Introduction to Abstract Group Theory
Module 07
Lecture 38-"Problems 7"

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In this video let us do some problems on group actions and stabilizers and orbits.
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So I want to do some problems in this video. Let us start with the following. So let us start with some calculation of stabilizers and orbits, okay. Let us consider the action of, let us take G, consider the action of a group G on itself by left multiplication. We know from viewing the action as an equivalence relation, we know that the group G, the set S when a group G acts on a set S, S is partitioned into orbits, S is a disjoint union of the orbits.

Find the orbit decomposition of G, and similarly find the stabilizer of an element s in G. This is the problem, because G is partitioned as a disjoint union of orbits, so I want you to find all orbits of G and see what are the orbits whose disjoint union is G. Similarly given an elements small s find it stabilizer. So recall that here the action is given by left multiplication g, s maps to gs, so this the action, lets compute the order.
So let us first take compute the orbit of an element, in order to

find the orbit decomposition, let us start with finding the orbit of a single element. So let us take s in S, what is the orbit of s? So orbit of s is gs as, by definition it is the set of the elements gs, as g varies in G.

Now this is something that I have implicitly done several times, but the point is this is equal to G, so of course the orbit is contained in G always, G being thought of the set here, but in fact orbit is equal to G. Because if t is in G, then g lets say s t inverse, sorry if t is in G, we want to find a group element such that g times s is equal to that, so what should that be? So how do we check this? We want gs is to be t, if t is in G, I want to show that t is in Os, but OSs consists of gs, so gs must be equal to t is for some g, so I want to solve for g, so g would be t(s inverse).

So we take t(s inverse) times s is t. So that means t is in Os, so orbit is G and this is remember for all s in G. So what is the orbit decomposition? So there is exactly one orbit for this action, in other words all orbits are identical to G, so this such an action is called “transitive action”, so I will define this may be formally again in some other video.
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But this is an example of transitive action, transitive means only one orbit, so this is the orbit decomposition, so we have done the first part of the problem, write the orbit decomposition so G is equal to Os, that is orbit decomposition, for any s. It is in general a disjoint union of orbits but there is only one orbit here.

Now on the other hand, let us take an element s in S in this case G, find the stabilizer. That was also the problem right, what is Gs? This is all elements of group now that fix s, all elements of group that fix s, what are these? If gs is equals s, a quick calculation tells you, in this also we have done earlier, so the stabilizer of any element is just {e}, okay.

And this illustrates the counting formula that we have done, so think for a moment that G is finite, remember that for any action we have, recall the counting formula .
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Counting formula says that order of the group is equal to number of elements of the orbit times the number of elements of the stabilizer, for a fixed elements, here again S is an arbitrary set and G acts on it. So remember the dynamics here, this left hand side is fixed, if for some element there is a large orbit, the stabilizer must be small, if the orbit is small the stabilizer must be big, because the product of these two is order of G. So large orbit, loosely speaking, this implies small stabilizer. Small orbit implies large stabilizer correct, small orbit means Os is small number of elements, so Gs must make up for it because the product is order of G, so very loosely speaking this what you this is a heuristic that you should remember.

In this case for the group action given by left multiplication, orbit is very big. As big as it can be, so stabilizer is very small, as small it can be, one element all elements, so that is what happens in this problem.

Let us do problem 2, do the same for the action of G on itself by conjugation when G is abelian. So let us assume in this problem that G is abelian, let us find out orbit decomposition and its stabilizers for this action. What are these? So now remember the action is given by gsg inverse, that is the action, conjugation action.
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So what is the orbit of an element? So orbit of an element, lets us fix s in S or s in G, what is the orbit? This is gsg inverse as g is inside G, but I am assuming G is abelian so that means gsg inverse is just s, so no matter what small g, this is just s. And what is the stabilizer? So here the reverse happens, as opposed to the pervious example problem. Here orbit is very small, orbit is as small it can be, it only consists of that element.

On the other hand, what is the stabilizer? This is g in G such that gsg inverse is s, things that fix s under the action gsg inverse is the action of small g and small s, that must be equal to s. But this is g in G such that s equals s, but there is no condition, in other words for all g in G this condition holds, so this is G.

So here orbits are small, stabilizer is big, stabilizer is as big as it can be, it is all of G. So orbit decomposition would be G would be union of singleton elements, earlier there is only one orbit in the previous example, so there is just one orbit, here there are as many orbits as there are elements of the group, so this the orbit

decomposition.
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Let us do problem 3. Show that so now again consider the action of G on itself by conjugation. So I am not going to assume now that G is abelian, G is an arbitrary group. Show that, two things, so let s be an element of G, show that stabilizer of s is the centralizer of s. I will recall in this problem what is the centralizer, so stabilizer of s is the centralizer of s and show that s belongs to the centre of G, I will also recall for you what is the centre of a group, if and only if stabilizer is all of G. So this the problem. So let us solve this, so I am asking you to show that the centralizer of s.
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So I am going to recall, let G be an aribtrary group, and let us take g in G, centralizer of g, which I have denoted by C of g I think, is all elements which commute with g, so a in G such that ag equals ga. Remember this is same as saying that a and g such that aga inverse is g. What is the centre of G, of the group G which I denoted by ZG, this is elements of the group that commute with everything else. This centre will always be contained in the centralizer of any element. So this is all elements such that aga inverse is a for all a in G, for all g in G, right.

