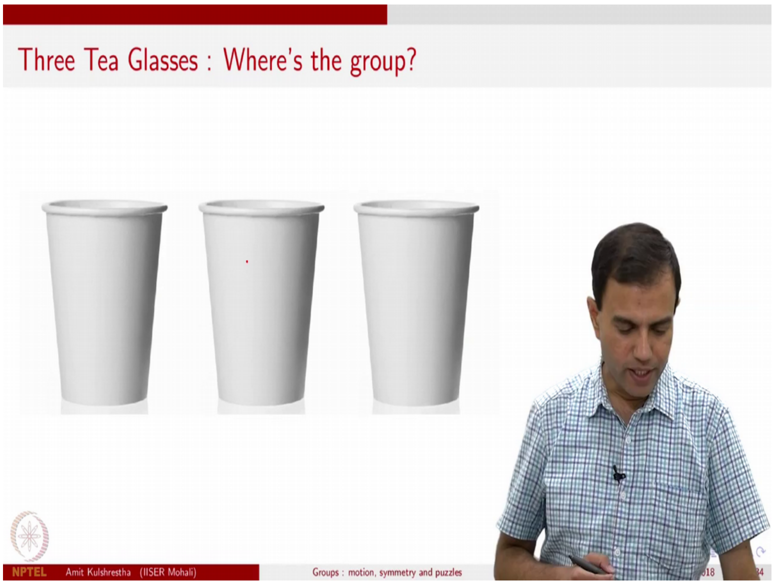

So, after all this, sp what did we do so far? We have kind of see examples of groups although I have not mentioned to you what a groupies. I will come to that, I will come to that after some time and before I do that I would like to show you an interesting experiment and this experiment as you can see involves 3 glasses, and I am saying that this interesting thing has something to do with groups.

So, let us see what happens here is an interesting puzzle for you and I am going to show its solution. So, I have these 3 glasses I have put them in inverted position. And what is the purpose? Purpose is to keep eventually all of them a primed, but there is a rule. What is a rule? At a given point of I have to pick any 2 of them and I have to change the position of them, and then I pick any 2 of them change their position. I take some other change their position like this. So, I hope the rules are clear. There are 3 glasses at a given point I am supposed to randomly pick 2 of them.

So, can I make all these 3 glasses in upright position after some time. If I do this couple of times it seems I cannot I am back to the same thing let us try more this, this, this, this I am back. So, it seems I cannot. How to prove that it is not possible? I am going to mention after couple of lectures concept of parity and also during the study of group action, during the discussion of group action we are going to learn the notion of orbits and using both the notions I will show that using group theory you can easily understand that why if I follow this rule where I am picking 2 and changing their state, changing their orientation then just by doing that again and again I will not be able to keep all 3 of them in upright position. So, I hope you enjoyed watching that puzzle with the 3 glasses, ok.

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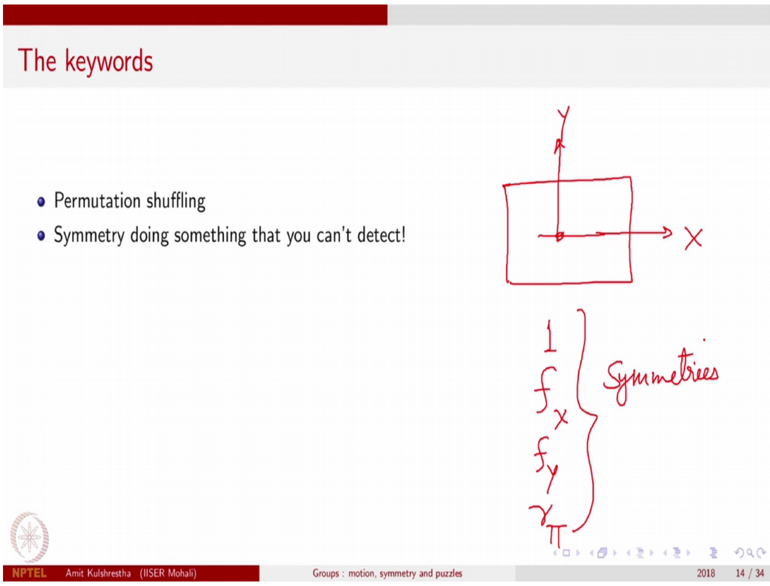

So, what are the key words that I have uttered in this lecture so far? Permutation, which is a shuffling, symmetry. So, what is symmetry? So, here is my definitions of symmetry is doing something that you cannot detect, sounds mysterious. So, here is something that I am going to do which you cannot be detect.

