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Introduction to Abstract
Group Theory
Module 08
Lecture 44- "Problems 9"
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So, in the last few videos, we looked at Sylow theorems, proved them, there are three Sylow theorems, remember that, they describe Sylow p-subgoups of a finite group G. The first one says there is always a Sylow p-subgroup, the second one says any two Sylow p-subgroups are conjugate, and the third one says the number of subgroups must satisfy, some conditions, so there is a range of possibilities.

So in this video, I am going to do some problems and these problems will illustrate how to apply Sylow theorems, so the first thing that I want to do
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let us called it a proposition. So I will show that any group of order 15 is cyclic okay. So this is a very strong statement, remember if you have a group of prime order, we already know it has to be cyclic, because you take any element that is not identity, its order must divide the prime number so it must be all of the, it must be that prime number, so the subgroup generated

by that element must be the whole group, but certainly that arguments fails for a group which has non-prime order like 15. So how do we show that a group of order 15 is cyclic, so in another words, we are saying that there is an element of order 15, and that is not clear.

All we know there is a element of order 5, and there is an element of order 3, but how do we know there is an element of order 5. So, and in this where Sylow theorems come in to play. So we will show that if G is a group of order 5, 15 rather, then G isomorphic to Z/3Z x Z/5Z, okay. So I do not remember whether I have talked about the products carefully before so, quickly we describe the product group.

It is very simple, so Z/3Z x Z/5Z, so remember that Z/3Z is a cyclic group order of 3, Z/5Z is a cyclic group of order 5. So we take elements (a, b), the first one in Z/3Z, the second one in Z/5Z, and we add them, so I am going to use additive notation here, so it does not matter, what notation you use, but we do component-wise okay.

So, in other words (a, b) + (c, d) simply is (a+ c, b+d) okay. And inverse of (a,b) will be (–a,-b), identity will be (0,0), so Z/3Z x Z/5Z is a group under this. So I am going to say that any group of order 15, this symbol stands for isomorphism, is isomorphic to this. Remember that there is always a group of any given order, namely a cyclic group of that order, so this statement applies to that cyclic group also, so every group is isomorphic to

this, hence to the cyclic group.
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So, now let us work with an arbitrary group, let G be a group of order 15. So let us look at how 15 decomposes, as a prime number, prime factoriazation, it is 3*5. So, G has at least one Sylow 3-subgroup and Sylow 5-subgroup right.

Let us say as s_3 is the number of, am going to use small s_3 number of Sylow 3-subgroups, and s_5 is the number of Sylow 5-subgroups. So now, the third Sylow theorem says what? The third Sylow theorems says, s_3 divides 5 and s_3 is 1+3a, right. This forces has s_3 to be 1, remember if s divides 5, it must be either 1 or 5, but 5 does not have this property. So, s_3 is 1.

Similarly s_5 divides 3, s_5 is 1+5a, remember a is some integer of course that is why 5 cannot be return as 1+3a, this forces s_5 to be 1 also, because only numbers are divided 3 or 1 and 3, 3 is not of this form.
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So, in other words, G, remember the whole point is that, we know nothing about G, other than that it is order of 15, but using only that, we say G has exactly one Sylow 3-subgroup and G has exactly, so, let us call this H, and G has exactly one Sylow 5-subgroup K. So it has exactly one Sylow 3-subgroup and exactly one Sylow 5-subgroup.

And now by second Sylow theorem, H and K are normal in, remember again I want to stress this again and again, we know nothing about G, that is whole point. We can conclude this for any group of order 15, of course G is order 15, nothing more.

So because there is only 1 Sylow 3-subgroup, any two Sylow subgroups, 3-sub-groups are conjugate, H must be normal in G; similarly K must be normal in G. So note that H is a Sylow 3-subgroup, this implies H has order 3, but that means H is isomorphic to Z/3Z, okay. This is a consequence of first isomorphism theorem way back. If you have a cyclic group of certain order, first of all H is a prime order group, so that must be cyclic and any cyclic group of order 3 must be isomorphic to Z/3Z.
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Similarly, K is the isomorphic to Z/3Z, Z/5Z. Now, we claim, consider the map from H x K to G. So what is this map, I send, so this is the cartesian product, (h,k), it goes to h times k, so h and k are elements of capital H, and Capital K right, but capital K, and capital H are subgroups of G so, small h and small k are elements of capital G, so I can multiply them, so I claim that this is an isomorphism.

And remember that, this proves that any group, any arbitrary group of order 15 is isomorphic to Z/3Z x Z/5Z. So, to prove that phi is an isomorphism, we need to show several things.
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First, phi is a homomorphism; so before that, we first show, I want to show that, H intersection K is e, why is this? This is easy, because H intersection K is a subgroup of K, of H or K, it is a subgroup of H, by Lagrange's theorem, order of H intersection K divides order of H, which is 3 okay, so in other words, order of H is intersection K is 1, or order of H intersection K is 3.
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If it is 1, H intersection K must be identity, because identity is certainly in H and K, so it is in the intersection. But if it is 3, then H intersection K is okay, but then, we also use that, now, assume this okay. We also know H intersection k is subgroup of K okay, so order of H intersection K divides order of K is 5, so this cannot happen right, so this cannot be 3, because 3 divides 5, so order of H intersection is a number divides 5 and 3, so it has be 1, okay. This is going to be useful to us. H intersection K is just the identity element.