So this is the centralizer and the center that we have defined earlier. Now for the action of conjugation, action of G on itself by conjugation show that the stabilizer is the centralizer, now what is the stabilizer? This is almost clear from the definition now, right, because Gs is by definition g in G such that gs g inverse is s, the stabilizer of s. But this is of course the centralizer as I have written here, if you just stare at the two things, it is all elements such that when you take the conjugate of g or this particular element, you get that element back, so I am of course confusing with the letters here, here am using a and g, here am using s and g but hopefully it's clear that stabilizer of an element s, is all g such that conjugate of s by g is equal to s, which is the centralizer of s. So this the first part of the problem.

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Now suppose, the second part of the problem is, an element is in the center if and only its stabilizer is G. Suppose s is in the center, if s is in the center, this means sx, rather I should write sg = gs for all g in G, this is the meaning of being in the center which is same as gsg inverse is s for all, these are all if and only if, right, they go back and forth. If s is in the center this happens, if this happens then s is in the center. This is clearly equivalent to this. But this means g belongs to stabilizer of s for all g in G, right. If gsg inverse is equal to s, that means the g is in the stabilizer so if this gsg inverse is s for all g then g is in the stabilizer for all g. Similarly if g is in the stabilizer, gsg inverse is s. But if g is in stabilizer for all g, that is equal to this, so s is in the stabilizer if and only if gs =g, this the second part of the problem, that also I have shown okay.

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So now let us look at a different action, so a different example. So let's look at problem 4. Here I am going to consider the action the group of invertible n by n matrices on the set of column vectors, here the action is given by A.V is AV, where AV is matrix multiplication okay. So here again find the orbit decomposition of R^N, and two find the stabilizer of the elementary vector E1, so that is (1, 0, ..,0), so two problems okay.

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Let us solve this find the orbit decomposition, so I have alluded to this when I did this example and did orbits, so let me quickly do this. First of all, if you take the 0 vector, the orbit of the 0 vector is just the 0 vector right, because any matrix times 0 vector is, this is a 0 vector, so this an orbit by itself. So this is one orbit. On the other hand, there is only one more orbit, because given any nonzero vectors V1 and V2, there exists an invertible matrix A such that AV1 is V2.

So this something that I said is a fact, that I don't want to prove this now. But you can check easily at least when N =2, so check this, I will leave this as an exercise for you for N = 2. It is very easy to check this because it is all 2 by 2 matrices and vectors of length 2, you can easily cook up a matrix which achieves this, given V1 and V2.

That means all nonzero vectors form an orbit, right. Is that clear? Because I am saying that any two vectors given any two nonzero vectors there is a matrix which sends one to the other, in other words any two nonzero vectors are equivalent.

So all nonzero vectors together form an orbit. So what is the orbit decomposition of RN? You have the 0 vector as one orbit, so there are, and all nonzero vectors as one orbit. So there are exactly two orbits for this action.

Note that all counting formula and so on don't make any sense here because the group is infinite and the set is infinite, so we cannot get anything from counting formula. However we can say that the set is a disjoint union of orbits, that always holds irrespective of the finiteness of the group or the set. And in this case the two orbits are 0 vector by itself and nonzero vectors by themselves. So this is the orbit decomposition modulo this fact which I will leave for you the verify, in general also it is not difficult and N =2 is especially easy.

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Now let us do the second part. What is the stabilizer of E1, so this is G(E1) is my notation for this. So these are all matrices in GL_N such that A E1 is E1. Let us work it out a little bit.

Let us write a matrix like this, it is an N by N matrix, so first entries, row are A11 and A12, A22 A2N finally the last row is

AN1 and AN2, ANN. And if you multiply this with E this is A, E is (1, 0, 0,…,0). What do you get?

If you are familiar with matrix multiplication, this is simply A11 times 1 + A12 times 0 and rest do not contribute so that is just A11. The second row times the vector here will give you A21 upto AN1, so this is the first row, first column rather, okay.
This is a property of matrix multiplication, when you multiply the matrix with the E1 elementary vector you get the first column. But what is the, if A is in the stabilizer then AE1 is E1. That means the first column of the matrix A must be (1, 0, 0,…,0), E1 itself. And there is no other condition, right. So stabilizer of E1 or G(E1) is matrices such that the first column of A is E1, okay.

More clearly, it would be all matrices of this form, the first column will be A and the rest is anything, this part is arbitrary. Of course, it must be something in GLN(R), right. So stabilizer of E1 is, because remember there is no condition on A12, A1N, A22, A2N, AN2, ANN, they do not contribute to the product at all, so they can be anything.

Only the condition is first column must be nonzero, must be E1, this is arbitrary, star means it can be anything, but in a way that this whole matrix forms an invertible matrix. So this is the stabilizer of E1. So this is the problem, right: we have shown that orbit decomposition of this is the 0 vector union nonzero vectors, and we have computed the stabilizer of E1.