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Introduction to Abstract

Group Theory

Module 06

Lecture 34- "Examples of Group actions"

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Let us do one more example, so again this is another example of G acting on itself. This is also an extremely important example of a group acting on itself. So what is this? So this is called conjugation, action by conjugation. So, given g in G and s in G defined g*s, I am going to use star here to be very clear about the action, this is star refers to the action. Not the product insides the group.

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Earlier I could be slightly vague because action is already the product inside the group.

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Here, I defined it to be g*s is gsg inverse. So gsg inverse is called conjugation of s by g. So this is called the conjugation of s by g, this is the conjugate of s by g. So I should not write conjugation, it is called conjugate. The process of doing this is called conjugation. So is this is an operation? Is this is an action? Let us check this, is this an action? So you have e dot or, e*s, it is by definition, so the action is defined to be this: e s e inverse which is ese which is s. So that is okay. What about g1 g2 *s by definition this is same as g1g2 times s times g1g2 inverse. Right this is the action.

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But this is same as (g1g2) s g2 inverse g1 inverse right, because that is inverse in a group, product of g1g2 inverse is g2 inverse times g1 inverse. This is g1 g2 s g2 inverse, by associativity of group operation. And this is g1 (g2*s) g1 inverse, right, because g2*s is by definition g2sg2 inverse. But this is same as g1*(g2*s), because g1*something is g1 times that thing times g1 inverse, right, this is the operation. So g1* this middle thing, g1 times the middle thing times g1 inverse, that g1* that middle thing.

Right so this says that g1g2*s is g1*g2s. So * is an operation, is an action. Again what I am doing is not a formality okay; it is, there is some content in this. * is an action, the identity is okay, that is very clear but the associativity required a small proof, right.

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So this is called, this action of G on itself, is called the conjugation action. Okay, so the two actions that we have studied by G on itself, or the left multiplication and conjugation and both are extremely important okay, both are important because of the results that we can prove using these actions later okay, so this is the, for now let me stop with the examples here. There are some other examples that I will come back to later.

But now let us see some properties of group action. So now let us continue with our study of, as I said this is actually an easy notion, group actions, initially if you are seeing it for the first time it is somewhat strange. So it will take some time get used to, that is only difficulty the unfamiliarity is the difficulty.

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So please understand the examples very carefully, so I have given five examples. First one being, how rotations operate on the vertices of equilateral triangle. Then our familiar symmetric group acting on in indices. Invertible matrices acting on column vectors. And finally the 4^{th} and 5^{th} examples were a group acting on itself first by left multiplication and then by conjugation.

So these are the important examples. And if you understand all the examples carefully, it is also illustrates the point I made earlier that group action supposed to be abstract, it is not in some specific context. In any context we have a group and a set S, and a function from G cross S to S satisfying some conditions we have a group action. So now there are some obvious, some important things that we can attach to group actions and they are the following.

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So let G, so the general situation is, let G be a group acting on a set S. So G is an abstract arbitrary group. S is an abstract arbitrary set. Let s be in S, the orbit of s, denoted by os, orbit o for orbit, is all, so you take gs okay, see the word orbit, you know the word orbit, it is how an object moves around right, the orbit of earth around the sun for example. It is how earth moves around the sun, you connect all the points when it is travels and you get the orbit. The same idea works here.

You take a small s, orbit is always defined for an element of the set S, and see where it travels under the action of G. So you take one group element and see what is gs. So I am going to use gs, to denote the action. So, here we have G cross S to S. I will take a small g, small s and I will denote its image by gs. I will not use * or (.) just for simplicity. So, I take small s see wherever it travels under the action of capital G. So, small g, one element small g, take another element g prime take g double prime. You do this for every element small g and see where it travels and you see call that orbit of s. Orbit of s is of course a subset of capital S.

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So, orbit of s should be thought of as the set where small s travels under the action of G, okay. Now I should quickly check, in all the examples orbits. For example, okay, so the first example G was the group of rotations and S was the vertices. What is the orbit of s, of let us say of A? Now if you go back and see the orbit, action, A will always be in the orbit because, identity A will be A. And r1 A was B, r2 A was C.

So orbit of A in this case happens to be S. Similarly this is also orbit of B and orbit of C. In the second one G was the symmetric group, and S was indices 1 through N. What is the orbit of 1? 1 is there certainly because the identity 1 is 1, 2 is there because (1,2) sends 1 to 2, 3 is also there because (1, 2) applied to 1 is 2, (1,3) applied to 1 is 3. (1 i) in general applied to 1 is i. So in this case also orbit of 1 is all of S, and S you can see this is orbit of 2, orbit of 3, orbit of n. So they are all S.

