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Introduction to Abstract Group Theory

Module 06

Lecture 33- "Group actions"

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In this video I am going to start with a very important notion about group theory and this is the topic that will help us to prove the remaining important theorems in this course, the final goal being the Sylow theorems. So the important operation that I want to define here, introduce to you in this video, is called group actions.

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Okay, so if you have not seen this before, this might seem abstract and strange. So, I am going to define this carefully and do a few examples, so that you are comfortable with this notion, it is very simple notion.

So, let me first say that it is not difficult at all, it is a fairly easy thing to understand, so group action, what is the situation? So, the setup is the following.

So, we are going to fix a group which we will denote by G. Okay, and we will fix set which we will denote by S. So G is a group and S is a set. This set may depend will change according to the context. It could even be the group G itself sometimes, the most important examples that we will study S would be G itself, but the important point to remember is S is just a set, even if it has some other structure as far as the group action is concerned it is irrelevant.

So, we want to understand the meaning of G acting on S. So, I am going to define for you what is the meaning of G acting on S. So, we say, I am going to write the definition which is very intuitive and clear and explain it after writing it. We say G acts on S if there is a function, okay, from G cross S to S. Okay, think of it like this. What is G cross S? This is the Cartesian product. So recall G cross S is simply all elements like this.

This is just a set, there are pairs the first one coming from G and the second one coming from S. I denote the image by either gs or g.s, or g*s, okay depending on the context, it is not important, sometimes I will just write gs. May be I write g.s or g*s. But what you have to remember is,

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this function is, this function does what? It takes an element of G and element of S, so this is the element of G I am calling g, this element of S I am calling small s, so in this case I am going to write small s okay, so I will try to keep this small s, to denote an element of capital S, takes a element of small g of capital G, and element s to capital S and outputs an element of capital S denoted by gs, or g.s, or g*s, depending on the context.

So, this function is telling me, so this function is to be thought of as, so G acts on S meaning G, capital G acts on capital S means, you take a small element, element small s of S, small g of capital G, and you tell how small g acts on small s and produce a second element of the set S. But this function must have some properties, it is not an action in general. This function has, this function must have the following properties.

Really not has, should have, in order for a proper action we should have the following properties, two properties. One is, if you apply e, to small s you must get small s for all s in capital S, okay, e being identity element, remember I am supposed to tell you what is small g times small s, whatever it is e must map to s to s, and it must be associative in the following sense: if you should take g and g prime and apply to s that must be same as first applying g prime to s and then applying g to s. So, this must be true for all small s in capital S and small g and small g prime in capital G.

Okay, so we must have a notion of capital G acting on capital S, but this notion must have some basic properties like the identity element must act, must not do anything, so the identity element must take an element to itself and how you apply for two distinct elements of the group, whether you first multiply them and then apply to s, or first apply one of the elements to s and then the other element, you should get the same result. So this is the action, so this is to be thought of as a very abstract situation, G is abstract to capital S is abstract set meaning, it is not, it does not have any properties, and S is just a set, and you have a function with these properties.

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I am going to give you a some examples here, to illustrate this action and to illustrate the point that, this should work in various situations and different contexts this happens.

Okay consider, so first example. Consider an equilateral triangle. Let's take S to be vertices of it, and this is {A,B,C}. So the set is {A,B,C}. So for every action we must have a set and a group and the group I take to be the group of rotational symmetries.

So, if you recall it has two elements, three elements, R1 is rotational by 120°, R2 is the rotation by 240°, okay. So, G is this. I claim that, in the above sense, so whenever I say ta group acts on a set, I always mean in the sense that I wrote in the beginning of the video. Okay, so we must have a group and a set and a way to attach, compose if you want to think like this, a way to combine an element of the group and an element of set to produce another element of the set, satisfying the two properties that I wrote here, the identity element must fix every s and you must have associativity.

For example what is R1? So what is e dot A? Identity means the identity rotation. So, you do not change anything. So, this is A, E dot B is B, E dot C is C right. So this is, the first property is satisfied. What is R1 dot A? So R1 dot A, if you rotate, if you can imagine by rotating by 120°, R1 of A is B, R1 of B becomes C, R1 of C is a A. Similarly, R2 of A is C, if you rotate by 240° A goes to C, B goes to A and C goes to B, okay.

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So this gives you a function from G, to S to S, so we get a function satisfying the required two properties okay. So you can check that, the second property is also satisfied. For example what is R1 R2 of A? It is R1R2 of A. So first of all R1, R2 of A, can be computed in two ways, so R1, R2 is remember rotation by 120 + 240 so that is identity. So this is A, if you first do R1 and R2 then you get this, but now

let's do R1 of R2 A, what is R2 of A?

In the function you see that R2 of A is C. So, this is R1 of C, and this is R1 of C, R1 of C is A, so R1R2 of A is same as R1 of R2 of A, so this as you can check with other examples, this will, the associativity will hold. So just think for a second about this example, so remember my goal right now is to explain to you what is a group action on a set, the definition that I wrote it may not be completely clear, so I want to give you three, four examples at least to illustrate the definition.

