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Introduction to Abstract Group Theory
Module 07
Lecture 35-"Orbits and stabilizers"

PROF.KRISHNA HANUMANTHU
CHENNAI MATHAMATICAL INSTITUTE

Okay, let's continue our study of group actions, in the last videos I have defined what it means for a group G to act on a set S. I have told you what orbit of an element is, orbit of an element is the, so you take an element of capital S, its orbit is, you apply all elements of the group to that fixed element, small s, and you consider the set.

Consider the set of where it travels, so orbit of s is collection of gs, where g varies over the group G, small g varies over the group G. And we saw several examples of group actions and in some of those examples we have seen orbits of specific elements. Okay, so today I am going to start with thinking of orbits as equivalence classes in the following way.
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Okay, so let us say our set-up is the following. So G is a group, S is a set, G acts on S right. This is our set-up always, a group acting on a set. Define an equivalence relation, first let us say a relation on S as follows: so I am going to define a relation denoted by this symbol as follows. I will say that s1 is related to s2, so we say s1 is related to s2, so remember we read this as “s1 is related to s2”, if there exists an element in the group G small g, such that gs1 is s2. Remember gs1 denotes the action of small g on small s1, so that is how we denote the action. So remember the action means there is a function from G cross S to S, which takes an element (g,s) to gs.

We just denote it simply by gs, instead of writing g star s, or g dot s, we, for convenience, denote the action by gs. So now this is the relation I am defining, s1 is related to s2, if they exists a group element such that gs1 is equal to gs2.

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So now the question is, is this an equivalence reaction?Remember an equivalence relation is a relation satisfying some conditions, right, what are they? We want every element to be related to itself, you want the relation to be reflexive.

So the first condition: is s related to s? Yes, because, why is s related to s? You can take es to be s, you have es is equal to s, this is by the definition of action, so you see that we need this, in order to make this an equivalence relation, we must have this. And this is guaranteed by the definition of group action itself. So identity element sends s to s, so s is related to s. In order for s1 to be related to s2, we want a group element such that gs1 is equal to gs2. We take the identity element of group which always exists in a group and es is equal to s, so s is related to s.

Suppose if s1 is related to s2, then is it true that s2 is related to

s1? This is the symmetry property right, if s1 is related to s2, is s2 related to s1, so let us see. If s1 related to s2, by definition, there exists g in G, such that by definition this happens, right?
If s1 is related to s2 there exists a group element such that gs1 is equal to gs2. Now let's apply the action properties, and let's in particular apply g inverse to both sides, right? If gs1 is equal to s2 remember these are elements of capital S, these are two elements of capital S that are equal to each other. Let's apply g inverse to both of them, then by definition of group action, this follows from as before definition of from group actions. What is the definition? Remember group actions must satisfy two condition, es is equal to s, and g1g2 of s is g1 of g2s.

These are properties of a group action these are the defining properties of a group action. So g inverse gs1 is same as g inverse g s1 but that is es1, g inverse g is e, but es1 again by the definition of action is s1, so s1 is equal to g inverse s2. So that means there is a group element such that that element times s2 is s1, so s2 is related to s1. Right, remember we say that s1 is related to s2, the first one is related to the second one, if a group element takes the first one to the second one, here s2 is taken by g inverse to s1, so s2 is related to s1.
So the question has an affirmative answer. If s1 is related to s2 then s2 is also related to s1.
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This is the symmetry. Finally we want to check transitivity of the group action. So suppose s1 is related to s2 and s2 is related to s3, so that means g1s1 is s2, g2s2 is s3, for suitable g1,g2 in capital G. Because s1 is related to s2, g1s1 is s2 for some group element, s2 is related to s3, so some group element takes s2 to s3.

Now again we apply g1, g2 to s1, by the definition of a group action, this is g1 of g2s1, but g2s1. Let's actually apply g2g1, g1 s1 by hypothesis s2, and g2 s2 by hypothesis again s3. So this implies s1 goes to s3 with a group element g2g1. So s1 is related to s3.
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So we conclude, we checked the three defining properties of an equivalence relation, so this is an equivalence relation. Right, so again let's recall what is the equivalence relation? It says that two elements are equivalent, if there is a group element that takes one to the other. So that is an equivalent relation I have checked so far.

But now I am going to use, the definition, the important property of equivalence relation, recall that, an equivalence relation in some sense the most important property for an equivalence relation is that it partitions the set S into disjoint equivalence classes. We take equivalence classes.

When I did equivalence relations and when we talked about cosets in a group, I did this in the detail, but so let us not repeat it again. But an equivalence relation partitions a set into disjoint equivalence classes, you take some element and you look at all elements that are equivalent to it. That is an equivalence class. And the most important property of equivalence class is that, two equivalence classes are either disjoint or they are identical to each other, right. They cannot be different and have something in common.

So, if you take disjoint distinct equivalence classes they are all disjoint and because every element is, its on equivalence, is in its own the equivalence class, equivalence classes certainly cover all of S. Okay, so now in this example of an equivalence relation, now that we have a group action on a set, let us take a small s in capital S, what is the equivalence class? Let us compute the equivalence class of s.