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In the third example we have G=GLn(R), and S=R^n. If you take the zero vector, okay, in the previous two examples you might think that orbit is always equal to all of the set. But that is not necessarily the case. So what is orbit of, if you take any matrix, A and multiply by 0, you get 0 right, so orbit of 0 is just the element 0, nothing else is there. And now on the other end to take a vector, this is an exercise that you can do using properties of matrix multiplication. If V is R^n and V is different from 0 then the orbit of V is all of R^n minus 0.

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So this is to say that, in other words, given any nonzero vectors, v and w in, so I do not have a bar here, w in R^n there exists an invertible nXn matrix A such that A.v=w. So what I am saying is that, let us take this as a fact okay, this fact can be independently proved. It is not part of what I am now taking about so I do not want to spend time on this. But what does this fact mean for the orbit of a vector v is what I am interested in. Suppose this statement is true, given any to nonzero vectors v and w there exits an invertible matrix A such that A.v=w.

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Then this implies that orbit of v, remember in order to get this w has to be nonzero also is all nonzero vectors. In R^n in other words it R^n minus the element 0. Okay, because if you recall what is orbit of v, by definition it is A.v as A varies inside the group, in this case GLn. So you take A.v as A varies but because of this fact you give me any w in R^n minus 0, there is an A such that A.v=w. So every w will be in the orbit. So in this case there are two orbits: 0 vector is an orbit by itself, and all nonzero vectors are an orbit by itself, one orbit together.

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So, now G acts on G by left multiplication right. In this case what happens? So take an element s in G. What is the orbit of s? Let us calculate this, orbit of s is gs, as, because gs is just in this case action by left multiplication, so when I write gs I mean actually the multiplication in the group.

It is gs has g varies over G. We claim that orbit of s, so again it is s, okay, is all of G. Why? So the proof of this. So a priori this is a subject of G right, it is a subset of S which is of course G, orbit is a subset of S. In our example, it is G. Why is orbit of s equal to G? So take any t in G okay, let t be any element of G.

Can we find a group element g such that gs=t? See again let me remind you, orbit is the subset of capital S through which small s travels under the action of G. So s fixed, whenever we talk about orbit the small s which is an element of S is fixed. And you are applying all group elements to it. And what you get is the orbit. So now in order to prove that Os=G, take any element of G which I am calling t and we want to find a group element g such that gs=t.

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Of course we can find such an element. What is g? g is simply ts inverse right. If you take that, because here G is playing two avatars here remember. G is the group as well as the set. If you ts inverse time s. What do you get? This is ts inverse s which t. So every element, so this means that every t is in the orbit. So Os=G. So again the orbit is all of G.

And finally let us look at the last example, G acts on itself by conjugation. And this is more interesting okay, here the action remember, is g*s, I am going to now use * because here it is not multiplication of group, gsg inverse. Right, so recall. Now let us look at the orbit, for example orbit of e, Oe, is I take arbitrary elements of g and apply *e. This is by definition of the action g e g inverse, as g is in G. But geg inverse is e for all g, so geg inverse is e for all g in G. So this is just e. So orbit of e is e.

Now let us go back to the pervious example, I said orbit of every element is equal to G. And in fact I should probably spend a minute more on this. Just to illustrate this. What is orbit of e in this example? It is ge has g varies in G. This is just G, okay, so I have proved that the orbit of every element is G, so this checks out here, the orbit of e is G, in particular orbit of e is G. But whereas conjugation behaves very differently. Orbit of e is just e.

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Okay, now suppose for a moment that G is abelian. Then what is the orbit of, let us take an arbitrary element s in G. Then Os is by definition g*s, g in G. This is by definition gsg inverse g in G. I have assumed that G is abelian. If G is abelian this is same as sgg inverse g in G. But then sgg inverse is s right, this is true for all g in G. So this is just s, so orbit of s is s. So if the group G is abelian, orbit of an element is just that element. So these are the five examples, in the fifth example we get something interesting if the group is abelian.

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So now coming back to orbits, hopefully it is clear by these examples, if G is a group an arbitrary group acting on a set S, if you take a s in S, small s always contain s right. So orbits are never empty and they always contain the elements.

Why is this? Because remember Os is gs as g varies, so in particular, can take g to be e. So es which is s belongs to Os always okay, orbits contain small s but they could be just small s or they could be all S. So in this example of abelian group and action by conjugation the orbit of an element is just that element, nothing else, whereas in the action by left multiplication orbit is all of the group, all of the set.

In the action GLn(R) on R^n, orbit of the zero vector is just the zero vector, but orbit of any other nonzero vector is the set of all nonzero vectors. And in these two actions the action of rotations on the three vertices or the action of Sn on the indices 1 to n, orbits are the entire sets.

So orbits are at least going to contain small s, orbit of small s contains at least small s and it can vary in size depending on the action.

So we have seen examples of different kinds of orbits. So in the next videos we will study further about orbits, and other important properties of actions and using these properties we will prove our main theorems in the rest of the course. Thank you.

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