Here is one A4 size sheet that is a rectangle right that is a rectangle, I am going to do something on the back and probably I am doing something probably I am not doing something and then I am showing you back. So, did I actually rotate it? Did I actually flip? Cannot make out right, ok.

Now, if I do something you can make out yes, I have rotated this rectangle by 90 degree. So, what is symmetry? So, given this rectangle if I flip across flip about x axis this is one x call this horizontal thing as x axis vertical thing as y axis, if I flip it about x axis that is a symmetry because hide I done it on the backside you would not have detected it, right. So, if I do it is it remains it looks the same by do like this it is clear that its it is different. So, flipping about x axis is symmetry flipping about y axis is also a symmetry what else, well doing nothing is also a symmetry I am not doing anything that is a symmetry and if I rotate by 10 degree that is also a symmetry. So, what I showed you was this. I have a rectangle I have these axis x axis y axis and then if I do not do anything I am denoting it by 1 or if I flip about x axis or if I flip about y axis or if I rotate by an angle of pi all these are symmetries of rectangle for me. So, that is what I said you do something which cannot be detected that is symmetry.

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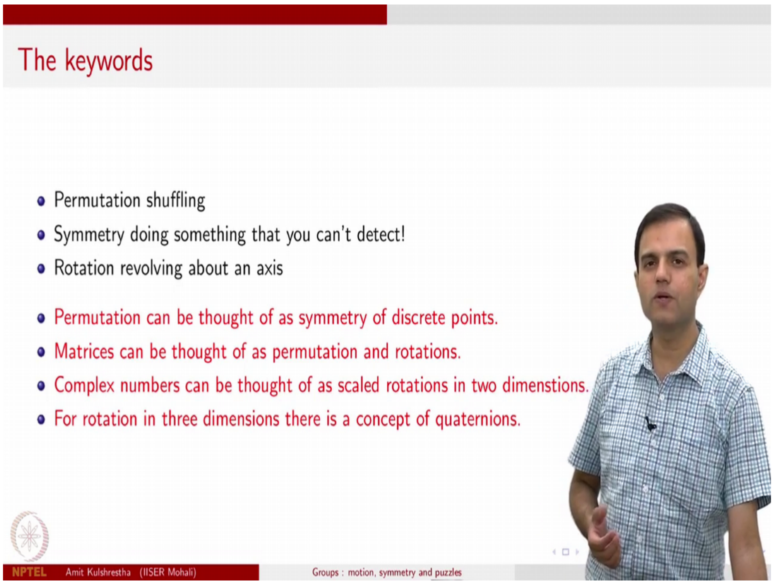

Rotation, of course rotation as we have says you have a sphere and I am rotating this sphere that is that that is an example of symmetry and rotation is also key word that I have been using here. So, revolving about the revolving about an axis is rotation.

And permutation is also kind of symmetry can be thought of as symmetry of discrete points. So, suppose I have these I am just using again these 3 glasses, there are 3 different points. They are not associated to each other they are not bound to each other discrete objects and when they permit them that is a symmetry. See if I just do it in the back and I show you have I done something you will not be able to detect right. So, permutation is a kind of symmetry by the symmetry of discrete points.

To understand rotations and permutations one of the important object that we have is matrix. It is not very relevant here even if you have not seen matrices do not be scared, and as far as rotations are concerned one can also think of rotations as multiplication by complex numbers the x rotation in 2 dimensions, but for rotation 3 dimensions there is a concept of quaternions, quaternion and algebras we going to be much worried about this in this lecture, ok.