Okay, in the earlier situation we have denoted equivalence classes by this symbol, by definition what is this? These are all t in S such that s is related to t. All elements of the set such that s is related to t. But now I am working with a specific equivalence relation.

So, these are all t in S such that t is gs for some g in G. Right, remember this is the equivalence relation. We say that s1 is related to s2 if there is a g such that gs1 is equal to s2. So, if t is supposed to be related to s then gs must be t for some g. So, this is simply can be written as gs for all g in G, right. I can eliminate the variable t here or as small g varies in capital G, I will take small gs. But we have another name for that.
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From the last video, but this is simply the orbit of s right, so this is simply the orbit of s. So the equivalence class of an element is precisely, in the notation of the last week, of the last video, is the orbit of s, right, so we have orbit of s remember is exactly this set.

It is where s travels under the action of G, so equivalence classes for the relation that we have defined today, is simply the orbits. So we conclude hence, after all this, we have, we conclude that, so the set-up is if G is a group acting on a set S, then the set S is partitioned, you take the distinct orbits under the G-action on S. This is an elaborate sentence and it is the conclusion of what we have done so far.

If G is a group acting on a set S, then S is partitioned into distinct orbits by the action G-action S. So in other words, S is a union of orbits, so that we know, because every element is in its own orbit, and two distinct orbits are disjoint. This is because being related under G action is an equivalence relation so distinct orbits are disjoint.

So we have the picture as I have drawn earlier, so you have a set S, so you have Os1 that’s one orbit, Os2 in general an equivalence relation partitions a set. In our situation equivalence classes are orbits, okay, and so on.

This could be infinite of course, I am trying to just illustrate this, orbits, number of orbits may be infinite, but each individual piece in this partition is a single orbit, and they have nothing in the common, two distinct orbits have nothing in common.

This is the first important observation about group actions that we will have exploit a lot, so this is an important fact. So this is an important observation as we will see later, we will make use of this observation in the future videos. Okay, so now let me introduce another important notion, object attached to a group action, so let, now consider a group; I will come back to this partition of a set into its orbits later.
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But now for the moment, consider a group G acting on a set S. Again let me repeat you, repeat for you, G is a group acting on a set S, so G is an abstract group, it can be any group, and S is just a set. Even when S is equal to its group itself, it is really is just a set.

So for, let us fix a small s in capital S, I have defined for you the orbit of s, Os, now I will define something else, the stabilizer of s is the set of elements, is the subset of G, defined as, okay, so stabilizer is denoted by G sub s, remember we have orbit of s was denoted by Os.

Now I am defining stabilizer as G sub s or sometimes I will use stab of s just for clarity, this is a subset of G, unlike the orbit, orbit is a subset of the capital S. Stabilizer as in the definition it's a subset of, it is more than a subset actually, but we first define it as a subset. It is all group elements that stabilize okay, think of this as stabilizing s. So g does nothing to us, g fixes s, so this is the set of group element that fix s, okay.

So this is group elements that fix s. So the immediate Lemma that I will prove immediately Gs is not merely a subset of the group, it is a subgroup of G.

What is the proof? We have now done so many examples and results to check a group a subset is a subgroup, so let us quickly do this, this is very easy. We have to check that the identity is there and it is closed under multiplication and it is closed under taking inverses.

First property: e times s is s, by the definition of group action. So e fixes s, so e is in the stabilizer, stabilizer is all g such that gs are equal to s. So stabilizer contains the identity element. Let's say g1& g2 are in the stabilizer of s, that means g1s is s, and g2s is s, right. This is the definition of being in the stabilizer, now what is g1 g2 of s, this is, by the definition of group action, is equal to g1, g2 of s, which is equal to g1 of s, because g2 of s is s, g1 of s is also s.

So this implies that g1g2 is in Gs, right. If two elements are in the stabilizer, their product is in the stabilizer, that's good. So the stabilizer is closed under multiplication.
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Let's say, you take an element g in stabilizer, then what is gs? That is s, by the definition of being in the stabilizer. Now if you multiply both sides by g inverse, I get this, but this is same as g inverse g of s is g inverse s, but that is es, is equal to g inverse s, this means s is equal to g inverse s, right. But this means that g inverse is in the stabilizer of s, because g inverse also fixes s, g inverse s is s, so g inverse is in the stabilizer. So this is the proof that stabilizer is a subgroup of G.

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So I want to make this important note here. So given an action of G on S and an element small s in capital S, we have two things, we have orbit of s, which we denote by Os which is all gs as g varies over G, this is inside S obviously. And we have stabilizer of s that I sometimes denote by G sub s, sometimes by Stab s, this is all g in G such that gs is equal to s. This is a subgroup of G, right. This is the situation now.

So we have attached to a single element of capital S something called the orbit which is a subset of capital S, remember capital S is just a set, so we cannot talk about a subgroup or anything like that, it is just a subset. Stabilizer is a subgroup of capital G. So there are two things, so stabilizer and I will do, in a later video, examples of stabilizers in more details.