Introduction to Abstract Group Theory

Module 06

Lecture 32 – "Alternating groups"

PROF. KRISHNA HANUMANTHU

CHENNAI MATHEMATICAL INSTITUTE

So this notion is well defined, because any two representations have either even number of transpositions or odd number of transpositions. So, now let us observe that even permutation have an interesting property.

(Refer Slide Time: 00:30)

So, before that, let me quickly give you an example. So you have, let us take S3, consider actually it is (12) (32). so this is in S3. Right, you can immediately conclude that, σ is even. Because, it is a product of 2 transpositions. Right, as we said in the theorem, it is possible that it is also some other, maybe it is a product of some other number of transpositions. But they must also have an even number. So, σ is even. On the other hand, another example is, any transposition is odd. Right, because if you take (12) it requires exactly 1 transposition. So it must be odd.

(Refer Slide Time: 01:49)

What about a 3-cycle? I claim a 3-cycle is even. Why is this? So example if you take (123) right, let us try to write this as a product of 2-cycles. And we saw this yesterday, it is same as (12) (23). Is this right? 2 goes to 3, so 2 goes to 3, 3 goes to 2 and 2 goes 1, so 3 goes to1, 1 goes to 2. So, this implies, 1 (123) is even. Of course this is true of any 3-cycle. More generally, a K-cycle is even if K is odd. And it is odd, if K is even.

So, this is a confusing situation. But it is clear right? We saw earlier that a 2-cycle is dd. Because, 2-cycle is transposition. A 3-cycle is even okay, 3 is odd but 3-cycle is even. Why is this?

Because if you take a K-cycle, and we have written this as a product of transpositions earlier, and this turns out to be (i1 i2) (i2 i3) (ik-1 ik), right, and again let me repeat once more that if you have one representation, as a product of transpositions, that will allow you to determined whether the permutation is even or odd. Because, every other permutation, must have a same number, same parity, the number of permutation must have same parity.

Now how many transpositions are there? There are K-1 transpositions here. Right, because 1, 2, up to K-1. So this tells me that if K is odd, implies K-1, so, this is σ, if K is odd, K-1 is even implies σ is even. Because it requires K-1 transpositions. Of course, if this is if and only, or you can write it like this, K is even means K-1 is odd, so σ is odd. So, a K-cycle has, K-cycle is odd precisely when K is even.

(Refer Slide Time: 04:38)

So, we can define now, sometimes it is useful to talk of sign of a permutation. So sign of σ is either 1 if σ is even, -1 if σ is odd. Okay, so this is the sign of σ. So, sign is just 1 if it even -1 if it is odd.

So, let us come back to the even permutations and look at them a bit more carefully. So, we want to prove a proposition, product of two even permutations is even, inverse of an even permutation is even.

So, if you take a couple of even permutations and take their product, it is even. If you take an even permutation, and take its inverse is also even. Why is this? If σ1 and σ2 are even, that means σ1 can be written as a product of τ1 up to τr. Right, τk where K is even. Similarly τ2 σ2 can be written as some ρ1 up to ρt then t is even. What is σ1 σ2 now? This is τ1 τk, ρ1 to ρt. Right, this is a product of transpositions.

So, if k and t are even, then k+t is even. Right, that is all we need. If k and t are even k+t is even. This implies σ1 σ2 is even, because the product decomposition of σ1 σ2 requires an even number of permutations. This is even, this is even.

(Refer Slide Time: 06:56)

Now on the other hand if σ is even and say σ is σ1 up to σk. So, k is even. What is σ inverse? I claim, σ inverse is σk inverse, σ(k-1) inverse up to σ2 inverse σ1 inverse, right, this is clear, because in general, in any group, recall in any group G, (ab) inverse is b inverse a inverse. So, σ1 to σk whole inverse is, you take the inverses and multiply in the reverse order. σk inverse … σ1 inverse. Now, if σ1 is 2-cycle, σ1 inverse is actually σ1. Because remember (1 2) times (1 2) for example, is e. so this is σ_k. σ_{k-1} … σ_2 σ_1. So, these are all transpositions, σ inverse is also even. Because, it requires again k many transpositions which is even. So, σ_1 inverse is even. Right, if you have a product of two even permutation it is even. Inverse of an even permutation is even. So because this is proof of the proposition.

(Refer Slide Time: 08:31)

So, now let us define a subset A_n. These are {σ in S_n, σ is even}. So this the subset consisting of even permutations. Right, I am taking all the even permutations. By the proposition okay, so actually almost by the proposition, A_n is a subgroup of S_n. Let see what is a subgroup of a group? It must include the identity element. So I didn't write it in the proposition. But I should say it now. Certainly, what is the identity permutation? It is even right, because it requires 0 transpositions.

So, this is 0 transpositions. We do not need any transpositions to write it. So, identity certain even okay, right, if you think about it. Identity is a certainly even. And if you take two elements of A_n. The product is again in A_n. That was the proposition. If 2 permutation are even, the product is even, if you take something in A_n, its inverse is also in A_n because it is even.

(Refer Slide Time: 10:08)

So, A_n is called the "alternating group". And it is subgroup of S_n. Okay, now let me just quickly mention one fact about A_n. Namely, what is its order?

So, now I am going to introduce a homomorphism, consider the group homomorphism φ from Sn to {1,-1}. So before I tell you what φ is, just a word about what is {1, -1}. {1, -1} it is group right, this is a group under multiplication. So, if you wish this is a subgroup of nonzero rational numbers under multiplication, if you wish. But abstractly, it is a cycle group of order to. Okay, so what is the group homomorphism φ?