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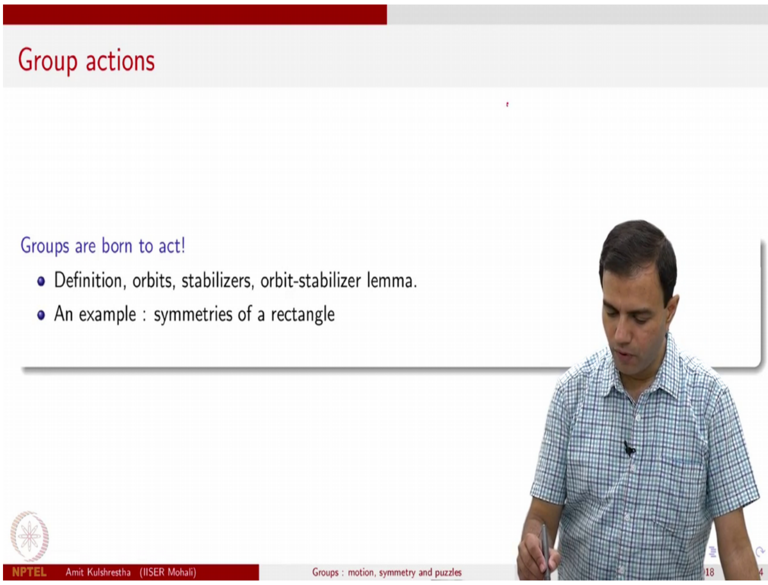

So, here comes now the formal part of the course, formal part of this lecture which is about definitions. So, what are groups finally? We have seen various examples, we have seen that groups are related to permutation groups are related to rotation and all that, but what exactly are groups. So, here it is time for us to learn all those things precisely, ok. So, for that let me come back to the symmetries of rectangle.

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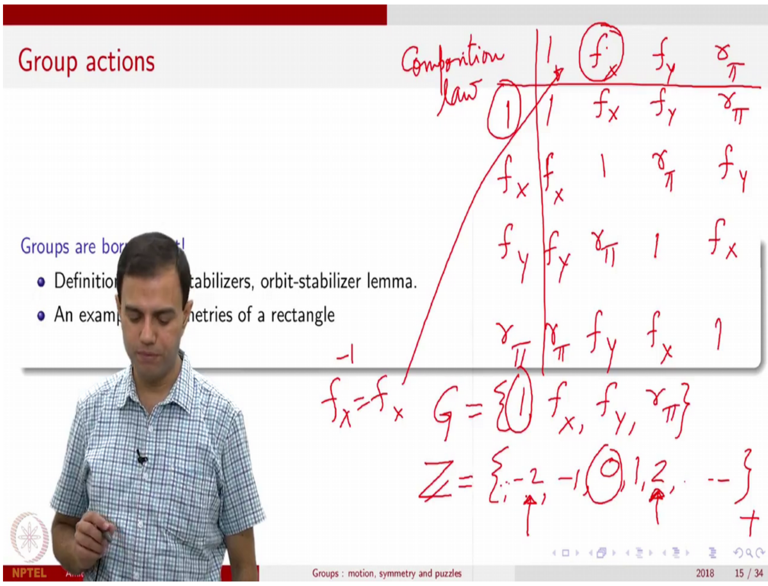

I had written 4 things 1, f x, f y and r pi, right. I will write it here as well 1, f x, f y, r pi. So, here is the thing. If we have a symmetry and composed like symmetry with another symmetry it again remains the symmetry. What do I mean? I mean that if you have this rectangle, I flip it by x axis and flip it by y axis and flipping it about both axis x and y, what I have is again a symmetry right.

So, let us see what is happening exactly. So, for that if you carefully look I have marked this as a front and 1 2 3 and 4. I am I am flipping about the x axis and after that I am flipping about y axis. Did you observe what has happened? When I am composing the 2 f x and f y what I am getting is actually rotation by pi. So, f x composed with that x y f x composed with f y is actually rotation by pi and if I compose doing nothing with anything what I get is that anything. So, 1 composed with f x is f x, similarly one composed with f y is f y one compose with r pi is r pi and same fashion I do it here. So, 1 composed with f x is f x, 1 composed with f y is f y, 1 composed with r pi is r pi.

What is f x composed with f x? That means, I am flipping about x axis and once more I am flipping about x axis I am back to identity. So, I am writing 1 for that. Similarly when I composed f y with f y I have 2, and if I rotate by pi and again rotate by pi I have rotation by 2 pi which is just identity position rotation by pi rotation by pi. So, I am again having 1 here. So, what about f x composed with r pi and that would be interesting. So, I have f x and I am composing it with r pi against rotation. Now, what do I have? I actually have do this experiment I have just f y I will just flip it once I will show you just f y. And similarly with this if you do experiment what you get is that f y composed with the f x is r pi, f y composed with r pi is f x r pi composed with the f x is f y and r pi composed with f y is f x.

