Introduction to Abstract Group Theory
Module 07
Lecture 39 – "Problems 8 and Class equation"

PROF. KRISHNA HANUMANTHU
CHENNAI MATHEMATICAL INSTITUTE

(Refer Slide Time: 00:16)

Let us do a related problem. I will write it as a separate problem, and problem number would be 5. Problem 5 is, for the same G as GLn(R) and S as Rn as in problem 4, okay, show that if A satisfies, if A is a, matrix invertible N by N matrix such that A dot V is V for all V in Rn, then A is the identity n by n matrix. Why is this? Okay, so any matrix which fixes every vector, so in other words, it is in the stabilizer of all vectors, all elements of the set S, then it must be the identity element.
Why is this? So we already saw that Ae1 equals to e1 implies, the first column, right, this we have seen already in the previous problem. And Ae1 is equal to e1 because AV is equal to V for all V. So now Ae2 is also equal to e2, right. e2 is the second elementary vector. So 0, second entry is 1 then followed by 0s. This implies so A is already we know this but second column can be anything at this point.

So this is a(12), a22, a 2n, a1n, a2n, ann, multiplied by 0, 1,0,…,0 must be 0, 1, 0,…, 0. And exactly the same calculation as before shows that, this implies, what is this product? This picks out the second column of the matrix. Because when you multiply it out, the second column is picked out ,this is 0,1,0,0,….,0. That means if Ae2 is e2 the second column of A is e2. Right, we already know the first column of Ae is e1. Now we know Aei is equal to ei, so the ith column of A is ei. So, now this is true for all numbers from 1 to n.

So the first column of is A is e1. Second column is e2, third column e3, fourth column is e4, nth column is en, but this means A is identity. Right, so A fixes, in fact the problem asked you to show that if AV is equal to V for all v in R^n, then A is identity matrix. It suffices to show, actually assume that if Ae1 is e1, Ae2 is e2, …, Aen is en.
(Refer Slide Time: 04:22)

But of course if you think about it a little bit, if Aei is equal to ei for all i, AV will be equal to V for all V. This is because the vectors e1, e2, …, en span the vector space R^n, if you know this, but it is not important for this problem.
(Refer Slide Time: 04:49)

So this action has the property that if a vector, if a group element is in the stabilizer in the all vectors, then it is the identity element. So this action is an example of “faithful action” okay. So I will define this now which I have not defined earlier. So definition: Let a group G act on a set S. We say that the action of G on S is faithful, we also say sometimes, or
(Refer Slide Time: 05:35)

“G acts faithfully on S”, so action is G on S is faithful or G acts faithfully on S, if g is an element such that gs is equal to s in S this implies that g is the identity. Right, so if g is an element of the group that fixes all elements in the set then g is the identity element okay. So this is an important class of actions, so G acts faithfully on S. So other examples include, action of G on itself by left multiplication is faithful.

And this is exactly what we used in the proof of Cayley’s theorem. On the other hand, action of G on itself by conjugation is not faithful in general. So for example, if G is abelian, then orbit of Gs is G, that is, the stabilizer is G, for all s, as we saw earlier. The orbit is G right, this is one of the problems we did earlier. So certainly, the intersection of the stabilizers is all of G.

On the other hand, this is a problem that I will eave for you as an exercise, I won’t do. But it is an immediate consequence of the problem on the relevant problem here. So problem 6.
(Refer slide Time: 08:06)

Show that the action of G on itself by conjugation is faithful if and only if the center is trivial. Okay, the center I have defined also in the previous problems. See if G is abelian it is very far from being faithful. It is in fact the diametrically opposite to being faithful because faithful means intersection of orbits is trivial, sorry intersection of stabilizers, I keep confusing between orbits and stabilizers, but faithful means intersection of stabilizers is trivial. But if it is abelian all stabilizers are the group G, so certainly it is not faithful.

So faithful and abelian are at the opposite ends of the spectrum as for as conjugation action of G on itself is concerned. So faithful and opposite spectrum of abelian is the center being trivial. Center being the whole thing is abelian, center being trivial is opposite thing, so if the center is trivial, the action of G on itself by conjugation is faithful. Let us finally do one more example. This is computational, to compute equivalence classes and orbits.

So consider, let G be S3. Compute the orbits for the action of S3 on itself by conjugation. Remember action of S3 on itself by left multiplication, orbits are only one. There is only one orbit. So that is not interesting, and stabilizers are all trivial okay, so operation, or action of S3 on itself by conjugation is more important, more interesting.

(Refer Slide Time: 10:28)

So let me use this representation of S3, cycle notation, so (12), (13), (23), (123), (132), okay. Let us use the, so a quick piece of definition. Orbits for the action of G on itself by conjugation are called "conjugacy classes".