So, the first example is this, and please pay close attention to this and all the subsequent examples and hopefully after these examples you will get a sense of what group actions are.

In this example we have rotations and we are only interested in the set of vertices of this equilateral triangle, given any element of the rotational symmetries, we apply it to the set of vertices and we get another vertex, so this complete description of the action is given here.

So, the group of rotations is acting on the set, so in some sense it is just permuting the vertices. Okay, if you think about this, this is related to the symmetric group because this is the way that symmetric group also acts on these three things. So, this is a group operation.

Let us look at one more example. S_n that we have studied in the past which is a symmetric group acts, on almost by definition, it acts on the set of indices, because how do you define σ? You take σ of an element of S_n and you take i so this is my S. Okay, and G is this. The set here is the set of indices and the group is the symmetric group. So, I take a σ in S_n and I take i in S. Then what is σ i, σ i is simply just σ of i, right.

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So, for example, you take G to be S3; it acts on, take G to be S4, it acts on S which is {1, 2, 3, 4}. If you take σ to be 1, 2, 3, let us say (1, 4, 3, 2), this is an S4. What does it do to 1? It sends it to 4. So, σ .1, this is a just a new notation but it is a same idea. The σ .2 is 1, σ.3 is 2, and σ.4 is 3.

Right, so σ, and remember the identity permutation fixes each i, right, this is the first condition and certainly σ 1, σ 2 apply to i, because σ 1σ 2 is a composition, is σ 1 applied to σ 2 of i. Okay, so this is the associativity. So, both properties are satisfied.

So, you can say S_n acts on the set S. Okay, so this is symmetric group acting on the set of indices from 1 to n is an important example of group actions.

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One more example. Let us take the group of invertible n by n matrices. So these are n by n invertible real matrices. Right, this is a group under multiplication that we have seen in the past and let us take S to be R^n, so these are all column vectors of length n. So, this is, for example, these are the things like so, this is in R^n where ai's are real numbers. How does this act? What is this action here? So, now I want to say $GL_n(R)$ acts on R^n in the following way.

Okay, so you take a matrix and you take a element. Let us have v in R^n. So, we need to tell what is A.v. But in this case there is a natural operation right, so we can simply take A times, so this is simply product of matrices. Right so this is an n by n matrix, this is an n by 1 matrix. So, you can multiply

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and what is the output? A.v is another vector or another element of R^n. Which is how it should be, right? If you take a group G and a set S, when you apply a group element to the set element you must get another set element. So, I have taken a group element A here, I have taken a set element v and I multiplied them to get another set element. And one can check that the two properties of group action are satisfied. What is the identity element? Of the group GL_n, it is the identity element, identity matrix.

And if you take identity times any vector any n by 1 column vector, it is just v obviously right, because identity times any matrix is itself. Similarly if you take AB times v, this is same as A times Bv. This is again because matrix multiplication is associative. So, we can say now that GL_n(R) acts on R^n. In all these things I will come back and do more, look at these examples again and again. At this point I am only giving you several examples of group operations. Okay we have so far looked at three examples. Now let us look at fourth example.

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Let G be any group and take the set also to be G. So, we will see how G acts on itself. So remember in general we talk about a group G, we will talk about how a group G acts on a set S. A group G acts on a set. In this example I am taking the group G acting on G itself, but the second copy of the group is really a set. When I am thinking of G as S the group operation of S is irrelevant. Okay, there are two ways.

Okay, let me first say I will define the action as follows. So let us take g in G and let us continue to use s in capital S but capital S is also again capital G. So I define g.s to be gs, just the multiplication.

So, this is the left multiplication by g. So, meaning you take g.s to be s multiplied by g on the left side. There is also a right multiplication that would be g.s to be defined as s times g, you multiply by g on the right hand side but now let us look at left multiplication.

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So, the function from G to S, G cross G to G to G, small g, small s goes to just the product. Okay, is this a group action? So this is a group action or not? This is a function right that is okay.

Now let us check the properties of group action. Remember a group action must satisfy this problem, e.s, what is e.s? Is by definition just es, in the left multiplication, so this is s. So this is okay. If you take g1g2.s, this is also okay because the multiplication in the group G is associative. So this is a group action. So in all these examples, please note that there is something I am doing. It is not just a formality.

In each example the property that I have to check is true because of some existing properties that you have learned. In the first example, because rotations are compositions, multiplication of rotations is a composition and composition of functions is associative, we can say this. It is not just a formality; it is not blind symbols I am writing here.

In the second example also product of permutation is by definition, composition of permutations which are functions. So composition is associative. So again it happens to be true, the associativity.

In the third example associatively holds because matrix multiplication is associative, that is an established fact from before. Okay, you have learned it in some other course or it is an established fact in some other course. So because of this existing known fact, these are true. Similarly here group multiplication is an associative operation by definition of a group.

So this is an existing fact which tells us that this is a group action. This action is called left multiplication of G on itself. Clearly enough right. This is an action of G on itself by left multiplication. This is an important action that we will see again in the course.

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