NPTEL

NPTEL ONLINE COURSE

Introduction to Abstract

Group Theory

Module 06

Lecture 31 – "Odd and even permutations II"

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Now I want to understand what σ star does when σ is a transposition, so now we want to understand,

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when σ is a permutation, a transposition. So let me remind you our goal in this video was to prove the theorem that I stated some time back, so maybe it is good time to recall the theorem.

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The goal of theorem is, the goal of video, is prove this theorem. If our permutation rho, and you have two different representations as a product of transposition, they are both, σ's are 2-cycles and tau's are 2-cycles then k and t are both even or both odd.

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And in order to this, let me recap what we done so far, we have defined a polynomial we have defined what is the modification of that polynomial via an element of the symmetry group, σ*f and I looked at various examples to explain what to σ*f, and then I commented that when I take the product of two permutations and apply * it is same as first applying first one,

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and to apply the second one to the resulting polynomial, σ* then apply to tau*f, okay, and I proved that. And hopefully it is clear to you as we have seen in several examples.

Now I want to understand what is σ*f, when σ is a permutation. Again we have seen this in our examples we got σ*f was – f, if recall σ *f was –f, whenever we in our examples σ was a 2-cycle, for example, in this example is σ is (13),

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σ *f is –f, okay. Is this is always true? That if you have a transposition,

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then its * is – f? Yes, this is always true. I am going to prove this, but let me first do an easy case of this first, and hopefully that you will be convinced, then I will give you a more general proof.

So first of all take a special case, as a special case consider σ is (1,2) and n is general here. So what is σ of F σ* of F? σ * of F remember what it is. So F remember, first of all recall what is F, in general now, f is (X1- X2), (X1-X3), …(X1 – Xn) then have (X2-X3)… (X2 – X n), and then you have a other terms.

I am only interested terms which involves X1, other terms, right, is this clear? We have X1 – X2, X1- X3, X1- Xn then X2- X3, X2- Xn, then X3 – X4, X3- X5 and so on, but remaining terms do not involve 1 and 2 right. So I do not care about that.

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So what is σ * f? σ is (1,2). I am only interested to (1, 2). σ * of F will be all, these first few terms will change now. So it will be X2 – X1, right then X2 – X3, up to X2 – X1, X2 – Xn, because the first term 1 and 2 in are interchanged.

So it becomes X2 – X1, second term 1 becomes 2 so X1 becomes X 2 and X3 is fixed so X2 – X3, X2 – X4, and then X1-Xn so that X2 – Xn. What happens to the terms involves X2?

X2 becomes X1 right, so it is X1 – X3, X1 – X4, all the way up to X1 – Xn, and other terms, other terms are fixed because other terms are terms whose subscripts are different from 1 and 2. And σ does not change them, so they remain the same, right. Other terms remain the same.

So now what is the change now? X1 – X2 became X2 – X1 so there is 1 minus sign, right, one interchange of the variables, order, so X1- X2 became X2 - X1. But if you now look at this no other interchanges happened. We just reordered them, X1–X3 comes here, X1 – X4 comes here, X1 – Xn comes here, X2 – X3 comes here, X2 – X4, X2 – Xn so all other terms, there is only one change of sign,

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right so this is – f. So remember I am claiming

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that if σ is a transposition σ * f is – f, and I have proved it at least in the special case that you have (1, 2) okay. So σ*f is – f.

Okay, now I am going to spend next the five minutes proving this general and you can, hopefully you have understood what happened in our examples and in this special case. This general thing is important to prove, but if you are not, I mean it is more important that you are convinced that is true based on these examples, okay. Let me now prove in more general cases.

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In general, we have σ * f = - f, if σ is a transposition, okay. This is what I want to show. σ*f is – f, so I am going to assume, let σ, it is a transposition right, in other words, it is a 2-cycle. So let us say σ is (i j) with i less than j. I can always assume that because (i j) is same as (j i), so i and j are different obviously, so I can put the smaller one first and the bigger one in the second position. It is a cycle so (i j) is same as (j i) so I can always arrange them so that the first one is the smaller one.

And now let us investigate what is σ * f, okay. So let us investigate what is σ * f? This involves, this requires us to investigate what is σ * of a particular term in f okay, so consider.

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a term of the form X u - X v right we have u < v remember these are, because I used i and j denote σ, I have now used u and v to denote a term, this is a typical term, right.

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F remember consists of products X u – X v where the first index is strictly smaller than the second index. In order to determine what is σ * f, I need to see what it does to a term like this. Really what we need to do is we have to figure out, to find σ * f, we have to determine how many terms of the form change sign, right, under σ, right, because if you know how many terms change the sign, then we know what happens to F. If the number of changes is even you have σ * f will be f, if the number of changes is odd σ * f will be – f. So when does, the question is, so the question we want to address is, the question is when does this change sign.

So when does X u – X v, change always under σ to X a – X b, with b strictly less than a, right, when does this change sign when you have a bigger thing minus a smaller thing. You start with a smaller thing – bigger thing, so here of course use u is less than v. You only start with such a term, u less than v, and see when it changes to b less than a, the second term less than the the first term. So in order to address this let us consider various cases.

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If u and v do not belong to the set i, j, if u is different from i and j and v is different from i and j, X u – X v does not change. That is clear enough right, because if σ, what is, so remember σ * of X u – X v is X σ u – X σ v. So we are now focused on when is σ u, for what σ, σ u is strictly bigger then σ v, so we are interested in, for which σu becomes strictly more than σv right, the first index strictly bigger than second index.

The trivial case would be if u an v different from i and J then σ u is u, σ v is v right, because if σ is (i j), if u and v both different from i and j, σ of u is u σ v is v so this does not happen. Now let us consider the other cases. So we have the remaining cases are the following.

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Case 1: so we can have u=i, and v=j, so remember u is less than v, and i is strictly less than v, so you can u=i, v=j then what is σ * of X u – X v, this is X σ u- Xσ v, but u is i, so this is X v- X u. So this is one case were σ u is of course bigger than σ v, so in this case the sign changes. So this is one situation where sign changes.

Let us take case 2, let take u to be i and j is something else okay, so v is something else. So if you take u to be i.

So what we have is, let's see, in this case, see remember I have already dealt with the case that u is different from i and j, v is different from i and j. So one of them must be u, and one of the must be j. If one of them is i the other is j both happen then that is case one. Other cases u=i, and v is not equal to j, let's say, okay.

In this case what do we have? We have u=i, which is less than, we have two cases, so let us suppose that, let us compare v and j.

So if v is less than j, so you have u=i right, so this is a sub-case of case 2. In this case what happens to X u – X v under σ *, σ * of X u – X v, this becomes, this is X of σ u – X of σ v. But u is i so this is x of σ i which is j, and σ v, remember, v is different from i and j so this is just X v right. In this case a change of sign happens.

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