Okay, so this is an important notion, so this is exactly as orbits, it is just a new word. Conjugacy classes is the word in the specific example of the action of a group on itself by conjugation okay. So this is useful to, because the action of a group on itself by conjugation is an extremely important example of group action.

And this is very important in proving various important theorems, so the orbits are given a special name in this case. They are called conjugacy classes.

So the problem 7 is simply asking you to find conjugacy classes of S3. That is just shortened version of saying find the orbits for the action of S3 on itself by conjugation. In other words, find the conjugacy classes of S3.
(Refer Slide Time: 12:04)
Okay, so what is the orbit of e? It is simply, remember orbit of e for the conjugation action is just e, because we have to do geg inverse, as g is in G. But this is just e, right. The action is by conjugation, so you vary small g and take geg inverse, this is okay.

Now find the orbit of let say (12). Wo we want to do g (12) g inverse as g varies in S3. Okay, so this is a quick calculation, so let us do one by one.

So e(12)e = (12), that is always there. So (12) is always in the orbit. What about (12) (12)(12) inverse, what is inverse of (12), so this is (12)(12)inverse again(12). This is e, that (12),(12) is identity, so that is not surprising. So the new thing will be if you take (13). So do (13),(12)(13) inverse, so what is (13) inverse it is again (13), so I have to vary g and compute g(12)g inverses and what is (13) (12) (13). If you do the calculation quickly (13), 1 goes to 3, 3 goes to 1 so 1 goes to 1, 2 goes to 1 and 1 goes to 3, and 3 goes to 1, 1 goes to 2, so 3 goes to 2.

Similarly (23) (12) (23), because inverse of (23) is again (23) will be I will let you calculate this quickly, it is (13). Because 1 goes to 2 and 2 goes to 3, 3 goes to 2, 2 goes to 1 so 3 goes to 1, 2 goes to 3, 3 goes to 2, okay. So now let us do (123)(12), inverse of (123) is (132), if you think about it for a minute. So this would be 1 goes to 3, 3 goes to 1, so that is 1; 2 goes to 1, 1 goes to 2, 2 goes to 3; 3 goes to 2, 2 goes to 1, 1 goes 2; so it is (23). And finally (132)(12) and (132) inverse is (123), and this will be 1 goes to 2, 2 goes to 1, 1 goes to 3; 3 goes to 1, 1 goes to 2, 2 goes to 1; 2 goes to 3, 3 goes to 2; so this is (12). So what is the conjugacy class of (12) or the orbit of (12)? This is equal to, (12) certainly will be there, (23) is there, and (13) is there. So, in fact, we get all 2-cycles or all transpositions, right this is.
(Refer Slide Time: 15:16)

So now recall the splitting of S as a union of disjoint orbits, so S3 remember must be a disjoint union of conjugacy classes, disjoint union of orbits, in this example we are calling them conjugacy classes, and so far we have found two of them. So, so far we have found e that is one conjugacy class and all two cycles form another conjugacy class, so this is one conjugacy class, this is second conjugacy class so, so far we have managed to account for four elements of the symmetric group on three letters.
The remaining two elements must be also part of some conjugacy classes. So find the orbit of (123), Let's say. So in

order to find the orbit of (123), let's find the conjugate of (123) by (12), (12) inverse as before is (12). So (12)(123)(12), let's compute this. 1 goes to 2, 2 goes to 3, so 1 goes to 3; 3 goes to 1, 1 goes to 2 so (132); 2 goes to 1,1 goes to 2, 2 goes to 1 so (13). So this means (123) is related to (132) and now you are able to conclude that so we can conclude that (123) and (132) is a conjugacy class, right, because so remember we have to find one more. That this must be the conjugacy class, why is that? These are already part of the conjugacy class, (123) and (132) are related.

So they are part of a conjugacy class, but can there be anything else in this conjugacy class? No, because S3 only has six elements and the remaining four elements are already accounted for, if any of them are in this conjugacy class this conjugacy class will have a nonempty intersection with them but conjugacy classes need to be disjoint so this is the conjugacy class and you can also check direct by calculation that all other conjugates will be equal to one of these but we do not need to do that.

(Refer Slide Time: 18:01)

So the conjugacy class decomposition of S3 would be e disjoint union with (12), (23) and (13) disjoint union with (123),(132). Now let's observe one thing. When I talked about counting formulas I noted that the cardinality of the set, of the group or of the set, which in this case is again the group, is the sum of cardinalities of orbits, right. And I also noted at that time that cardinalities of orbits divides the order of the group, so in this cases what is that formula this is called the "class equation" okay.