So, let us look at it carefully, what do I have? I have 4 objects I have a set let me call this set as g, this set has 4 objects, 4 elements and there is naturally way to compose these elements right. And composition goes in very natural way, I am composing these elements I am able to write composition table I am able to give a composition law which is motivated by doing one operation after another operation.

So, let us try to see this at with the larger perspective. So, I am able to compose 2 objects well we have been composing these object various objects since our school when this set was given to us. The set of integers you were composing is not it, and the set of integers was given to us we are adding integers. And what was the law for, what were the properties of those at the addition? If we take 2 elements from this set add them answer is again of the same type right. So, here composition law is there, what type the properties of this composition law? There is a neutral element which does not do anything.

And then for every symmetry there is an inverse symmetry what is inverse symmetry you do this and after that you do something. So, that you have 1. You are doing this you do something. So, that it is 1. So, here I would say it like I would say that f x inverse is again f x because after f x if you perform f x again you are back to identity ok, same is true with f y here same is true with r pi here.

However, when we have integers doing nothing is 0 as far as addition is concerned 2 is here what should you compose with 2 in order to obtain 0 minus 2, right. So, when we observe all these properties we can write the definition of a group.

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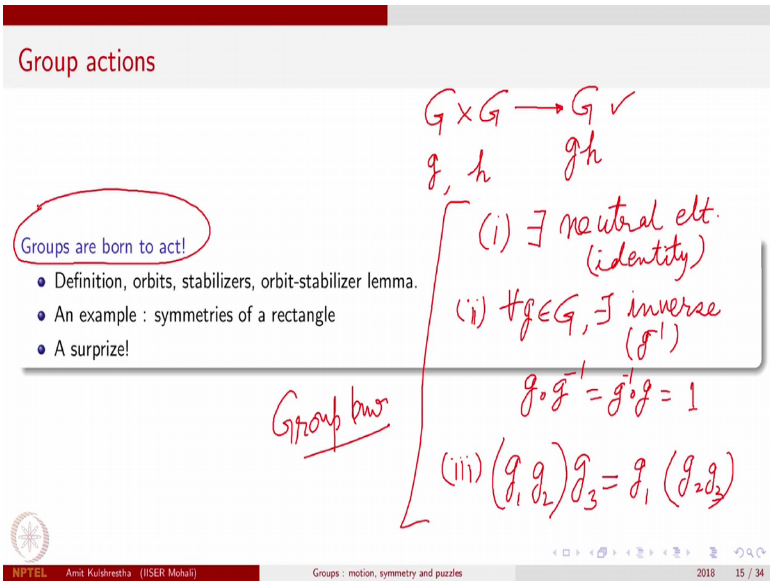

We take a set and you pick 2 elements from this set how do I pick when I pick one element from G cross G right. So, I take 2 elements from this set call them g and h and to that I assign something in the same G and I just simply call it g h.

And then there should be properties. What are the properties? One, there should exist a neutral element identity element which does nothing for every element in this set there should exist an inverse that means, I am just denoting the inverse by g inverse, and inverse what is the meaning of inverses. So, that if I compose that element in either of the directions left or right what I get is the neutral element that was a property that our symmetry group for a rectangle head.

And third thing is what is called associativity property. So, if I take 3 elements. So, suppose I take g 1, g 2 I compose them according to this operation this law and then I take g 3 or I could have done g 2, g 3 I could have composed it according to this law and then I could have taken g 1. The answer should be same. So, if on a set I define an operation like this which has these 3 properties then I call it a group law. And we have seen examples of group laws the set of rectangles set of symmetries of rectangle is a group its following group law all integers along with addition again is following group law because 0 is identity element additive identity, minus is inverse and the associativity is also followed in that law.

Quite important property of groups is group action. As I say groups are born to act unless they have an action, unless they have nice applications they are not interesting. So, we are also going to see what group action is and again for that group action I will take the example of symmetries of a rectangle.

So, what have we seen in this lecture we have seen various examples of groups without actually realizing that they were groups, and then we saw formal definition which came through the example of symmetries of rectangle and as I am saying groups are born to act group action is what we are going to learn in the next lecture. So, group action is a key word and that will remain the key word for quite some time we are going to learn orbits and stabilizers, and orbit stabilizer lemma it is quite interesting and we will see very interesting applications of these things. So, stay tuned. We will see you again.