Now, let us show one by one, that it is a homomorphism, then it is 1-1 and then it is onto. Why is it a homomorphism, what we need to check here, to check, what we need to check to, check something is a homomorphism, we need to check that, phi of (H_1,K_1) times (H_2 ,K_2),

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we want to check this is equal to phi of (H_1,K_1) times phi of (H_2,K_2) , right. So we want to check that, in order for something to be a homomorphism you multiply in the left hand side group, apply the map, or your apply the map and then multiply, you get the same answer, multiply first, apply phi or apply a phi first and multiply.

So what is this? Remember the product is component-wise, and this is H_1K_1 phi of (H_1,K_1) is that, phi of (H_2,K_2) is H_2K_2 , this is $H_1H_2K_1K_2$ and this is $H_1K_1H_2K_2$. So now the question is are this equal? So now, you must remember that I am not necessary assuming that, G is abelian, if G is abelian, of course it will follow immediately that these are equal, but I am not assuming that, however I claim that,
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we have, if h is in H, h is in K, then, they commute. I claim that they commute, why is this? Let us right two elements as follows, so we have (hkh inverse) times k inverse, is equal to h(kh inverse k inverse). So, I am not doing anything here, I am just grouping them differently in a group, that gives you the same answer, so h k hinverse, k inverse, hk h inverse k inverse. But now, remember that, this is in K, why is that because K is normal in G, small k is an element of Capital K right, and small h is in H, that is irrelevant, it is in capital G.

So hk h inverse will be in K, so, here is where you know, both second and third Sylow theorems are being used. To show that K is the only Sylow 5 sub-group, we needed Sylow third theorem and to know that, it is normal, we needed second Sylow theorem.

So this is in K of course, this is also in K, so this wholee thing in K. But now, let us look at this, this because H is normal that is in H, right, because H is in normal, h inverse is in H, h is in G, this is in H, this is in H, the product is in H, but we have already argue that,
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K intersection H is in empty, sorry just the identity element. This means this element, is the identity element, but that means, hk is kh, because you can multiply by first k and then by h. So, while it is true that G is not necessary abelian, elements of H, and elements of K commute with each other.

Now let us come back here, we have H_1 H_2 K_1K_2. This can be written as H_1 interchange these two, which gives this, okay.

So, phi is a homomorphism, if you can inter change H_2, K1, you are done. So this shows that, phi is a homomorphism, we are

claiming that is an isomorphism, so let us prove that phi is 1-1.

Suppose that, phi of H_1 K_1,sorry, phi of (H,K), so can do this, phi of (H_2 ,K_2). But what is this, this implies, so if two things map to the same thing, I want to conclude those two things are equal, so this is H_1K_1= H_2K_2 . That is the map phi, it takes (H_1,K_1) to H_1 times K_1, but that means, H_2 inverse H_1 is K_2K_1 inverse, but this is in H, this in K, again use that H and K have nothing in common.

This implies that, if this element seems to be in both H and K, so this is e, this means, similarly K_2 K_1 inverse e. This implies from H_1 is H_2 , this implies K_1 =K_2 okay so phi is 1-1.
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Now why is phi is onto? Remember again phi is a map from H x K to G, it sends (h,k) to h times k, okay. So what we have shown is, I claim that H x K has 15 elements.

So suppose for the moment you grant me this claim, if it has 15 elements, it is a 1-1 map to G, which is also 15 elements, it must be bijective, it must be onto. In fact, we claim two, why does it follow that it has 15 elements? This follows, this implies first of all phi is onto right, because phi is 1-1, and G has also 15 elements. So the claim completes the argument that phi is an isomorphism. You have a 15 element set mapping to a 15 element set, but it is 1-1, so it must be onto also so, this proves

the proposition.

The claim that phi is an isomorphism, so why does H cross K have 15 elements?

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So first we show that, we can show that H x K is a subgroup okay. I will leave this an exercise for you, the proof is exactly as before, using the same idea that, if you have HK=KH, you can use that, okay, so it is an easy exercise, so this is easy, for you. Then I claim that order of H x K, order of H x K, not claim, this follows, this is Lagrange's theorem, this divides 15, Lagrange's theorem says that it divides 15.

But certainly order of H x K is greater than 5, because K has already 5 elements and you are multiplying by H, and these are different elements, it must be at least 5, so the only number that divides 15 and bigger than 5 is 15, so that shows that, H x K is 15, has 15 elements and hence, H x K.