I will define φ(σ) to be sign (σ). Okay, if σ is odd, it goes to -1, if σ is even, it goes to 1. So, first all of, φ is a group homomorphism.

Right, this is because, φ is a group homomorphism because of the proposition. That we just did above. Right, because if φ(σ1 σ2), so almost by the proposition actually, we need to be careful. So, what is σ1 σ2. So this is sign σ1 σ2, right. Which is another way of saying, you write σ1 as a product of transpositions. And see how many there are. If it is odd sign is -1. If it is even sign is 1.

But by the same idea that we used in the proposition, we can first write σ1 is a product of transpositions. And then write σ2 as a product of transpositions. Put them together. If σ1 is I think I used τ1 … τ k, σ2 is ρ1. … ρt, then σ1 σ2 is right, now if k is odd, t is odd, k, t are both odd.

First let's start with k, t are both even. Then k+t is even. k, t are both odd. Then k+t is even right, two odd numbers add up to an even number. One odd, one even right, exactly one of the 2 numbers k and t is odd. The other is even. Then k+t is odd. This is the proof of this.

Because sign of σ1 is determined by whether k is odd or even. Sign of σ2 is determined by whether t is odd or even. So in this case you have sign σ1 and sign σ2 are both 1 and the product is also 1. In this case we have sign of σ1 and sign of σ2 are both -1. So, -1 times -1 is 1, which is the sign of σ1 σ2, because the σ1 σ2 is even. And here one is even, one is odd. So, it is 1 times minus 1, which is -1.

(Refer Slide Time: 14:29)

Okay, so φ is a group homomorphism. I am going fast here may be, but I hope this is clear enough. So, φ is a group homomorphism. What are the other properties of φ.

So φ is onto. Right, because certainly both, because Sn contains even permutations as well as odd permutations. Right, because it contains both even and odd permutations. Clearly it is onto. Some even things, for example, 2-cycles are odd 3-cycles are even. And let us take, so here I am taking n at least. So I am going to let us say n is at least, in all these considerations n is at least 2. If I take n=1 it is not interesting. So always assume, because if you take n=1, then there is only one permutation that is even.

So, now I claim that kernel of φ. What is kernel of φ? These are the things that go to the identity element of {1, -1}. What is identity element of {1, -1}? It is 1. So, all permutations which go to 1. In other word, all permutation sthat are even. So, this the precisely An. Right kernel of φ is An.

(Refer Slide Time: 16:11)

This is because An is precisely the set of even permutations. Now the first isomorphism theorem says, Sn mod An is isomorphic to {1, -1}. So, in particular, from the counting formula, says that |Sn| is equal to |An| times cardinality of this group, which is 2. Okay, and we know this is n factorial.

(Refer Slide Time: 16:59)

So, we have n factorial equals 2|An|. So this implies |An| is n factorial by 2. So, upshot of this is, exactly half of the elements of Sn are even the other half are odd. So, of the n factorial many elements of Sn, half are even, half are odd. And even permutation is are nice because they form a group. So, now it is just a quick check for you. Can you also take odd permutations and see i they form a group.

(Refer Slide Time: 17:57)

So, a quick question for you. Do odd permutations also form a group? The answer to this is an oblivious no. For several reasons, first identity is not there right, because identity is even. So identity is not odd. This is one reason, that is good enough to conclude that they are not a group.

But also remember if you take (12) and (34) are both odd. But the product is even. So this amounts to saying that sum of two odd numbers is even. So just like even numbers in integers form a subgroup but not odd integers, similarly even permutations form a subgroup, but odd permutation do not.

Okay, so this the end of my video about symmetric groups. So, just for recap what we have done is, what we have learned how to use cycle notation for permutations. We learned how to write a given permutation as a product of disjoint cycles. Using that we can figure out the order, the order of a K-cycle is K and order of a product of disjoint K-cycles is the LCM of the orders of those disjoint cycles.

Then we also saw that any permutation can be written as a product of the transpositions, not necessarily disjoint and not necessarily unique, but we proved a big theorem saying that the number of transpositions required is always even or always odd. Using this we have defined the notion of odd and even permutations and we saw that even permutations form of subgroup of the symmetric group. We called it the alternating group An. And we finally saw that order of An is exactly half of the order of Sn. So, An has order n factorial divided by 2. That means of the n factorial permutations in Sn, exactly half are even, the other half are the odd. Thank you.

Online Editing and Post Production

Karthik

Ravichandran

Mohanarangan

Sribalaji

Komathi

Vignesh

Mahesh Kumar

Web Studio Team

Anitha

Bharathi

Clifford

Deepthi

Dhivya

Divya

Gayathri

Gokulsekar

Halid

Heamvathy

Jagadeeshwaran

Jayanthi

Kamala

Lakshmipriya

Libin

Madhu

Maria Neeta

Mohana

Mohana Sundari

Muralikrishnan

Nivetha

Parkavi

Poonkuzhale

Poornika

Premkumar

Ragavi

Raja

Renuka

Saravanan

Sathya

Shirley

Subash

Suriyaprakash

Vinothini

Executive producer

Kannan Krishnamurty

NPTEL Co-ordinators

Prof.Andrew Thangaraj

Prof. Parthap Haridoss

IIT Madras Production

Funded by

Department of Higher Education

Ministry of Human Resource Development

Government of India

HYPERLINK "http://www.nptel.ac.in" www.nptel.ac.in