So in the case of a group, a finite group, I will define this more formally later when I come to this later in a video. In the case of finite group acting on itself by conjugation this statement that cardinality of S or the cardinality of the group which is also the set in this same example, is equal to order of one element orbit of it is equal to sum of sizes of orbits, is called the "class equation" of finite and group G acting on itself by conjugation is called the class equation.

This is a very, very important piece of data attached to a group okay, as I said the action of G on itself by conjugation is a very very important group action and the usual things that we have in all group actions, are given special names in this case because of the importance of this example. So, orbits are called conjugacy classes and this formula here is called the class equation.

So, the class equation of S_3 is, so you have 6 is the cardinality order of S_3, one orbit is just containing the identity, that is 1. The other orbit contains all 3 cycles. There are three of them and the final orbit contains two 2-cycles. So, this is one element, this is 3 elements; 1 element, 3 elements, and 2 elements.

So, the class equation is 6 equals 1 plus 3 plus 2 ($6=1+3+2$) and remember another point that we made earlier checks out. Each number that appears in the class equation must divide the order of the group which is 6 in this case. 1 divides 6, 3 divides 6, 2 divides 6. Because these are orbits and the counting formula says that orbit size divides the orbit of the group. Now using this

you can also compute the stabilizers of these elements, because remember
(Refer Slide Time: 21:25)

orbit of (12) is (12), (23) and (13). So, remember the stabilizer of (12) or size of that and orbit of (12) size of that must equal 6, this is the order of S_3. And this is 3. So, stabilizer of, must be 2. What is it? Let us compute this. Stabilizer of this would be, remember there are some obvious elements in the stabilizer. $G_{(12)}$ so, I am doing more than what is in the problem. So, really we have finished the problem.

Problem was compute the conjugacy classes of S3 which I have done. But let us go and do, exploit whatever we have found, in more detail. The stabilizer of (12) is elements of S3 such that g (12) g inverse is (12). Right that means this is all group elements that commute with 12. So, definitely e belongs to $G_{(12)}$ and (12) belongs to $G_{(12)}$. Right because e certainly commutes with (12) and (12) commutes with (12)
(Refer Slide Time: 23:14)

But order of $G_{(12)}$ the stabilizer is 2 and it already contains two elements, namely e and (12). So, the stabilizer of (12) must be e and 12. The counting formula tells us that we do not need to now check for other elements because the stabilizer must contain exactly 2 elements and it contains these two elements, so no need to check make further checkings. This is the stabilizer of (12).

Similarly, what is the stabilizer of (13)? It also must be of size 2 and contain e and (13). Similarly, stabilizer of (23) must contain identity and (23) and must be of size 2. So, this is the stabilizer. Finally, what is this stabilizer of (123)? The size of the stabilizer must be six by two because the orbit of (123) is 3 right? Is 2, orbit of (123) that we calculated has two elements (123) and (132). So, the stabilizer must have three elements. Stabilizer must have 3 elements and
(Refer Slide Time: 24:52)

certainly e and (123) are contained in the stabilizer and in fact since stabilizer is a group, is a subgroup of S_3 and (123) is in it, its inverse is also there. So, inverse of that is (132), (132) is in the stabilizer. So $G_{(123)}$ contains e, contains (123) and contains (132). But it is group of size 3 so, that must be it. This is of course also equal to the stabilizer of (132). Okay, so this extended problem here hopefully gave you a clear idea of the power of the counting formula and the class equation that we have computed, we have proved in the previous video.
So, again just to recap, the action of a group on itself by conjugation is a very important example of a group action and we have worked out in this problem fully all there is to work out in the case of the symmetry group on 3 letters.

We first computed the conjugacy classes by looking at orbits of elements, there happened to be three orbits. Orbit of e which is one element, orbit of (12) is 3 2-cycles, orbit of (123) is two 3-cycles.

So, the class equation which is simply the expression of the order of the set as the sum of sizes of orbits. In this, it is 6 equals 1 plus 3 plus 2 (6=1+3+2). They are three orbits of sizes 1, 3 and 2, so they add up to 6 and we noted that 1 divides 6, 3 divides 6, 2 divides 6, as we would expect because orbit size divides the orbit of the order of the group.

And then using the counting formula we guessed or rather we computed the stabilizers by first noting that stabilizers must have either orbit or size 2 or 3 depending on what element we are taking and by picking obvious elements in this stabilizer, we are able to compute the stabilizer of (12), (13), (23) and also of the 3-cycle (123). Stabilizer of (123) happens to be the stabilizer of (132).

Okay, so hopefully all these problems in this video give you clarity on group actions and the remaining content of the course would be using the class equation and conjugacy classes to prove some important results about finite groups called Sylow theorems which we will do in the next video. Thank you.