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Thus phi from H x K to G is an isomorphism right, so it is an isomorphism. Now, we have, so we showed that, we recall that H is isomorphic to Z/3Z, and K is isomorphic to Z/5Z, so G is

isomorphic to Z/3Z, isomorphisms are preserved by composition. So G is isomorphic to Z/3Z x Z/5Z for any group of order 15.
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So in particular Z/15Z, which is a group of order 15, is isomorphic to Z/3Z x Z/5Z, and which is further isomorphic to any group of order of G.

So any group, being an isomorphism is a transitive relation so, G is isomorphic to Z/15Z, and hence G is cyclic. This proves the proposition. We showed that any group of order 15 is cyclic. Hence, this is a very good application of Sylow theorems.

We have no way of proving that an arbitrary 15 order group is cyclic, I mean how do we do that, there could be lots of groups, we do know the structure of those groups but the strength and power of the Sylow theorem is that, we can conclude that any group of order 15 is cyclic.
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Now, let me as further examples of Sylow theorems, let me look at some problems. Show that any group, a group of order 100 has a normal subgroup of order 25. So, this is an easy

application of the Sylow theorems again, so we recall that, we know that order of G is 100, which is 5^2 times, 4, so if you want to look at, first of all there is, a Sylow 5-sub –group of G has order what? has order 5^2, because 5^2 is the largest power of 5 appearing in 100 so, we know by the first Sylow theorem, we know that there is a subgroup of order 5, but why is there normal subgroup of order 5. By first Sylow theorem, we know that G has a subgroup of order 25, but why normal? Why should there be a normal subgroup of order 25? For use, to prove that we have to use the remaining two Sylow theorems.
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Let's say S_{25} is the number of Sylow 5-subgroups, so I am using S_{25} because to emphasise the fact that Sylow 5-subgroups have order 25.

By Sylow third theorem, by Sylow III, S_{25} divides 4, and $S_{25}=1+5a$. Remember that, the prime plays a role here, 5 is the prime that is relevant over here, so S_{25} divides 4, and $S_{25}= 1 +5a$, but S_{25} dividing 4 can be 1 or 2 or 4. 1 of course is possibility, but if it is 2, it is not of this form right, it is not when you divide by 5 the remainder is not 1, similarly it cannot be 5, 4. So S_{25} has to be 1, so there is only 1 Sylow 5-subgroup and by Sylow II, it must be a normal right so, that solves the problem.
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Just as an aside, let us look at Sylow 2-subgroups; 2 is the other prime dividing 100, have order 4 right, 100 is 4 times 25, so Sylow 2-subgroups have order 4.

So let us say S_4 is the number of Sylow 2-subgroups so, S_4 divides 25 and S_4 is 1+2a okay.

Now, what are the possibilities? If it divides 4, it is 1, sorry, it divides 25, either 1 or 5 or 25. Now 1 is certainly a possibility but even 5 is a possibility or 25 is a possibility. So without knowing further about the group, we cannot restrict choices any more. So the number of Sylow 2-subgroups is either 1 or 5 or 25. So we do know which one actually happens.

So this tells you both the strength and the limitation of Sylow theorems, in the case of Sylow 5-subgroups here, we are able to conclude one is the only option and it must be normal, but for Sylow 2-subgroups, there are three possibilities. And Sylow theorems themselves do not tell us how to eliminate any of those possibilities.
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Let me do one more problem along the same lines, okay. Let us say p and q are distinct primes and suppose p strictly less than Q. Let G be a group of order pq, show that, G has a normal

subgroup of order q, okay.

So I will solve it very quickly, the proof is very simple, solution is very simple and similar to previous problem. What is the idea? We want show that there is a exactly one Sylow q-subgroup, so, Sylow q-subgroups of G have order q, it is not q^2 or q^3 right, because q is the largest power of q appearing in order of G. So they have, Sylow q-subgroups have order q and always we denote S_q to be the number of Sylow q-subgroups. We know that S_q divides, S_q divides p and S_q is 1+aq, right.

But let us see. If S_q divides p, S_q must be 1 or p, but can it be 1? Of course it can be 1, but it can be p? Can p be written as 1+aq? It cannot be, because p is smaller than q, p is smaller than q means p cannot be return as 1+ aq, because 1+aq is more than q.

So G has only one Sylow q-subgroup right. We are using the third Sylow theorem here. G has only one Sylow q-subgroup, let us say H. And Sylow II implies H is normal in G, because any two Sylow q-subgroups are conjugate, so if you take H and conjugate it, it is another Sylow q-subgroup because it is a subgroup and it has q elements. But because there is only one Sylow q-subgroup, that must be H, so all conjugates of H are H, so H is normal in G. So this is the solution.

Okay, these two problems gave you some idea of how to apply these Sylow theorems and how why they are very powerful, okay. So, I will stop this video here, in one more video I will do some more examples and problems which illustrate how to use

Sylow theorems. Thank you.

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