NPTL ONLINE COURSE
Introduction to Abstract Group Theory
Module 08
Lecture 42-“Sylow Theorem 11”
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So let us now, in this video, do the second Sylow theorem.
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We have done the first Sylow theorem last video, so second Sylow theorem. The first Sylow theorem remember said that if you have a group, a finite group G and prime number p dividing the order of the group then G has a Sylow p-subgroup. Second Sylow theorem says that, let G be a finite group and let, this is the same assumption always, let p be a prime that divides order of G, then any two Sylow p-subgroups are conjugate, okay.

So let me, maybe I mentioned this usage earlier, but we say that, I am going to say it again, we say that two subgroups H and K are conjugate, if H equals some gKg inverse, for some g in G, okay. So conjugation is this operation right, g sometimes, something times g inverse. So we say the two subgroups are conjugate if gK g inverses is H.

So the Sylow theorem says that any two Sylow subgroups are conjugate, so all the Sylow subgroups in other words are conjugate, because being conjugate is an equivalence relation. If H1 and H2 are conjugate, H2 and H3 are conjugate, H1 and H3 are also conjugate. And as we will see later in applications, this is a very important statement that, Sylow subgroups, Sylow p-subgroups are always conjugate.

So let us go ahead and take two Sylow subgroups, let H and K be two Sylow p-subgroups of G, remember Sylow I guarantees the existences of a Sylow p-subgroup, but there can be more right, we are not saying there is exactly one, so let us take two of them, our goal is to show that they are conjugate, so now I am going to consider the following.
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So we will consider, the actions of, let's say, K on G/H by conjugation. So as I have been saying repeatedly all Sylow theorems and their proofs are essentially done by playing with various group actions, different groups, different sets, different actions.

So here my goal is to, my focus will be on the action of K on G/H by conjugation. What is G/H by the way, these are all cosets of H, left coset by our conversion, so these are gH where g is in G.

So how do we define the action, so let say b is in K, so I am going to use a for consistence, a as an element of capital G and

aH is an element of G/H, remember that in order to define the action of a group K on a set G/H, I need to tell you what is a group element times the set element. So b times aH, no surprise here, it is simple ba times H, okay b times aH is define to be ba times H, and easy to check that this an action. This is an action of K on G/H, because identity element acts as identity and the associativity naturally holds okay .
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So what we have is counting formula. Says that, this is not the counting formula actually, this is the remember G/H is a union of disjoint K-orbits, I am going to use the notations K-orbits to denote that it is actually action of K, usually we talk about the action of G, so here I am looking at action of K, so I am going to stress that by talking about K-orbits, so G/H is a union of disjoint K-orbits, so that means we can write the order of G/H as order of O1, order of O2, order of Ok. Let me use some other letter here let say Or, here O1,…, Or are the distinct K-orbits of G/H, okay.
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Now let's see, what is G/H, order of G/H, so remember always we will use this notation, order of G is p^em and what is the order of H? H is a Sylow p-subgroup right, so order of H is p power e, this implies order of G/H is m. This is our original counting formula, so this is of course not divisible by p, that was how we write this. p does not divide m okay. So now in other words, if you look at this equation, this sort of thing happened exactly as it is, exactly like this, in the first Sylow theorem. We have G/H order is a sum of some numbers, p does not divide the order of G/H, so p cannot divide one of the orbits sizes.

So p does not divide the order of some OI, for some I, because if p divides order of each OI, then p divides the sum which means p divides order of G/H, which is not possible. So p does not divide OI, order of OI, for some I. So say OI is the orbit, remember what is OI is, it is convenient to write it like O1, O2, O3, but they all orbits of elements of G/H. So say OI is the orbit of sum aH. Remember the set in question here is G/H, so orbits are element, orbits of elements of G/H. So say OI is the orbit of aH, so in other words, what we have is that, p does not the divide the orbit of aH.

So now let's apply the counting formula. Now the counting formula applied to the K-action on G/H, what does the counting formula and this particular element, counting formula remember is applied to a specific element of the set, which I am taking to be aH.

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So it says that the order of K is equal to order of stabilizer of aH times order of orbit of aH. Again what is the order of K? Order of K is p power e, that is because K is a Sylow p-subgroup so it must have order p^e. So this is equal to order of stabilizer of aH times orbit of aH. But remember what we have here, p does not divide the order of orbit of aH, but p does not divide the orbit, the order of orbit of aH, because that is how we chose this, p does not divide order of some orbit and we called it orbit of aH.

So now let us look at this equation: p power A is equal to the product of stabilizer order of stabilizer of aH and order of orbit of aH, but p does not divide orbit of aH.
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So stabilizer of aH must have order p power e, so again the uniqueness of factorization of integers, p power e is equal to some number times another number, the second number cannot be divisible by p and it of course cannot have any other prime factors because p is only prime factor, on the left hand side, so that means all of p power e must be present in the order of stabilizer of aH, so order of orbit of aH is 1, order of stabilizer of aH is p^e.

But remember stabilizer of aH, where is it living? It is living in K because we are looking at K-actions. Stabilizers are subgroups of K but K has already order p^e, stabilizer also has order p^e, so stabilizer of aH is exactly equal to K right. The entire group is the stabilizer of aH, but what does this mean, this means that b times aH.

So we have the following, b times aH is equal to aH for all b in K. b times aH is equal to aH for all b in K. This is the exactly the meaning of every element of K being the, being in the stabilizer of aH. This means (ba)H is equal to aH, for all b in K right. But this means (ba) H contain ba so ba belongs to aH, for all b in K right.

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So I am just going step by step here, hopefully each step is clear, so ba is in aH. That means, b is in aH a inverse for all b in K, right. Because small b times small a is equal to, small a times small h, I can now multiply by a inverse on the right for both sides to get b is equal to a small h, a inverse. So b is in aH a inverse. This is true for every b in K, that means K is a subset of aH a inverse right, because every small b in capital K is in aH a inverse. So all of K is in aH a inverse. But this must mean that K must have the same order K, remember what is the order of, okay.

So let me right it like this. Note, order of K is p^e, order of H is p^e because both are Sylow p-subgroups, but order of H is p^e order of aH a inverse is also p power e, because conjugation also does not change the order. This is a simple exercise for you, but K is inside aH a inverse, but both have the same number of elements. So K is equal to aH a inverse. Remember this is the exactly what we wanted to show.

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What did we want to show, any two Sylow p-subgroups are conjugate. So this completes the proof of the second Sylow theorem.

So in other words, hence the second Sylow theorem shows that all Sylow p-subgroups are conjugate to each other. So you take any two Sylow p-groups, apply the theorem to show that they are conjugate to each other.

This is a very strong statement and I will just give you quickly two remarks here. So first of all, as an example let us take S3 and I mentioned in an earlier video that G has 3 Sylow -2 subgroups okay.
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So if you use the cycle notation for S3, remember it is { e (12) ,(13), (23), (123), (132)} and the Sylow 2-subgroups are { e,(12)} , H2 will be {e,(13)} , and H3 is {e,(23) }. These are the Sylow 2-subgroups because remember Sylow 2-subgroups must have order 2 because 6 is 2 times 3.
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So now the Sylow, second Sylow theorem says that H1, H2, H3 are all conjugate to each other, okay. This I will leave as an exercise for you to specifically choose an element which conjugates H1 to give you H2, okay. As an exercise, may be I will just ask this, find an element, let’s say a in S3 such that a H1 a inverse is H2. By the Sylow’s theorem we know that the H1, H2 are conjugate. In this example I want to explicitly find

such an a.
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And an important corollary I will now mention is the following. Suppose if a group G, a finite group of course always, has only one Sylow p-subgroup, H let's say. So G is a finite group and H is the only Sylow p-subgroup of G. Then H is normal G, okay.
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Why is this? So the proof is the following. Recall what is a normal subgroup? To prove H is normal in G, we must show, what do we need to show, we need to show that if g belongs to G, h belongs to H then gfg inverse is in H. Equivalently we have to show, show that g capital H g inverse is equal to H, for all g in G. That is what one has to show to prove some subgroup is normal.

Now if H is a Sylow p-subgroup then H will have order p power e, of course again order of G contain p power e as the largest power of e. So now if g is, small g is in capital G then as I have said before the order gH g inverse is p power e and it is easy to show that gH g inverse is a subgroup of G. See if H is a subgroup gH g inverses is a subgroup. See the cosets are not subgroups but gH g inverse, conjugate are always subgroups because identity is there. Remember g times e, small e is in H so g times e times g inverse is e and any two things like this are, their product is also a thing like this, inverse is also a thing like this.

So it is an easy exercise to show that gH g inverse is a subgroup of G and it has p power e elements. So gH g inverse is a Sylow p-subgroup, remember Sylow p-subgroup is a fancy term for just a subgroup for order p power e, gH g inverses is a Sylow p-subgroup because it is a subgroup and it has a order p power e. But what was our assumption in the corollary, G has only one Sylow p-subgroup.
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By hypothesis, there is only one Sylow p-subgroup, namely H. So gH g inverse is H and that proves that H is normal.

As an example, so if you somehow conclude that there is exactly one Sylow p-subgroup, that Sylow p-subgroup must be normal. If you take S3 and p to be 3, so 6 is 3 times 2, and we know that and I can, I mean I mentioned this earlier. We know that there is only one Sylow 3-subgroup, namely H will be {e, (123) and (132)}.
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Of course, in this case we can directly check that H is normal, but it follows from the by corollary that H is normal. Note that any conjugate of H is a subgroup of order 3, hence it is a Sylow 3-subgroup. But Sylow 3-subgroups there is only one such thing, so H is normal. In this case it is also clear that H is normal directly, but I wanted to give a simple exercise, example to show that, and we will see more examples later, it is very useful know that is there is one Sylow p-subgroup, in which case it will automatically be normal, okay.

I will stop the video here. In this video we talked about Sylow, second Sylow theorem, which said that any two Sylow subgroups are conjugate. In the next video I will talk about the 3^{rd} Sylow theorem and which talks about the number of Sylow subgroups. As a corollary here, as the corollary here shows, if you know that there is only one Sylow p-subgroup we know it is normal. So that is useful to know. So the third Sylow theorem tells something about the number of Sylow p-subgroups. So I will stop the video here, thank you.

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